

# Testing Juntas

Eldar Fischer, Guy Kindler, Dana Ron, Shmuel Safra, Alex Samorodnitsky

Sublinear Time Algorithms Seminar

January 20, 2009

Reut Levi

## 1 Preliminaries

- Juntas
- Variation

## 2 Size Test

- Algorithm
- Soundness

## 3 Adaptive Test

- Algorithm
- Soundness

## Definition (Juntas, dominating sets)

A boolean function  $f : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  is called a *J-junta* if there exists a set  $\mathcal{J} \subseteq [n]$  of size at most  $J$ , such that  $f(x) = f(y)$  for every two assignments  $x, y \in \mathcal{P}([n])$  that agree on  $\mathcal{J}$ , namely that satisfy  $x_i = y_i$  for all  $i \in \mathcal{J}$ . In this case it is said that  $f$  is *dominated by*  $\mathcal{J}$ .

## Definition (Juntas, dominating sets)

A boolean function  $f : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  is called a *J-junta* if there exists a set  $\mathcal{J} \subseteq [n]$  of size at most  $J$ , such that  $f(x) = f(y)$  for every two assignments  $x, y \in \mathcal{P}([n])$  that agree on  $\mathcal{J}$ , namely that satisfy  $x_i = y_i$  for all  $i \in \mathcal{J}$ . In this case it is said that  $f$  is *dominated by*  $\mathcal{J}$ .

## Informally

A  $J$ -junta is a function that depends on  $\leq J$  variables.

## Definition (Juntas, dominating sets)

A boolean function  $f : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  is called a  $J$ -junta if there exists a set  $\mathcal{J} \subseteq [n]$  of size at most  $J$ , such that  $f(x) = f(y)$  for every two assignments  $x, y \in \mathcal{P}([n])$  that agree on  $\mathcal{J}$ , namely that satisfy  $x_i = y_i$  for all  $i \in \mathcal{J}$ . In this case it is said that  $f$  is *dominated by*  $\mathcal{J}$ .

## Informally

A  $J$ -junta is a function that depends on  $\leq J$  variables.

## Example (Dictatorship)

Dictatorship is a 1-junta.

## Definition $((\epsilon, J)$ -juntas)

Let  $f : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  be a boolean function,  $f$  is said to be an  $((\epsilon, J)$ -junta if there exists a boolean  $J$ -junta  $g : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  s.t:

$$\Pr_{x \in \mathcal{P}([n])}[f(x) = g(x)] \geq 1 - \epsilon.$$

## Definition $((\epsilon, J)$ -juntas)

Let  $f : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  be a boolean function,  $f$  is said to be an  $(\epsilon, J)$ -junta if there exists a boolean  $J$ -junta  $g : \mathcal{P}([n]) \rightarrow \{-1, 1\}$  s.t:

$$\Pr_{x \in \mathcal{P}([n])}[f(x) = g(x)] \geq 1 - \epsilon.$$

## Example $((\epsilon, 1)$ -juntas)

$$f(x_1, \dots, x_n) = x_1 \vee (x_2 \wedge \dots \wedge x_J)$$

$f$  depends on  $x_1, \dots, x_J \Rightarrow f$  is a  $J$ -junta.

If  $J > \log \frac{1}{\epsilon} \Rightarrow f$  is an  $(\epsilon, 1)$ -junta since  $f$  depends on  $x_2 \wedge \dots \wedge x_J$  w.p  $< \epsilon$ .

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Example

$$I = \{1, 3\}, w = (0, 0)$$

$$H = \{2, 4\}, z = (1, 1)$$

$$w \sqcup z = (0, 1, 0, 1)$$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Definition (variation of boolean function)

$$Vr_f(I) \stackrel{\text{def}}{=} 2Pr_{\substack{w \in \mathcal{P}([n] \setminus I) \\ z_1, z_2 \in \mathcal{P}(I)}}} [f(w \sqcup z_1) \neq f(w \sqcup z_2)]$$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Definition (variation of boolean function)

$$Vr_f(I) \stackrel{\text{def}}{=} 2Pr_{\substack{w \in \mathcal{P}([n] \setminus I) \\ z_1, z_2 \in \mathcal{P}(I)}}} [f(w \sqcup z_1) \neq f(w \sqcup z_2)]$$

