Testing H-minor freeness

all graphs have max degree $\leq d$

def. • H is "minor" of G
if can obtain H from G via
vertex removals, edge removals, edge contractions

• G is "H-minor-free" if H not minor of G

• G is "ε-close to H-minor-free" if
can remove $\leq \varepsilon dn$ edges to make it
H-minor-free

• minor closed property P –
if $G \in P$ then all minors of G are in P

Really Cool Theorem [Robertson + Seymour]
Every minor-closed property is expressible
as a constant # of excluded minors.

Some minor-closed properties:
planar graph, bounded tree width, ...

Goal: Testing H-minor freeness
Pass H-minor free graphs
Fail if far from H-minor free

more definitions

- G is "(ε, k)-hyperfinite" if

  can remove $\leq \varepsilon n$ edges

  & remain with connected components of size $\leq k$

- G is "ρ-hyperfinite" if

  $\forall\ \varepsilon > 0$, G is $(\varepsilon, \rho(\varepsilon))$-hyperfinite

Useful Thm

constant that depends only on H

Given H

$\exists\ C_H$ s.t. $\forall\ 0 < \varepsilon < 1$, every H-minor free graph of deg $\leq d$

is $(\varepsilon d, C_H^2/\varepsilon^2)$-hyperfinite.

(ie. remove $\leq \varepsilon d n$ edges & components of size $O(1/\varepsilon^2)$)

note

Subgraphs of H-minor free graphs also H-minor free

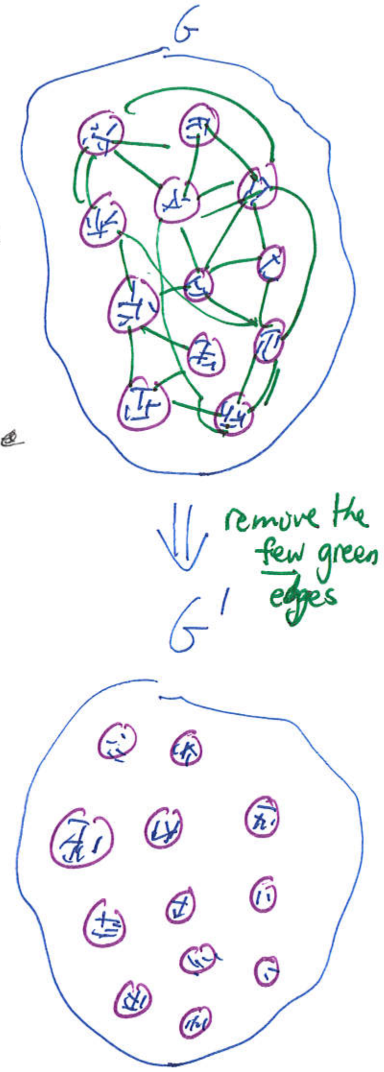
& so also hyperfinite

but, only remove #edges in proportion to #nodes in subgraph

Why is hyperfiniteness useful?

Partition graph G into G'

how in sublinear time? }
- only const size connected components remain
- removed only few edges ($\le \varepsilon dn$)
- if can't do this, G is <u>not</u> H-minor free



If G' is close to having property, so is G

Constant time } so test G' by picking random components & seeing if they have the property

Assume we have "partition oracle" P

(with parameters $\frac{\varepsilon d}{4}$, k)

↗ fraction edges removed ↖ component size

<u>input</u>: vertex v
<u>output</u>: $P[v]$ (v's partition name)

s.t. $\forall v \in V$ (1) $|P[v]| \le k$
(2) $P(v)$ connected

& if G is H-minor free
with prob $\ge \frac{9}{10}$ $|\{(u,v) \in E \mid P(u) \ne P(v)\}| \le \frac{\varepsilon dn}{4}$

Algorithm given partition oracle P :

- estimate number $\hat{f}$ of edges (u,v)

  s.t. $P[u] \neq P[v]$ to additive error $\leq \frac{\varepsilon d n}{8}$ with prob failure $\leq \frac{1}{10}$

- if $\hat{f} \geq \frac{3}{8} \varepsilon d n$, output "fail" + halt

- else choose $S' = O(1/\varepsilon)$ random nodes

  if for any $s \in S'$

  $P[s]$ not H-minor free, reject & halt

- Accept

Analysis (assume P always correct)

if G H-minor free:

$$E[\hat{f}] \leq \frac{\varepsilon d n}{4}$$

Sampling bounds $\Rightarrow \hat{f} \leq \frac{\varepsilon d n}{4} + \frac{\varepsilon d n}{8} = \frac{3}{8} \varepsilon d n$

with prob $\geq 9/10$

$\forall s \in V$, $P[s]$ is H-minor free

if G ε-far from H-minor free:

Case 1 P's output doesn't satisfy $|\{(u,v) \in E : P(u) \neq P(v)\}| \leq \frac{\varepsilon d n}{2}$

Sampling bnds $\Rightarrow \hat{f} \geq \frac{\varepsilon d n}{2} - \frac{\varepsilon d n}{8} \geq \frac{3}{8} \varepsilon d n$

$\Rightarrow$ output "fail" with $\underbrace{\text{prob} \geq 9/10}_{}$

make mistake only if additive estimate is off by $\geq \frac{\varepsilon d n}{8}$

Case 2 P's output satisfies $|\{(u,v)\in E : P(u)\neq P(v)\}| \leq \frac{\varepsilon dn}{2}$

$G' \leftarrow G$ with "cross" edges removed

$\uparrow$

(u,v) s.t. $P[u]\neq P[v]$

if G' is $\frac{\varepsilon}{2}$ - far from having property,

third step likely to fail ✵ why?

else, G' is $\frac{\varepsilon}{2}$-close to property & G is $\frac{\varepsilon}{2}$-close to G',

so G is ε-close to having property ▨

✵ if G' is $\frac{\varepsilon}{2}$-far, need to remove at least εdn edges. εdn edges touch at least εn nodes. Therefore, with prob $\geq \varepsilon$, will pick a node in a component which is not minor-free.

So the main remaining issue is

how do we implement the partitioning oracle?