Selection and Uniformization Problems in the Monadic Theory of Ordinals: A Survey

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Dedicated with deepest appreciation and respect to Boris Abramovich Trakhtenbrot whose inspiration as a teacher, a researcher and a role model has been guiding us and many others for many years.

Abstract. A formula $\psi(Y)$ is a *selector* for a formula $\varphi(Y)$ in a structure $\mathcal{M}$ if there exists a unique Y that satisfies ψ in $\mathcal{M}$ and this Y also satisfies φ. A formula $\psi(X,Y)$ *uniformizes* a formula $\varphi(X,Y)$ in a structure $\mathcal{M}$ if for every X there exists a unique Y such that $\psi(X,Y)$ holds in $\mathcal{M}$ and for this Y, $\varphi(X,Y)$ also holds in $\mathcal{M}$. In this paper we survey some fundamental algorithmic questions and recent results regarding selection and uniformization, when the formulas ψ and φ are formulas of the monadic logic of order and the structure $\mathcal{M} = (\alpha, <)$ is an ordinal α equipped with its natural order. A natural generalization of the Church problem to ordinals is obtained when some additional requirements are imposed on the uniformizing formula $\psi(X,Y)$. We present what is known regarding this generalization of Church's problem.

1 Introduction

The aim of this paper is to survey recent results on the selection and uniformization problems for monadic (second-order) logic of order. These results are well-known for the standard discrete time model of natural numbers. When selection and uniformization problems are considered over countable ordinals new and interesting phenomena appear. Our exposition focusses on methodological issues rather than providing the technical details. No proofs are offered, though we sometimes indicate the main ideas of the proofs.

1.1 Selection

Definition 1 (Selection). *Let $\varphi(Y)$, $\psi(Y)$ be formulas and $\mathcal{M}$ a structure. We say that ψ* selects *(or, is a* selector *for) φ* in *$\mathcal{M}$ iff:*

1. either *both formulas are not satisfied in $\mathcal{M}$,* or
2. *ψ defines in $\mathcal{M}$ a unique P and this P satisfies φ in $\mathcal{M}$.*

We say that ψ selects *φ* over *a class $\mathcal{C}$ of structures iff ψ selects φ in every $\mathcal{M} \in \mathcal{C}$.*

A. Avron et al. (Eds.): Trakhtenbrot/Festschrift, LNCS 4800, pp. 571–588, 2008.

Generally, once some logic $\mathcal{L}$ and a class $\mathcal{C}$ of structures for $\mathcal{L}$ have been fixed, three basic questions may be raised concerning selection for $\mathcal{L}$-formulas over $\mathcal{C}$. First,

(1) Selection property: Does every $\mathcal{L}$-formula have a selector over $\mathcal{C}$?

When this is the case we shall say that $\mathcal{C}$ has the *selection property* (with respect to $\mathcal{L}$-formulas). When $\mathcal{C}$ lacks the selection property, the following algorithmic question naturally arises:

(2) Deciding selectability: Can we decide, given an $\mathcal{L}$-formula φ, whether it has a selector over $\mathcal{C}$?

Finally, whether or not $\mathcal{C}$ has the selection property, it seems interesting to ask:

(3) Synthesis of a selector: If φ has selectors over $\mathcal{C}$, can one be computed for it?

When both Questions (2) and (3) are answered affirmatively, we say that the *selection problem* over $\mathcal{C}$ is solvable.

We consider the above questions when the logic $\mathcal{L}$ is either the *second-order monadic logic of order* (MLO) or its first-order fragment and $\mathcal{C} = \{(\alpha, <)\}$ for some ordinal α or $\mathcal{C}$ is a class of countable ordinals.

MLO extends first-order logic by allowing quantification over *subsets* of the domain. The binary relation symbol '$<$' is its only non-logical constant. Since our structures are ordinals, we shall assume that '$<$' is interpreted as a well-order of the domain. In short, MLO uses first order variables $s,\ t, \ldots$ interpreted as elements and monadic second-order variables $X,\ Y, \ldots$ interpreted as subsets of domain. The atomic formulas are $s < t$ and $t \in X$; all other formulas are built from the atomic ones by applying Boolean connectives and quantifiers $\forall, \exists$ for both kinds of variables. An MLO formula is *first-order* if it does not use quantification over set variables.

An MLO formula $\varphi(Y)$ defines in an ordinal α the family of sets which satisfy $\varphi(Y)$ in α. If this family is non-empty, then a selector $\psi(Y)$ for φ defines one set from this family.

MLO plays a very important role in mathematical logic and computer science. The fundamental connection between this logic and automata was discovered independently by Büchi, Elgot and Trakhtenbrot [1, 6, 20–22] and the logic was proved to be decidable over the class of finite chains. Büchi [2] proved the decidability of MLO in $(\omega, <)$ and later [4] that the monadic theory of every ordinal $\leq \omega_1$ is decidable. Shelah [18] showed that the MLO-theory of any ordinal $\alpha < \omega_2$ is decidable. Rabin proved that the MLO theory of the full binary tree $T_2 := (D, <, \mathrm{Left}, \mathrm{Right})$ is decidable [13, 14]. Here D is the set all finite strings over $\{0, 1\}$; the relation symbol '$<$' is interpreted as the prefix relation and the unary predicate 'Left' (respectively, 'Right') is interpreted as the set of strings whose last symbol is '0' (respectively, '1').