## Example (variation of Dictatorship)

$$f(x_1, x_2) = x_1$$

$$Vr_f(\{1\}) = Vr_f(\{1, 2\}) = 2Pr_{z_1, z_2 \in \{0,1\}} (z_1 \neq z_2) = 1$$

$$Vr_f(\{2\}) = 0$$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Definition (variation of boolean function)

$$Vr_f(I) \stackrel{\text{def}}{=} 2Pr_{\substack{w \in \mathcal{P}([n] \setminus I) \\ z_1, z_2 \in \mathcal{P}(I)}} [f(w \sqcup z_1) \neq f(w \sqcup z_2)]$$

## Variation Properties

Monotonicity:  $Vr_f(B) \leq Vr_f(A \cup B)$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Definition (variation of boolean function)

$$Vr_f(I) \stackrel{\text{def}}{=} 2Pr_{\substack{w \in \mathcal{P}([n] \setminus I) \\ z_1, z_2 \in \mathcal{P}(I)}}} [f(w \sqcup z_1) \neq f(w \sqcup z_2)]$$

## Variation Properties

Monotonicity:  $Vr_f(B) \leq Vr_f(A \cup B)$

Sub-additivity:  $Vr_f(A \cup B) \leq Vr_f(A) + Vr_f(B)$

## Notation

If  $w \in \mathcal{P}(I)$  and  $z \in \mathcal{P}(H)$  are two partial assignments, and  $I$  and  $H$  are disjoint, let  $w \sqcup z \in \mathcal{P}(I \cup H)$  denote the partial assignment whose  $i$ th coordinate is  $w_i$  if  $i \in I$ , and  $z_i$  if  $i \in H$ .

## Definition (variation of real valued function)

$$Vr_f(I) \stackrel{\text{def}}{=} E_{w \in \mathcal{P}([n] \setminus I)} [V_{z \in \mathcal{P}(I)} [f(w \sqcup z)]]$$

## Definition (variation of boolean function)

$$Vr_f(I) \stackrel{\text{def}}{=} 2Pr_{\substack{w \in \mathcal{P}([n] \setminus I) \\ z_1, z_2 \in \mathcal{P}(I)}}} [f(w \sqcup z_1) \neq f(w \sqcup z_2)]$$

## Variation Properties

Monotonicity:  $Vr_f(B) \leq Vr_f(A \cup B)$

Sub-additivity:  $Vr_f(A \cup B) \leq Vr_f(A) + Vr_f(B)$

Sub-modularity:  $Vr_f(A \cup B) - Vr_f(B) \geq Vr_f(A \cup B \cup C) - Vr_f(B \cup C)$

## The Independence Test

Choose a random  $w \in \mathcal{P}([n] \setminus I)$ , and choose  $z_1, z_2 \in \mathcal{P}(I)$ .  
Verify that  $f(w \sqcup z_1) = f(w \sqcup z_2)$ .

## The Independence Test

Choose a random  $w \in \mathcal{P}([n] \setminus I)$ , and choose  $z_1, z_2 \in \mathcal{P}(I)$ .  
Verify that  $f(w \sqcup z_1) = f(w \sqcup z_2)$ .

### Notice

$f$  passes the Independence Test w.p  $1 - Vr_f(I)/2$

## The Independence Test

Choose a random  $w \in \mathcal{P}([n] \setminus I)$ , and choose  $z_1, z_2 \in \mathcal{P}(I)$ .  
Verify that  $f(w \sqcup z_1) = f(w \sqcup z_2)$ .

### Notice

$f$  passes the Independence Test w.p  $1 - Vr_f(I)/2$

### Example

$$f(x_1, \dots, x_n) = x_1$$

$f$  passes the independence w.p 1 if  $x_1 \notin I$  and w.p  $\frac{1}{2}$  otherwise.

## Birthday Problem

There are  $n$  people in a room. What is the probability that all  $n$  birthdays are different?

## Birthday Problem

There are  $n$  people in a room. What is the probability that all  $n$  birthdays are different?

$$1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right), n < 365$$

## Birthday Problem

There are  $n$  people in a room. What is the probability that all  $n$  birthdays are different?

$$1 \times \left(1 - \frac{1}{365}\right) \times \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right), n < 365$$

## Buckets

We would like to divide  $n$  balls with  $J$  special balls into buckets without special balls collisions. If number of buckets is  $O(J^2)$  collision is very unlikely.

## The Size Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

For  $j : 1 \dots r$

    Apply  $h$  iterations of the independence test to  $I_j$ .

The test rejects if  $f$  is dependent on more than  $J$  subsets.

## The Size Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

For  $j : 1 \dots r$

    Apply  $h$  iterations of the independence test to  $I_j$ .

