HAMPATH $\leq_p$ UHAMPATH – sketch of proof

Input: <directed G=(V,E),s,t>
Output <undirected G',s_{in},t_{out}>

Construction:

1. For every vertex $v \in V$ generate 3 vertices and 2 edges: v_{in}, v_{mid}, v_{out}, $(v_{in}, v_{mid}), (v_{mid}, v_{out})$
2. For every edge $(u,v) \in E$ generate an edge (u_{out}, v_{in})

The reduction is polynomial: <explanation>

Correctness:

➔ If <G,s,t>∈HAMPATH then the following directed hamiltonian path exists
$s \rightarrow v_1 \rightarrow \dots \rightarrow v_{|V|-2} \rightarrow t$
From the construction we can see that the following undirected path exists in G'
$S_{in}\text{-}s_{mid}\text{-}s_{out}\text{-}v_{1in}\text{-} \dots \text{-}v_{|V|-2out}\text{-}t_{in}\text{-}t_{mid}\text{-}t_{out}$
Since the path is simple, contains all the vertices in G', starts with s_{in} and ends with t_{out} it is a Hamilotnan path in G'. Therefore <G',s_{in},t_{out}>∈UHAMPATH

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Lemma: A Hamiltonian path that starts at s_{in} and ends at t_{out} does not contain an edge (v_{in}, u_{out})
Proof: By contradiction. Assume such an edge (v_{in}, u_{out}) exists and it's the first of that form in the Hamiltonian path. We have two options:

1. we already visited v_{mid} but this means we reached v_{mid} from v_{out} (since the path is Hamiltonian and we reached v_{out} from some vertex z_{in} – contradicting the fact that (v_{in}, u_{out}) is the first of that form
2. We will visit v_{mid} – but then we will reach v_{mid} from v_{out} and will not be able to proceed to t_{out}, contradicting the fact that the path is Hamiltonian.

Corollary: Since we start from s_{in}, we must traverse any triplet u_{in}, u_{mid}, u_{out} in a Hamiltonian path in this exact order.

So if <G',s_{in},t_{out}>∈UHAMPATH, the Hamiltonian path must be of the form:
$S_{in}\text{-}s_{mid}\text{-}s_{out}\text{-}v_{1in}\text{-} \dots \text{-}v_{|V|-2out}\text{-}t_{in}\text{-}t_{mid}\text{-}t_{out}$
From the construction it's easy to see that that this path can be easily mapped to a path in G:
$s \rightarrow v_1 \rightarrow \dots \rightarrow v_{|V|-2} \rightarrow t$
This is a Hamiltonian path in G and <G,s,t>∈HAMPATH

QED