The Rabin basis theorem states that if $T_2 \models \exists Z \varphi(Z)$ then there is a regular subset $S \subseteq D$ such that $T_2 \models \varphi(S)$. Since a subset of T_2 is regular iff it is definable, the Rabin basis theorem can be restated as following: the full binary tree has the selection property.

1.2 Uniformization

A *uniformizer* for a binary relation R is a function $f \subseteq R$ such that $\mathrm{dom}(f) = \mathrm{dom}(R)$. That every binary relation has a uniformizer is a statement equivalent to the Axiom of Choice. Existence of a uniformizer becomes mathematically interesting when we place certain restrictions on the uniformizing function f. A *uniformization context* is a pair $\langle \mathcal{R}, \mathcal{F} \rangle$, where $\mathcal{R}$ is a class of binary relations and $\mathcal{F}$ a class of functions. We call $\mathcal{R}$ the *challenge* class and $\mathcal{F}$ the *response* class of the context. Given such a pair, one may ask whether a particular (resp. every) $R \in \mathcal{R}$ has a uniformizer $f \in \mathcal{F}$.[1]

This paper focuses on two uniformization contexts – *definable* and *causal* uniformization – in which the challenge class $\mathcal{R}$ is taken to be the class of relations definable in some ordinal α, when α is viewed as a structure for MLO. Our aim is to survey the current state of research into these two contexts, to report recent developments, and to indicate what seem to us the most interesting questions still left open.

1.3 Definable Uniformization

The first uniformization context we explore we call *definable uniformization* (in the literature, this is referred to simply as "uniformization"; see, for instance, [8] and [11]). Here, the response class $\mathcal{F}$ is taken to be the class of functions $\mathcal{P}(\alpha) \to \mathcal{P}(\alpha)$ definable in the ordinal α (or, strictly speaking, in the structure $(\alpha, <)$ where α is equipped with its natural order):

Definition 2 (Definable uniformization). *Let $\varphi(X, Y)$, $\psi(X, Y)$ be formulas and $\mathcal{M}$ a structure. Say that ψ* uniformizes *(or, is a* uniformizer *for) φ* in *$\mathcal{M}$ iff:*

1. $\mathcal{M} \models \forall X \exists^{\leq 1} Y \psi(X, Y)$,
2. $\mathcal{M} \models \forall X \forall Y (\psi(X, Y) \to \varphi(X, Y))$, *and*
3. $\mathcal{M} \models \forall X \big(\exists Y \varphi(X, Y) \to \exists Y \psi(X, Y)\big)$.

Here "$\exists^{\leq 1} Y \ldots$" stands for "there exists at most one…".
We say that ψ uniformizes *φ* over *a class $\mathcal{C}$ of structures iff ψ uniformizes φ in every $\mathcal{M} \in \mathcal{C}$.*

Note that $\varphi(Y)$ has a selector if and only if $X = X \wedge \varphi(Y)$ has a definable uniformizer. Thus, selection is a special case of uniformization. Accordingly, Questions (1)–(3) above can be generalized to the latter case.

(1′) Uniformization property: Does every formula have a uniformizer over $\mathcal{C}$?
(2′) Decidability of uniformization: Can we decide, given a formula φ, whether it has a uniformizer over $\mathcal{C}$?
(3′) Synthesis of a uniformizer: If φ has uniformizers over $\mathcal{C}$, can one be computed for it?

[1] Some famous examples are found in descriptive set theory, where one proves, for instance, that for every Π^1_1 relation there is a Π^1_1 uniformizer.

Again, when both Questions (2′) and (3′) are answered affirmatively, we say that the *uniformization problem* over $\mathcal{C}$ is solvable.

Gurevich and Shelah [8] proved that the full binary tree does not have the uniformization property. In [11], Lifsches and Shelah characterize the trees which have the uniformization property.[2] For ordinals they show:

Theorem 3. *An ordinal α has the uniformization property iff $\alpha < \omega^\omega$.*

This answers Question (1′) for ordinals. Question (3′) was answered in the affirmative when $\mathcal{C} = \{(\alpha, <)\}$ for $\alpha < \omega^\omega$. However, for an ordinal $\alpha \geq \omega^\omega$, Questions (2′) and (3′) remain open.

In Section 10 we consider a restricted version of uniformization problem which we call bounded uniformization, and show that the bounded uniformization problem is solvable in every ordinal $\leq \omega_1$.

1.4 Church Uniformization

The second uniformization context we look at is that of *causal uniformization*, better known as the *Church uniformization.* While definable uniformization makes sense in any structure, causal uniformization is only relevant in a linear order.

Definition 4 (Causal operator). *Let $(A, <)$ be a linear order and $f : \mathcal{P}(A) \to \mathcal{P}(A)$. We call f* causal *iff for all $P, P' \subseteq A$ and $\alpha \in A$,*

$$\textit{if } P \cap [0, \alpha] = P' \cap (-\infty, \alpha], \textit{ then } f(P) \cap (-\infty, \alpha] = f(P') \cap (-\infty, \alpha].$$

That is, if P and P' agree up to and including *α, then so do $f(P)$ and $f(P')$.*

When discussing causal uniformization, we fix some ordinal α. Again, we take as challenge class $\mathcal{R}$ the class of relations definable in $(\alpha, <)$. Note that in MLO variables range over subsets of the domain. Thus, relations definable in $(\alpha, <)$ are relations on $\mathcal{P}(\alpha)$. It therefore makes sense to ask, whether a definable relation can b e uniformized by a causal function. Accordingly, in causal uniformization the response class $\mathcal{F}$ consists of all causal functions $f : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha)$ (whether definable or not).