The test rejects if  $f$  is dependent on more than  $J$  subsets.

## Query Complexity

We set  $r \stackrel{\text{def}}{=} 16J^2$  and  $h \stackrel{\text{def}}{=} 4er(\ln(J+1) + 2)/\epsilon$ .

Overall  $2rh = \theta(J^4 \ln(J+1)/\epsilon)$  queries.

## The Size Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

For  $j : 1 \dots r$

Apply  $h$  iterations of the independence test to  $I_j$ .

The test rejects if  $f$  is dependent on more than  $J$  subsets.

## Query Complexity

We set  $r \stackrel{\text{def}}{=} 16J^2$  and  $h \stackrel{\text{def}}{=} 4er(\ln(J+1) + 2)/\epsilon$ .

Overall  $2rh = \theta(J^4 \ln(J+1)/\epsilon)$  queries.

## Completeness & Soundness

Completeness - perfect.

Soundness -  $f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.

## Theorem (Size Test Soundness)

*$f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

## Theorem (Size Test Soundness)

*$f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

## Proof Outline

## Theorem (Size Test Soundness)

*$f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

## Proof Outline

- Let  $\mathcal{J}$  be the set of coordinates on which  $f$  has variation larger than some threshold  $t$ :  $\mathcal{J} = \{i \mid \text{Var}_f(i) \geq t\}$ , where  $t \stackrel{\text{def}}{=} \frac{\epsilon}{2er}$

## Theorem (Size Test Soundness)

*$f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

### Proof Outline

- Let  $\mathcal{J}$  be the set of coordinates on which  $f$  has variation larger than some threshold  $t$ :  $\mathcal{J} = \{i \mid \text{Var}_f(i) \geq t\}$ , where  $t \stackrel{\text{def}}{=} \frac{\epsilon}{2er}$
- Show  $f$  passes the test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$

## Theorem (Size Test Soundness)

*f passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

### Proof Outline

- Let  $\mathcal{J}$  be the set of coordinates on which  $f$  has variation larger than some threshold  $t$ :  $\mathcal{J} = \{i \mid \text{Var}_f(i) \geq t\}$ , where  $t \stackrel{\text{def}}{=} \frac{\epsilon}{2er}$
- Show  $f$  passes the test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$
- Show  $f$  passes the test w.p  $\geq 1/2 \Rightarrow \text{Var}_f(\overline{\mathcal{J}}) < 2\epsilon$

## Theorem (Size Test Soundness)

*$f$  passes the test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.*

### Proof Outline

- Let  $\mathcal{J}$  be the set of coordinates on which  $f$  has variation larger than some threshold  $t$ :  $\mathcal{J} = \{i \mid Vr_f(i) \geq t\}$ , where  $t \stackrel{\text{def}}{=} \frac{\epsilon}{2er}$
- Show  $f$  passes the test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$
- Show  $f$  passes the test w.p  $\geq 1/2 \Rightarrow Vr_f(\overline{\mathcal{J}}) < 2\epsilon$
- Show  $Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$*

## Proposition

*f* passes the test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$

## Proof Idea

Show that otherwise w.h.p there are **many subsets** that **intersect**  $\mathcal{J}$  and therefore have **high variation**

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$*

## Proof.

Assume on the contrary  $|\mathcal{J}| > J$ :

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$*

## Proof.

Assume on the contrary  $|\mathcal{J}| > J$ :

- $Pr(\text{test not detects } I_j \mid (I_j \cap \mathcal{J}) \neq \Phi) \leq (1 - t/2)^h$

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$*

## Proof.

Assume on the contrary  $|\mathcal{J}| > J$ :

- $\Pr(\text{test not detects } I_j \mid (I_j \cap \mathcal{J}) \neq \Phi) \leq (1 - t/2)^h$
- Since  $r = 16J^2$ ,  $\Pr(\#\{I_j \mid (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > 3/4$

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$*

## Proof.

Assume on the contrary  $|\mathcal{J}| > J$ :

- $\Pr(\text{test not detects } I_j \mid (I_j \cap \mathcal{J}) \neq \Phi) \leq (1 - t/2)^h$
- Since  $r = 16J^2$ ,  $\Pr(\#\{I_j \mid (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > 3/4$
- $\Pr(\text{test detects } < J + 1 \text{ subsets} \mid \#\{I_j \mid (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1)$   
 $\leq (J + 1)(1 - t/2)^h \leq (J + 1)e^{-\ln(J+1)-2} = e^{-2} < 1/4$