We speak of a causal function f uniformizing a formula φ in $(\alpha, <)$, meaning that f uniformizes the *relation* defined by φ in $(\alpha, <)$. In this context, Question (1′) above becomes the question whether any formula φ has a causal uniformizer in $(\alpha, <)$. For any $\alpha \geq 2$, the answer is easily seen to be negative. For example, the formula saying "if $X = \varnothing$, then $Y =$ All; otherwise, $Y = \varnothing$" has no causal uniformizer in $(\alpha, <)$ for any $\alpha \geq 2$. Question (2′) is already more interesting.

Definition 5 (Church uniformization problem). *Let α be an ordinal. Given a formula $\varphi(X, Y)$, decide whether there is a causal uniformizer for φ in $(\alpha, <)$.*

[2] A *tree* for them is a poset $(T, <)$ such that for every $a \in T$, $\{b \in T \mid b \leq a\}$ is a linear order.

Church [5] was the first to formulate this problem for the case $\alpha = \omega$. Some restricted versions of this problem were solved by Church and Trakhtenbrot. Church's Problem for ω was solved by Büchi and Landweber [3] building on McNaughton's game-theoretical interpretation of this problem [12]. Under this game-theoretical interpretation the causal operators correspond to the strategies of the players.

It would seem that Question (3′) is irrelevant to causal uniformization. In what sense can we speak of *computing* a general causal uniformizer? Already in ω, there are $2^{\aleph_0}$ such operators. It becomes relevant once more, when we examine the uniformization context where the response class consists of all operators $f : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha)$ which are *both definable and causal*.

(1″) Does every formula which has a causal uniformizer in $(\alpha, <)$ also has a *definable* causal uniformizer?

When this fails, we have an analogue of Question (2′).

(2″) Can we decide, given a formula φ, whether it has a definable causal uniformizer in $(\alpha, <)$?

And, of course,

(3″) If φ has definable causal uniformizers in $(\alpha, <)$, can we compute a formula defining one?

Question (1″) will be answered here for all ordinals. Questions (2″) and (3″) are yet unsolved for $\alpha \geq \omega^\omega$.

Note that $\varphi(Y)$ has a selector if and only if $X = X \wedge \varphi(Y)$ has a definable causal uniformizer. Thus, selection is also a special case of definable causal uniformization. Indeed, looking at selection would turn out to be the key for answering Question (1″).

1.5 The Structure of the Paper

In Sect. 2, we fix our notations and terminology. We also recall some fundamental theorems about the monadic theories of countable ordinals. Section 3 surveys the selection property in an ordinals. In Sect. 4 the selection problem in an ordinal are considered. Section 5 investigates the selection property and the selection problem over classes of countable ordinals. The logic considered in Sect. 3–5 is MLO, while in Sect. 6 we consider the first-order fragment of MLO and other logics with expressive power between first-order MLO and MLO. In Sect. 7 we assign to each formula φ a *selection degree* which measures "how difficult it is to select φ". We show that in a countable ordinal all non-selectable formulas share the same degree.

If a structure $\mathcal{M}$ lacks the selection property, it is natural to ask whether there is a finite expansion of $\mathcal{M}$ which has the selection property. This question is investigated for a countable ordinal in Sect. 8. In Sect. 9 the Church uniformization problem for countable ordinals is considered. In Sect. 10 we treat a restricted version of the definable uniformization problem. Finally, Sect. 11 contains some open problems.

As mentioned above, this paper offers no proofs and only occasionally indicates their main ingredients. For results having to do with selection in a particular ordinal (Sects. 3, 4, 6 and 7) proofs can be found in [16]. Sections 5, 8 and 10 are covered in [17]. All results having to do with Church (= causal) uniformization (Sect. 9) are in [15]. It is perhaps worth mentioning that almost all proofs relay on what is known as the “composition method” (originating in [7] and adapted and ingeniously applied to the case of MLO in [18]). In [16] use is also made of Büchi’s translation of MLO-formulas into automata over ordinal words (see, for instance, [4]).

From now on, “uniformization” *simpliciter* would mean “definable uniformization”. When we intend to refer to causal uniformization, this would be stated explicitly.

Finally, for the sake of notational simplicity, we state our results for formulas $\varphi(X, Y)$ with free variables X and Y. All results generalize in a straightforward manner to formulas $\varphi(\bar{X}, \bar{Y})$ where $\bar{X}$ and $\bar{Y}$ are finite tuples of (distinct) variables.[3]

2 Preliminaries

2.1 Notations

We use n, k, l, m, p for natural numbers, $\alpha, \beta, \gamma, \delta, \zeta$ for ordinals. The set of natural numbers is $\omega := \{0, 1, 2, \ldots\}$. ω_1 is the first uncountable ordinal. We write $\alpha + \beta$, $\alpha\beta$, α^β for the sum, multiplication and exponentiation, respectively, of ordinals α and β.

We use standard notation for sub-intervals of a chain: if $(A, <)$ is a chain and $b < a$ are in A, we write $(b, a) := \{c \in A \mid b < c < a\}$, $[b, a) := (b, a) \cup \{b\}$, etc.

2.2 MLO

The vocabulary of MLO consists of first-order variables $t_0,\ t_1,\ t_2, \ldots$ interpreted as elements of the domain and monadic second-order variable $X_0, X_1, \ldots$ interpreted as subsets of the domain. The atomic formulas are $t_i < t_j$ and $t_i \in X_j$; the MLO formulas are built from atomic ones by applying Boolean connectives and quantifiers $\forall, \exists$ for both kinds of variables. An MLO formula is first-order if it does not use quantification over set variables; note however, that such formula may contain free set variables.

The *quantifier depth* of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

We use lower case letters s,t, … to denote the first-order variables and upper case letters X, Y, … to denote second-order set variables.

[3] To fit the general notion of uniformization, the relation defined by $\varphi(\bar{X}, \bar{Y})$ must be thought of as consisting of pairs $(\bar{P}, \bar{Q})$ of tuples of subsets of the domain, where $\lg(\bar{P}) = \lg(\bar{X})$ and $\lg(\bar{Q}) = \lg(\bar{Y})$.

A *structure* is a tuple $\mathcal{M} := (A, <, \bar{a}, \bar{P})$ where: A is a non-empty set, $<$ is a binary relation on A, and $\bar{a} := \langle a_0, \ldots, a_{m-1} \rangle$ (respectively, $\bar{P} := \langle P_0, \ldots, P_{l-1} \rangle$) is a *finite* tuple of elements (respectively, sub*sets*) of A.

Suppose φ is a formula with free-variables among $t_0, \ldots, t_{m-1}, X_0, \ldots, X_{l-1}$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as usual.

The *monadic theory* of $\mathcal{M}$, $\mathrm{MTh}(\mathcal{M})$, is the set of all formulas satisfied by $\mathcal{M}$. When $\bar{a}$ and $\bar{P}$ are the empty tuple (as is most often the case for us), $\mathrm{MTh}(\mathcal{M})$ is a set of *sentences*.

2.3 The Monadic Theory of Countable Ordinals

Büchi (for instance [4]) has shown that there is a *finite* amount of data concerning any ordinal $\leq \omega_1$ which determines its monadic theory:

Theorem 6. *Let $\alpha \in [1, \omega_1]$. Write $\alpha = \omega^\omega \beta + \zeta$ where $\zeta < \omega^\omega$ (this can be done in a unique way). Then the monadic theory of $(\alpha, <)$ is determined by:*

1. *whether α is countable or $\alpha = \omega_1$,*
2. *whether $\alpha < \omega^\omega$, and*
3. *ζ.*

We can associate with every $\alpha \leq \omega_1$ a finite *code* which holds the data required in the previous theorem. This is clear with respect to (1) and (2). As for (3), if $\zeta \neq 0$, write

$$\zeta = \textstyle\sum_{i \leq n} \omega^{n-i} \cdot a_{n-i}, \text{ where } n, a_i \in \omega \text{ for } i \leq n \text{ and } a_n \neq 0$$

(this, too, can be done in a unique way), and let the sequence $\langle a_n, \ldots, a_0 \rangle$ encode ζ. The following is then implicit in [4]:

Theorem 7. *There is an algorithm that, given a sentence φ and the* code *of an $\alpha \in [1, \omega_1]$, determines whether $(\alpha, <) \models \varphi$.*

Agreement In this paper, whenever we say that an algorithm is "given an ordinal..." or "returns an ordinal...", we mean the *code* of the ordinal.

3 The Selection Property in an Ordinal

In [11], Lifsches and Shelah characterize the trees which have the uniformization property.[4] For ordinals they show that an ordinal α has the uniformization property iff $\alpha < \omega^\omega$. It follows, in particular, that $\alpha < \omega^\omega$ has the selection property. On the other hand, it does not immediately follow that all ordinals above ω^ω lack the selection property. Indeed, the selection property is known *not* to imply the uniformization property. As mentioned in Sect. 1, Rabin proved that the full binary tree has the selection property[14], while Gurevich and Shelah proved that the full binary tree lacks the uniformization property [8]. But, in fact, for selection, too, we have:

[4] A *tree* for them is a poset $(T, <)$ such that for every $a \in T$, $\{b \in T \mid b \leq a\}$ is a linear order.

Proposition 8 (Selection property). *An ordinal α has the selection property iff $\alpha < \omega^\omega$.*

The proof that in any ordinal $\alpha \geq \omega^\omega$ there are non-selectable formulas, reduces to the cases $\alpha = \omega^\omega$ and $\alpha = \omega_1$. The key to handling these, in turn, is the notion of a *periodic* subset.

If $(A, <, P)$ is a structure and $D \subseteq A$, we write $(A, <, P)_{\restriction D}$ for the *restriction* of $(A, <)$ to D, that is, $(A, <, P)_{\restriction D} := (D, <, P \cap D)$.

Definition 9. *Let $\alpha \in \{\omega^\omega, \omega_1\}$ and $P \subseteq \alpha$. We say that P is* periodic *iff there are $\alpha_0, \alpha_1 < \alpha$ and $P_1 \subseteq \alpha_1$ such that $(\alpha, <, P)_{\restriction[\alpha_0, \alpha)}$ is the "concatenation" of α copies of $(\alpha_1, <, P_1)$, i.e., for every $\beta < \alpha$,*

$$(\alpha, <, P)_{\restriction[\alpha_0 + \alpha_1\beta, \alpha_0 + \alpha_1(\beta+1))} \text{ is isomorphic to } (\alpha_1, <, P_1).$$

The notion of a periodic subset enters our discussion through the following lemma.