## Proposition

*f* passes the test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$

## Proof.

Assume on the contrary  $|\mathcal{J}| > J$ :

- $Pr(\text{test not detects } I_j \mid (I_j \cap \mathcal{J}) \neq \Phi) \leq (1 - t/2)^h$
- Since  $r = 16J^2$ ,  $Pr(\#\{I_j \mid (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > 3/4$
- $Pr(\text{test detects } < J + 1 \text{ subsets} \mid \#\{I_j \mid (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1)$   
 $\leq (J + 1)(1 - t/2)^h \leq (J + 1)e^{-\ln(J+1)-2} = e^{-2} < 1/4$
- By union bound *f* passes the test w.p  $< 1/2$



## Proposition

$Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proposition

$Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{D}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

## Proposition

$Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

## Proposition

$Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,  
 $2Pr_x[f(x) = h(x)] - 1$

## Proposition

$Vr_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,  
 $2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x))$

## Proposition

$\forall r_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

$$2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$$

## Proposition

$\forall r_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,  
 $2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$   
 $= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}]$

## Proposition

$\forall r_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,  
 $2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$   
 $= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}] \geq E_y[(E_z[f(y \sqcup z)])^2]$

## Proposition

$\forall f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

$$2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$$

$$= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}] \geq E_y[(E_z[f(y \sqcup z)])^2]$$

$$= E_y[1 - \underbrace{E_z[(f(y \sqcup z))^2]}_1 + (E_z[f(y \sqcup z)])^2]$$

## Proposition

$\forall f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

$$\begin{aligned}
 2Pr_x[f(x) = h(x)] - 1 &= E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]] \\
 &= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]|}] \geq E_y[(E_z[f(y \sqcup z)])^2] \\
 &= E_y[1 - \underbrace{E_z[(f(y \sqcup z))^2]}_1 + (E_z[f(y \sqcup z)])^2] \\
 &= E_y[1 - V_z[f(y \sqcup z)]]
 \end{aligned}$$

## Proposition

$\forall f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

$$\begin{aligned}
 &\text{Let } x \in \mathcal{P}([n]), y \in \mathcal{P}(\mathcal{J}), z \in \mathcal{P}(\overline{\mathcal{J}}) \text{ be random elements,} \\
 &2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]] \\
 &= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}] \geq E_y[(E_z[f(y \sqcup z)])^2] \\
 &= E_y[1 - \underbrace{E_z[(f(y \sqcup z))^2]}_1 + (E_z[f(y \sqcup z)])^2] \\
 &= E_y[1 - V_z[f(y \sqcup z)]] = 1 - Vr_f(\overline{\mathcal{J}})
 \end{aligned}$$

## Proposition

$\forall f_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

$$2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$$

$$= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}] \geq E_y[(E_z[f(y \sqcup z)])^2]$$

$$= E_y[1 - \underbrace{E_z[(f(y \sqcup z))^2]}_1 + (E_z[f(y \sqcup z)])^2]$$

$$= E_y[1 - V_z[f(y \sqcup z)]] = 1 - V_{f_f}(\overline{\mathcal{J}}) > 1 - 2\epsilon$$

## Proposition

$\forall r_f(\overline{\mathcal{J}}) < 2\epsilon \Rightarrow f$   $\epsilon$ -close to junta dominated by  $\mathcal{J}$

## Proof.

$f$  is  $\epsilon$ -close to  $h(x) = \text{sign}(E_{z \in \mathcal{P}(\overline{\mathcal{J}})}[f((x \cap \mathcal{J}) \sqcup z)])$

Let  $x \in \mathcal{P}([n])$ ,  $y \in \mathcal{P}(\mathcal{J})$ ,  $z \in \mathcal{P}(\overline{\mathcal{J}})$  be random elements,

$$2Pr_x[f(x) = h(x)] - 1 = E_x(f(x)h(x)) = E_y[E_z[f(y \sqcup z)h(y \sqcup z)]]$$

$$= E_y[\underbrace{E_z[f(y \sqcup z)] \cdot \text{sign}(E_z[f(y \sqcup z)])}_{|E_z[f(y \sqcup z)]}] \geq E_y[(E_z[f(y \sqcup z)])^2]$$

$$= E_y[1 - \underbrace{E_z[(f(y \sqcup z))^2]}_1 + (E_z[f(y \sqcup z)])^2]$$

$$= E_y[1 - V_z[f(y \sqcup z)]] = 1 - Vr_f(\overline{\mathcal{J}}) > 1 - 2\epsilon$$

$$\Rightarrow Pr_x[f(x) = h(x)] \geq 1 - \epsilon$$



## Proposition

*f passes the test w.p  $\geq 1/2 \Rightarrow \forall r_f(\overline{\mathcal{J}}) < 2\epsilon$*