Lemma 10. *Let $\alpha \in \{\omega^\omega, \omega_1\}$. Any definable subset of α is periodic.*

Now, no unbounded ω-sequence in ω^ω is periodic (in fact, an unbounded periodic subset of ω^ω has order-type ω^ω). Note that there is a formula $\theta_{\omega\,\mathrm{ub}}(Y)$ that in every countable limit ordinal α defines the set of all unbounded ω-sequences in α. This formula $\theta_{\omega\,\mathrm{ub}}(Y)$ is the conjunction of the following two formulas:

$$\text{"}Y \text{ is unbounded"}: \forall t_1 \exists t_2 (t_2 > t_1 \wedge t_2 \in Y), \text{ and}$$
$$\text{"no point is a limit point of } Y\text{"}:$$
$$\forall t_1 \exists t_2 \Big(t_1 > 0 \rightarrow \big(t_2 < t_1 \wedge \forall t_3 (t_2 < t_3 < t_1 \rightarrow t_3 \notin Y)\big)\Big).$$

Therefore,

Corollary 11. *The formula $\theta_{\omega\,\mathrm{ub}}(Y)$ saying "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$.*

To handle ω_1 recall the following definitions:

Definition 12 (Clubs and stationary sets)

1. *Let $C \subseteq \omega_1$. C is called:*
   closed *iff for every limit $\beta < \omega_1$, if* $\sup(C \cap \beta) = \beta$*, then $\beta \in C$.*[5]
   a club *iff C is closed and unbounded in ω_1.*
2. *$S \subseteq \omega_1$ is called* stationary *iff for every club $C \subseteq \omega_1$, $S \cap C \neq \varnothing$.*

Note that being a club and being stationary are definable properties of a subset of ω_1 and that ω_1 itself is definable. It is also easy to show that any unbounded periodic subset of ω_1 contains a club. From this and Lemma 10, one derives:

Corollary 13. *Let $\theta_{stat}(Y)$ say: "Both $Y \cap \omega_1$ and $\omega_1 \setminus Y$ are stationary in ω_1". Then θ_{stat} has no selector in $(\alpha, <)$ for* every *$\alpha \geq \omega_1$.*

[5] That is, C is closed under taking sup.

Note that if two ordinal have the same monadic theory, then ψ selects φ in the first ordinal iff ψ selects φ in the second. By Theorem 6, ω^ω and $\omega^\omega\beta$ have the same monadic theory for every countable ordinal $\beta > 0$. Therefore, $\theta_{\omega\,\mathrm{ub}}(Y)$ is not selectable in $\omega^\omega\beta$ for every countable $\beta > 0$.

For a countable ordinal $\alpha > \omega^\omega$, a formula $\psi_\alpha(Y)$ which is unselectable in α can be constructed as follows. Write $\alpha = \omega^\omega\beta + \zeta$ where $\zeta < \omega^\omega$. If $\zeta = 0$, then $\theta_{\omega\,\mathrm{ub}}(Y)$ is not selectable in α. Otherwise note that since $0 < \zeta < \omega^\omega$ there is a formula $\Psi(t)$ such that $\alpha \models \Psi(\mu)$ iff $\mu = \omega^\omega\beta$. Hence, $\omega^\omega\beta$ is definable in α. The formula $\psi_\alpha(Y)$ saying "Y is unbounded ω-sequence in the interval $[0, \omega^\omega\beta)$" is unselectable in α.

Note that in Corollary 13 a formula not *selectable* in every $\alpha \geq \omega_1$ was presented. We sketched a construction of a formula ψ_α not selectable in a countable ordinal $\alpha \geq \omega^\omega$. However, ψ_α depends on (the code of) α. Is there a single formula not selectable in *every* $\alpha \in [\omega^\omega, \omega_1)$? The answer turns out to be negative:

Proposition 14. *For every $n \in \omega$, we can compute $\xi(n) < \omega^\omega$ such that for every formula $\varphi(Y)$ with $\mathrm{qd}(\varphi) \leq n$ and $\zeta \in [\xi(n), \omega^\omega)$, φ is selectable in $\omega^\omega + \zeta$.*

4 The Selection Problem in an Ordinal $\alpha \leq \omega_1$

In [11], issues of decidability and computability are not discussed. However, from the proof of Proposition 6.1 there, one can extract an algorithm as follows (a detailed proof is given in [17]):

Proposition 15 (Uniformization below ω^ω). *There is an algorithm that, given (the code of) an ordinal α and $\varphi(X, Y)$, computes a $\psi(X, Y)$ that uniformizes φ in α.*

In the case of *selection*, we are able to go beyond ω^ω.

Proposition 16 (Solvability of the selection problem). *There exists an algorithm that, given $\alpha \in [\omega^\omega, \omega_1]$ and a formula $\varphi(Y)$, decides whether φ has a selector in $(\alpha, <)$, and if so, constructs one for it.*

Roughly speaking, the proof breaks into three steps. One shows that:

1. If $\alpha \in \{\omega^\omega, \omega_1\}$, then any formula $\varphi(Y)$ satisfied by a periodic predicate in $(\alpha, <)$ is selectable in $(\alpha, <)$.

By Lemma 10, this means that being satisfied by a periodic predicate (or not being satisfied at all) is a necessary and sufficient condition for selectability in these ordinals.

2. It is decidable whether φ is satisfied by a periodic predicate.
3. Selection in any countable ordinal is reducible to the case of ω^ω.

The full uniformization problem turns out to be trickier. There is currently no proof of the solvability (or insolvability) of the uniformization problem in $(\omega^\omega, <)$. A restricted case of this problem is treated in Sect. 10.

5 Selection over Classes of Countable Ordinals

Here we discuss the selection property and problem over classes of countable ordinals.

5.1 The Selection Property for Subclasses of ω^ω

By Proposition 8, any class of ordinals which has an $\alpha \geq \omega^\omega$ as a member does not have the selection property. What can we say about subclasses of ω^ω? It turns out that there is a simple combinatorial criterion for a class $\mathcal{C} \subseteq \omega^\omega$ to have the selection property.

Notation (Trace). *Let $0 \neq \alpha < \omega^\omega$. Write $\alpha = \omega^{n_r} a_r + \omega^{n_{r-1}} a_{r-1} + \cdots + \omega^{n_0} a_0$ where $r \in \omega$ and $n_r > n_{r-1} > \ldots > n_0$ and a_i (for $i \leq r$) are positive integers (this presentation is unique). Let* $\mathrm{trace}(\alpha) := \{n_r, \ldots, n_0\}$.

Proposition 17. *A class $\mathcal{C} \subseteq \omega^\omega$ has the selection property iff*

$$\forall p \in \omega \exists N(p) \in \omega \forall \alpha \in \mathcal{C}\big(\alpha > \omega^{p+N(p)} \to [p, p + N(p)] \cap \mathrm{trace}(\alpha) \neq \varnothing\big).$$

If in addition, $N(p)$ is computable form p, then selectors are computable over $\mathcal{C}$.

Therefore, $\{\omega^k \mid k \in \omega\}$ does not have the selection property. On the other hand, both of the following classes have the selection property and selectors are computable over them (for both, let $N(p) := 0$ for all $p \in \omega$):

1. $\{\omega, \omega^2 + \omega, \omega^3 + \omega^2 + \omega, \ldots\}$.
2. The class of $\alpha < \omega^\omega$ whose trace is a prefix of ω, that is, such that $\mathrm{trace}(\alpha) = \{0, 1, ..., n-1\}$ for some $n \in \omega$.

Note that the first of these classes has order-type ω while the second has order-type ω^ω.

5.2 The Selection Problem over Definable Classes of Countable Ordinals

In [17] we proved that the selection problem is solvable over every MLO definable class of countable ordinals. Thus, given a formula $\varphi(Y)$, we may decide, for instance, whether φ has a selector over the class of all countable ordinals, of countable limit ordinals, etc. In fact, something slightly more general holds.

Proposition 18. *There is an algorithm that, given formulas $\pi(t)$ and $\varphi(Y)$ and an ordinal $\delta \leq \omega_1$:*

1. *decides whether φ has a selector over the class definable by π in $(\delta, <)$, namely over $\{(\alpha, <) \mid \alpha \in \delta \setminus 1 \wedge (\delta, <) \models \pi(\alpha)\}$, and*
2. *if a selector exists constructs it.*

This is indeed more general. For example, ω^ω is not a definable ordinal, but $\{\omega^\omega\}$ *is* definable in $(\omega^\omega + \zeta, <)$ for any $\zeta < \omega^\omega$. The proof of the last proposition is based on a reduction of this problem to the bounded uniformization problem discussed in Sect. 10.

Note that Proposition 17 provides sufficient and necessary conditions for the selection property. However, the definability conditions of Proposition 18 are sufficient but are not necessary conditions for solvability of the selection problem over a class of countable ordinals. The class $\{\omega^k \mid k \in \omega\}$ is not definable (in any ordinal), however, the selection problem over this class is solvable.

6 Selection between First-Order and Second-Order Logics

Let φ be an MLO-formula. Recall that φ is a *first-order formula* iff all quantifiers appearing in φ are first-order quantifiers. Note, however, that φ can contain second-order free variables which range over subsets of the domain.[6] Let us call the set of first-order MLO formulas the *first-order fragment* of MLO. Then all results concerning the selection property and selection problem in *countable* ordinals carry through from MLO to its first-order fragment. This follows from:

Proposition 19. *If α is a countable ordinal and $\varphi(Y)$ is an* MLO *formula selectable in $(\alpha, <)$, then there is a* first-order *$\chi(Y)$ that selects φ in $(\alpha, <)$. Furthermore, χ is computable from α and φ.*

From the last proposition, we can infer a little more.

Let $\mathcal{L}_2$ and $\mathcal{L}_1$ be logics. We say that a structure $\mathcal{M}$ has the $\mathcal{L}_2 - \mathcal{L}_1$ *selection property* iff for every $\mathcal{L}_2$-formula φ there is an $\mathcal{L}_1$-formula such that ψ selects φ in $\mathcal{M}$. We say that the $\mathcal{L}_2 - \mathcal{L}_1$ *selection problem* for a structure $\mathcal{M}$ is *solvable* iff there is an algorithm which for every $\varphi \in \mathcal{L}_2$ decides whether there is $\psi \in \mathcal{L}_1$ which selects φ in $\mathcal{M}$, and if so, constructs such a ψ. Using this terminology Proposition 19 can be rephrased as "The MLO $-$ FOMLO selection problem is solvable for ordinal $\alpha \leq \omega_1$". More generally,

Corollary 20. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be logics such that:*

1. *For every first-order ϕ, there is an $\mathcal{L}_1$-formula Λ equivalent to it.*
2. *For every $\mathcal{L}_2$-formula Λ, there is an* MLO *formula φ equivalent to it.*

Then a countable *ordinal α has the $\mathcal{L}_2 - \mathcal{L}_1$ selection property iff $\alpha < \omega^\omega$.*

If furthermore, in (1) Λ is computable from ϕ and in (2) φ is computable from Λ, then the selection problem in $(\alpha, <)$ is solvable for all $\alpha \leq \omega_1$.