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow \text{Var}_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof Idea

Show that otherwise w.h.p there are **many subsets** that have **high variation** by showing that  $\overline{\mathcal{J}}$  variation is **well spread** on the subsets

## Proposition

*f passes the test w.p.  $\geq 1/2 \Rightarrow Vr_f(\overline{\mathcal{J}}) < 2\epsilon$*

## Proof.

Assume on the contrary that  $Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow Vr_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow Vr_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $Vr_f(I_j) \geq 2t$  a **detectable subset**.

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow V_{r_f}(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $V_{r_f}(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $V_{r_f}(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, \Pr(V_{r_f}(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $V_{r_f}(I_j) \geq 2t$  a **detectable subset**.
- $3r/4 \leq E(\#\{\text{detectable subsets}\})$

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow \text{Vr}_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $\text{Vr}_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $\text{Vr}_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, \text{Pr}(\text{Vr}_f(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $\text{Vr}_f(I_j) \geq 2t$  a **detectable subset**.
- $3r/4 \leq E(\#\{\text{detectable subsets}\}) \leq qr/4 + (1 - q)r$   
where  $q \stackrel{\text{def}}{=} \text{Pr}(\#\{\text{detectable subsets}\} < r/4)$

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow \forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $\forall r_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $\forall r_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, \Pr(\forall r_f(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $\forall r_f(I_j) \geq 2t$  a **detectable subset**.
- $3r/4 \leq E(\#\{\text{detectable subsets}\}) \leq qr/4 + (1 - q)r$   
 where  $q \stackrel{\text{def}}{=} \Pr(\#\{\text{detectable subsets}\} < r/4)$
- $\Rightarrow q \leq \frac{1}{3} \Rightarrow \Pr(\#\{\text{detectable subsets}\} \geq \underbrace{r/4}_{>J+1}) \geq \frac{2}{3}$

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow V_{r_f}(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $V_{r_f}(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $V_{r_f}(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, \Pr(V_{r_f}(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $V_{r_f}(I_j) \geq 2t$  a **detectable subset**.
- $3r/4 \leq E(\#\{\text{detectable subsets}\}) \leq qr/4 + (1 - q)r$   
 where  $q \stackrel{\text{def}}{=} \Pr(\#\{\text{detectable subsets}\} < r/4)$
- $\Rightarrow q \leq \frac{1}{3} \Rightarrow \Pr(\#\{\text{detectable subsets}\} \geq \underbrace{r/4}_{> J+1}) \geq \frac{2}{3}$
- $\Pr(\text{test detects } < J + 1 \text{ subsets} \mid \#\{\text{detectable subsets}\} > J + 1) \leq (J + 1)(1 - t)^h < 1/16$

## Proposition

$f$  passes the test w.p  $\geq 1/2 \Rightarrow \forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

## Proof.

Assume on the contrary that  $\forall r_f(\overline{\mathcal{J}}) \geq 2\epsilon$ :

- Lemma:  $\forall r_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, \Pr(\forall r_f(I_j) \geq 2t) \geq 3/4$
- Let's call a subset  $I$  s.t  $\forall r_f(I_j) \geq 2t$  a **detectable subset**.
- $3r/4 \leq E(\#\{\text{detectable subsets}\}) \leq qr/4 + (1-q)r$   
where  $q \stackrel{\text{def}}{=} \Pr(\#\{\text{detectable subsets}\} < r/4)$
- $\Rightarrow q \leq \frac{1}{3} \Rightarrow \Pr(\#\{\text{detectable subsets}\} \geq \underbrace{r/4}_{> J+1}) \geq \frac{2}{3}$
- $\Pr(\text{test detects } < J+1 \text{ subsets} \mid \#\{\text{detectable subsets}\} > J+1) \leq (J+1)(1-t)^h < 1/16$
- By union bound  $\Pr(f \text{ passes the test}) < 1/3 + 1/16 < 1/2$



## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity

## Lemma

$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity  
 $E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{J}) = \sum_{i=1}^n (Vr_f([i] \cap I) \setminus \mathcal{J}) - Vr_f([i-1] \cap I \setminus \mathcal{J})$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{J}) = \sum_{i=1}^n (Vr_f([i] \cap I) \setminus \mathcal{J}) - Vr_f([i-1] \cap I) \setminus \mathcal{J})$
- All summands are bounded by  $t$  but they are **dependent**

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{J}) = \sum_{i=1}^n (Vr_f((I_j \cap I_i) \setminus \mathcal{J}) - Vr_f((I_{j-1} \cap I_i) \setminus \mathcal{J}))$
- All summands are bounded by  $t$  but they are **dependent**

## Example

$$f(x_1, x_2) = x_1 \wedge x_2$$

$$Vr_f(\{1\}) = Vr_f(\{2\}) = \frac{1}{2}$$

$$Vr_f(\{1, 2\}) = \frac{3}{4}$$

$$Vr_f([2]) - Vr_f([1]) = \frac{1}{4} \neq Vr_f(\{2\})$$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{J}) = \sum_{i=1}^n (Vr_f([i] \cap I) \setminus \mathcal{J}) - Vr_f([i-1] \cap I) \setminus \mathcal{J})$
- All summands are bounded by  $t$  but they are **dependent**

## Definition (Unique-variation)

$$Ur_f(i) = Vr_f([i] \setminus \mathcal{J}) - Vr_f([i-1] \setminus \mathcal{J})$$

$$Ur_f(I) = \sum_{i \in I} Ur_f(i)$$

## Lemma

$$Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J}) \geq Vr_f(\overline{\mathcal{J}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{J})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{J})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{J}) = \sum_{i=1}^n (Vr_f([i] \cap I) \setminus \mathcal{J}) - Vr_f([i-1] \cap I) \setminus \mathcal{J})$
- All summands are bounded by  $t$  but they are **dependent**

## Definition (Unique-variation)

$$Ur_f(i) = Vr_f([i] \setminus \mathcal{J}) - Vr_f([i-1] \setminus \mathcal{J})$$

$$Ur_f(I) = \sum_{i \in I} Ur_f(i) \text{ summands are } \mathbf{independent}$$

## Lemma

$$Vr_f(\overline{\mathcal{I}}) \geq 2\epsilon \Rightarrow \text{Fix } j, Pr(Vr_f(I_j) \geq 2t) \geq 3/4$$

## Proof Outline

- For **any partition**:  $\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{I}) \geq Vr_f(\overline{\mathcal{I}}) \geq 2\epsilon$  by sub-additivity
- $\sum_{j=1}^r E[Vr_f(I_j \setminus \mathcal{I})] = E[\sum_{j=1}^r Vr_f(I_j \setminus \mathcal{I})] \geq 2\epsilon$
- $\Rightarrow$  Fix  $j$ ,  $E[Vr_f(I_j \setminus \mathcal{I})] \geq 2\epsilon/r$
- $Vr_f(I_j \setminus \mathcal{I}) = \sum_{i=1}^n (Vr_f([i] \cap I) \setminus \mathcal{I}) - Vr_f([i-1] \cap I) \setminus \mathcal{I})$
- All summands are bounded by  $t$  but they are **dependent**

## Definition (Unique-variation)

$$Ur_f(i) = Vr_f([i] \setminus \mathcal{I}) - Vr_f([i-1] \setminus \mathcal{I})$$

$$Ur_f(I) = \sum_{i \in I} Ur_f(i) \text{ summands are } \mathbf{independent}$$

- Show  $Ur_f(I) \leq Vr_f(I)$  and  $Pr(Ur_f(I_j) < 2t) < 1/4$

## Proposition

$$Ur_f(I) \leq Vr_f(I)$$

## Proof.

$$\begin{aligned} Ur_f(I) &= \sum_{i \in I} Ur_f(i) \\ &= \sum_{i \in I} (Vr_f([i] \setminus \mathcal{J}) - Vr_f([i-1] \setminus \mathcal{J})) \\ &\leq \sum_{i \in I} (Vr_f(([i] \cap I) \setminus \mathcal{J}) - Vr_f(([i-1] \cap I) \setminus \mathcal{J})) \\ &= \sum_{i=1}^n (Vr_f(([i] \cap I) \setminus \mathcal{J}) - Vr_f(([i-1] \cap I) \setminus \mathcal{J})) \\ &= Vr_f(I \setminus \mathcal{J}) \leq Vr_f(I) \end{aligned}$$