[6] This is significant. For instance, there is a first-order $\phi(Y)$ such that the only subset of ω satisfying ϕ in $(\omega, <)$ is the set of even numbers. On the other hand, there is no first-order formula $\phi(y)$, with y an *individual* variable, such that for any $n \in \omega$, $(\omega, <) \models \phi(n)$ iff n is even.

A famous example of a logic $\mathcal{L}$ as in the corollary is *weak* MLO (WMLO), where the second-order quantifiers range over *finite* subsets of the domain. Therefore, MLO – WMLO, WMLO – WMLO and WMLO – FOMLO selection problems are solvable for every $\alpha \leq \omega_1$.

When we turn to ω_1, the first-order fragment of MLO no longer behaves like full MLO.

Proposition 21. *$(\omega_1, <)$ has the* FO *order selection property, but not the* MLO *selection property.*

In fact, an interesting dichotomy holds. Let $\phi(Y)$ be first-order.

1. If ϕ is selectable in $(\omega^\omega, <)$, then ϕ is also selectable in $(\omega_1, <)$, and we can compute for it a (first-order) selector that works in both;
2. If ϕ is not selectable in $(\omega^\omega, <)$, then it is not even *satisfied* in $(\omega_1, <)$ (hence, is trivially selectable).

7 Selection Degrees

We know that the formula $\theta_{\omega\,\mathrm{ub}}(Y)$ saying "Y is an unbounded ω-sequence" has no selector in $(\omega^\omega, <)$. Now, let us look at the formula $\theta_{\omega^2\,\mathrm{ub}}$ saying "Y is unbounded and of order type ω^2". It is immediate from Lemma 10, that $\theta_{\omega^2\,\mathrm{ub}}$, too, has no selector in $(\omega^\omega, <)$. But are there any other interesting relations between these two formulas? Can we say, for instance, that $\theta_{\omega^2\,\mathrm{ub}}$ is even "harder" to select than $\theta_{\omega\,\mathrm{ub}}$ (whatever that might mean)? Or, perhaps the other way round?

To turn this admittedly vague question into a mathematical one, we require a notion of comparing formulas and perhaps an equivalence relation on them. But, as our example shows, semantical equivalence seems not to be the right notion. Note, however, the following. For any unbounded ω^2-sequence $S_2 \subseteq \omega^\omega$, the set of limit points of S_2 (i.e., those $\alpha < \omega^\omega$ such that $\sup(S_2 \cap \alpha) = \alpha$) is an unbounded ω-sequence. Also, this set is *definable* from S_2. On the other hand, given an unbounded ω-sequence $S_1 \subseteq \omega^\omega$, the set $\{\alpha + n \mid \alpha \in S_1, n \in \omega\}$ is an unbounded ω^2-sequence. And, again the latter set is definable from S_1. The example suggests the following definition:

Definition 22 (Reduction). *Let $\varphi_0(Y)$, $\varphi_1(X)$ be formulas and $\mathcal{M}$ a structure. We say that φ_0 is* easier than φ_1 to select in $\mathcal{M}$ *(in symbols: $\varphi_0 \preceq_{\mathcal{M}} \varphi_1$) iff there exists a formula $\psi(X, Y)$ such that:*

1. *if φ_1 is not satisfied in $\mathcal{M}$, neither is φ_0, and*
2. *if P satisfies φ_1 in $\mathcal{M}$, then $\psi(P, Y)$ selects φ_0 in $\mathcal{M}$, i.e., there is a unique Q which satisfies $\psi(P, Y)$ in $\mathcal{M}$ and this Q satisfies φ_0 in $\mathcal{M}$.*

We call ψ a reduction *of φ_0 to φ_1 over $\mathcal{C}$.*

It is clear that $\preceq_{\mathcal{M}}$ is a partial preorder on the formulas. The corresponding equivalence classes of $\preceq_{\mathcal{M}}$ are called *selection degrees* in $\mathcal{M}$.

A formula which has a selector in $\mathcal{M}$ is easier to select than any other formula. A non-selectable formula is never easier than a selectable one. Thus, *the* minimal selection degree is the set of selectable formulas. It turns out that in a countable ordinal, all non-selectable formulas also form a single degree:

Proposition 23. *Every $\alpha \in [\omega^\omega, \omega_1)$ has two selection degrees:*

1. *the class all formulas selectable in $(\alpha, <)$, and*
2. *the class of all non-selectable formulas.*

Furthermore, given α and two non-selectable (in α) formulas φ_0 and φ_1, we can compute a reduction ψ of φ_0 to φ_1.

Phrased somewhat differently, this becomes:

Corollary 24. *Let $\alpha \in [\omega^\omega, \omega_1)$ and $P \subseteq \alpha$. Suppose P satisfies some non-selectable formula in $(\alpha, <)$. Then for every formula $\varphi(Y)$, there is a formula $\psi(X, Y)$ such that $\psi(P, Y)$ selects φ in $(\alpha, <)$.*

For the case $\alpha = \omega^\omega$, the proof proceeds by showing that any formula is easier than the formula $\theta_{\omega\,\mathrm{ub}}(Y)$ which says "Y is an unbounded ω-sequence." Then one shows that, conversely, $\theta_{\omega\,\mathrm{ub}}$ is easier than any non-selectable formula in $(\omega^\omega, <)$. Finally, one reduces every other countable $\alpha \geq \omega^\omega$ to the case $\alpha = \omega^\omega$.