## Proposition

$$\Pr(Ur_f(I_j) < 2t) < 1/4$$

## Proof.

$$E[Ur_f(I_j)] = \frac{1}{r} \sum_{i \in [n]} Ur_f(i) = Vr_f(\overline{\mathcal{J}})/r \geq 2\epsilon/r$$

## Proposition (Chernoff-like bound)

Let  $X = \sum_{i=1}^l X_i$  be a sum of non-negative independent random variables  $X_i$ ,  
if for all  $i$ ,  $X_i \leq t$  then:

$$\Pr[X < \eta E[X]] < \exp\left(\frac{E[X]}{et}(\eta e - 1)\right) \text{ for every } \eta > 0.$$

$$\Pr(Ur_f(I_j) < \underbrace{\frac{\epsilon}{er}}_{2t}) < \exp\left(-\frac{\epsilon}{ert}\right) = e^{-2} < 1/4 \quad \square$$

## The Adaptive Test

The idea - speed up the finding of the subsets of non-negligible variation:

- apply the independence test to blocks
- If a block passes the independence test - all its elements declared “variation free”.
- else divide the block into two equally sized sub-blocks and repeat.

## Definition (Blocks)

Fix a partition  $I_1, \dots, I_r$ . A block is a union of positive number of subsets in the partition. The size of the block is the number of subsets that take part in this union.

## The Adaptive Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

Initialize  $S = I_1 \cup \dots \cup I_r$

While not all blocks in  $S$  of size one:

    Choose a block  $B \in S$  of maximal size, remove  $B$  from  $S$ .

    Partition  $B$  to half  $B = B' \cup B''$ .

    Apply  $h$  iterations of the independence test to  $B'$ .

    If  $f$  is dependent on  $B'$ , insert  $B'$  into  $S$ , otherwise discard it. repeat for  $B''$ .

    If size of  $S$  is greater than  $J$  reject.

## The Adaptive Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

Initialize  $S = I_1 \cup \dots \cup I_r$

While not all blocks in  $S$  of size one:

    Choose a block  $B \in S$  of maximal size, remove  $B$  from  $S$ .

    Partition  $B$  to half  $B = B' \cup B''$ .

    Apply  $h$  iterations of the independence test to  $B'$ .

    If  $f$  is dependent on  $B'$ , insert  $B'$  into  $S$ , otherwise discard it. repeat for  $B''$ .

    If size of  $S$  is greater than  $J$  reject.

## Query Complexity

We set  $h \stackrel{\text{def}}{=} 4er \cdot \ln(32J(1 + \log_2 r))/\epsilon$ . Overall  $2rh = \theta(J^3 \ln(J+1)^2/\epsilon)$  queries.

## The Adaptive Test

Choose a random partition  $I_1, \dots, I_r$  of  $[n]$ .

Initialize  $S = I_1 \cup \dots \cup I_r$

While not all blocks in  $S$  of size one:

    Choose a block  $B \in S$  of maximal size, remove  $B$  from  $S$ .

    Partition  $B$  to half  $B = B' \cup B''$ .

    Apply  $h$  iterations of the independence test to  $B'$ .

    If  $f$  is dependent on  $B'$ , insert  $B'$  into  $S$ , otherwise discard it. repeat for  $B''$ .

    If size of  $S$  is greater than  $J$  reject.

## Query Complexity

We set  $h \stackrel{\text{def}}{=} 4er \cdot \ln(32J(1 + \log_2 r))/\epsilon$ . Overall  $2rh = \theta(J^3 \ln(J+1)^2/\epsilon)$  queries.

## Completeness & Soundness

Completeness - perfect.

Soundness - If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow f$  is an  $(\epsilon, J)$ -junta.

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

### Proof.

Assume on the contrary that this is not the case:

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

### Proof.

Assume on the contrary that this is not the case:

- If  $|\mathcal{J}| > J \Rightarrow \Pr(\#\{I_j | (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > \frac{3}{4}$

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\forall r_f(\overline{\mathcal{J}}) < 2\epsilon$

### Proof.

Assume on the contrary that this is not the case:

- If  $|\mathcal{J}| > J \Rightarrow \Pr(\#\{I_j | (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > \frac{3}{4}$
- If  $\forall r_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \Pr(\#\{I_j | \forall r_f(I_j) \geq 2t\} \geq \underbrace{r/4}_{> J+1}) \geq \frac{2}{3}$

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\text{Vr}_f(\overline{\mathcal{J}}) < 2\epsilon$

### Proof.

Assume on the contrary that this is not the case:

- If  $|\mathcal{J}| > J \Rightarrow \text{Pr}(\#\{I_j | (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > \frac{3}{4}$
- If  $\text{Vr}_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Pr}(\#\{I_j | \text{Vr}_f(I_j) \geq 2t\} \geq \underbrace{r/4}_{> J+1}) \geq \frac{2}{3}$
- In both cases:  $\text{Pr}(\#\{I_j | \text{Vr}_f(I_j) \geq t\} \geq J + 1) \geq \frac{2}{3}$

## Theorem (Adaptive Test Soundness)

If  $f$  passes the adaptive test w.p  $\geq 1/2 \Rightarrow |\mathcal{J}| \leq J$  and  $\text{Vr}_f(\overline{\mathcal{J}}) < 2\epsilon$

### Proof.

Assume on the contrary that this is not the case:

- If  $|\mathcal{J}| > J \Rightarrow \text{Pr}(\#\{I_j | (I_j \cap \mathcal{J}) \neq \Phi\} \geq J + 1) > \frac{3}{4}$
- If  $\text{Vr}_f(\overline{\mathcal{J}}) \geq 2\epsilon \Rightarrow \text{Pr}(\#\{I_j | \text{Vr}_f(I_j) \geq 2t\} \geq \underbrace{r/4}_{> J+1}) \geq \frac{2}{3}$
- In both cases:  $\text{Pr}(\#\{I_j | \text{Vr}_f(I_j) \geq t\} \geq J + 1) \geq \frac{2}{3}$
- In order to accept test must **discard a block whose variation  $\geq t$ .**

## Proof cont.

- $\Pr(\text{test discard block whose variation} \geq t)$   
 $\leq (1 - \frac{t}{2})^h \cdot \#\text{blocks the test encounters}$

## Proof cont.

- $\Pr(\text{test discard block whose variation} \geq t)$   
 $\leq (1 - \frac{t}{2})^h \cdot \text{\#blocks the test encounters}$
- $\text{\#blocks the test encounters} = 2 \cdot \text{\#rounds}$

## Proof cont.

- $\Pr(\text{test discard block whose variation} \geq t)$   
 $\leq (1 - \frac{t}{2})^h \cdot \#\text{blocks the test encounters}$
- $\#\text{blocks the test encounters} = 2 \cdot \#\text{rounds}$
- **If after round  $T$  maximal size is  $m \Rightarrow$  after round  $T + J$  maximal size  $\leq \lceil \frac{m}{2} \rceil \Rightarrow \#\text{rounds} \leq 2J(1 + \log_2 r)$**

## Proof cont.

- $\Pr(\text{test discard block whose variation} \geq t)$   
 $\leq (1 - \frac{t}{2})^h \cdot \# \text{blocks the test encounters}$
- $\# \text{blocks the test encounters} = 2 \cdot \# \text{rounds}$
- **If after round  $T$  maximal size is  $m \Rightarrow$  after round  $T + J$  maximal size  $\leq \lceil \frac{m}{2} \rceil \Rightarrow \# \text{rounds} \leq 2J(1 + \log_2 r)$**
- $(1 - \frac{t}{2})^h \cdot \# \text{blocks the test encounters} \leq 1/8$   
 $\leq \underbrace{\frac{1}{32J(1 + \log_2 r)}}_{\leq \frac{1}{32J(1 + \log_2 r)}} \cdot \underbrace{\leq 4J(1 + \log_2 r)}_{\leq 4J(1 + \log_2 r)}$

## Proof cont.

- $\Pr(\text{test discard block whose variation} \geq t)$   
 $\leq (1 - \frac{t}{2})^h \cdot \# \text{blocks the test encounters}$
- $\# \text{blocks the test encounters} = 2 \cdot \# \text{rounds}$
- **If after round  $T$  maximal size is  $m \Rightarrow$  after round  $T + J$  maximal size  $\leq \lceil \frac{m}{2} \rceil \Rightarrow \# \text{rounds} \leq 2J(1 + \log_2 r)$**
- $(1 - \frac{t}{2})^h \cdot \underbrace{\# \text{blocks the test encounters}}_{\leq 4J(1 + \log_2 r)} \leq 1/8$   
 $\leq \frac{1}{32J(1 + \log_2 r)}$
- By union bound  $\Pr(f \text{ passes the test}) < 1/3 + 1/8 < 1/2$

## Homework

Try to think how to construct a two-sided test based on ideas presented in both test. Can you improve the query complexity?