Note that if $\varphi_0(Y)$ and $\varphi_1(X)$ are both satisfiable in $\mathcal{M}$, then φ_0 is easier than φ_1 to select in $\mathcal{M}$ if and only if $\varphi_0(Y) \wedge \varphi_1(X)$ has a uniformizer in $\mathcal{M}$ (indeed, a uniformizer for the latter formula and a reduction of φ_0 to φ_1 are one and the same thing). Thus, Proposition 23 actually solves a special case of the uniformization problem in a countable ordinal, namely, where $\varphi(X, Y)$ has the form $\varphi_0(Y) \wedge \varphi_1(X)$.

8 Labeled Ordinals

Corollary 24 leaves open an interesting question. It tells us that if P satisfies some non-selectable formula in $(\omega^\omega, <)$, then with P as a parameter, we can select all formulas in $(\omega^\omega, <)$. But, the formulas we select using P do not themselves "mention" P. In other words, the proposition does not tell us that $(\omega^\omega, <, P)$ has the selection property.

Let P_ω be an unbounded ω-sequence $\{\omega^k \mid k \in \omega\}$. Then for every $\varphi(Y)$ there is a formula $\psi(X, Y)$ such that $\psi(P_\omega, Y)$ selects $\varphi(Y)$ in ω^ω. However, let $\varphi(X, Y)$ says: "If $x < x'$ are successive elements of X, then $Y \cap [x, x')$ is an ω-sequence unbounded in $[x, x')$". Then it is easy to show $\varphi(P_\omega, Y)$ has no selector in $(\omega^\omega, <, P_\omega)$. But, is this fact an artifact of the specific choice of P? That is, could $(\omega^\omega, <)$ be expanded by finitely many subsets of ω^ω to have the selection property?

Proposition 25. *Let $P := \{\omega, \omega^2 + \omega, \omega^3 + \omega^2 + \omega, \ldots\}$. Then:*

(a) $(\omega^\omega, <, P)$ has the selection property,

(*b*) *for any formula* $\varphi(X, Y)$*, a selector for* $\varphi(P, Y)$ *in* $(\omega^\omega, <, P)$ *is computable, and*
(*c*) *the monadic theory of* $(\omega^\omega, <, P)$ *is decidable.*

This proposition can be extended to all $\alpha < \omega^{\omega^2}$.

9 The Church Uniformization Problem

McNaughton [12] observed that the Church uniformization problem can be equivalently phrased in game-theoretic language. This phrasing is easily generalizable to all ordinals.

Definition 26. *For an ordinal* α *and a formula* $\varphi(X, Y)$*, the* McNaughton game $\mathcal{G}^\alpha_\varphi$ *is a game of perfect information of length* α *between two players,* X *and* Y*.*

At stage $\beta < \alpha$*,* X *either accepts or rejects* β*; then,* Y *decides whether to accept or to reject* β*.*

For a play π*, we denote by* X_π *(resp.* Y_π*) the set of ordinals* $< \alpha$ *accepted by* X *(resp.* Y*) during the play. Then,*

$$Y \text{ wins } \pi \text{ iff } (\alpha, <) \models \varphi(X_\pi, Y_\pi).$$

What we want to know is: Does either one of X and Y have a *winning strategy* in $\mathcal{G}^\alpha_\varphi$? If so, which of them? That is, can X choose his moves so that, whatever way Y responds we have $\neg\varphi(X_\pi, Y_\pi)$? Or can Y respond to X's moves in a way that ensures the opposite?

Since at stage $\beta < \alpha$, Y has access only to $X_\pi \cap [0, \beta]$, a winning strategy for Y is one and the same thing as a causal uniformizer for φ. Thus, we may rephrase Definition 5 as follows.

Definition 27 (Game version of the Church uniformization problem). *Let* α *be an ordinal. Given a formula* $\varphi(X, Y)$*, decide whether* Y *has a winning strategy in* $\mathcal{G}^\alpha_\varphi$*.*

In their seminal [3], Büchi and Landweber prove the decidability of the Church uniformization problem in $(\omega, <)$. While in defining the problem, we did not require that the winning strategy (= causal uniformizer) be definable, Büchi and Landweber have shown that in the case of $(\omega, <)$ we can indeed restrict ourselves to definable winning strategies (compare Question ($1''$) in Sect. 1).

Theorem 28 (Büchi and Landweber [3]). *Let* $\varphi(X, Y)$ *be a formula. Then:*

- Determinacy*: One of the players has a winning strategy in the game* $\mathcal{G}^\omega_\varphi$*.*
- Decidability*: It is decidable* which *of the players has a winning strategy.*
- Definable strategy*: The player who has a winning strategy, also has a* definable *winning strategy.*
- Synthesis algorithm*: We can compute a formula* $\psi(X, Y)$ *that defines (in* $(\omega, <)$*) a winning strategy for the winning player in* $\mathcal{G}^\omega_\varphi$*.*

It seems that Büchi and Landweber believed their theorem would generalize to all countable ordinals. Indeed, after stating the theorem just quoted they write:

> "We hope to present elsewhere an extension of [the theorem] from ω to any countable ordinal."

But, from Proposition 8 it follows that for every $\alpha \geq \omega^\omega$ there are formulas φ such that Y wins $\mathcal{G}^\alpha_\varphi$, but has no *definable* winning strategy. Indeed, fix any $\alpha \geq \omega^\omega$. Pick a formula $\varphi'(Y)$ not selectable in $(\alpha, <)$ and let $\varphi(X, Y)$ denote $X = X \wedge \varphi'(Y)$. If $\psi(X, Y)$ defined a winning strategy for Y in $\mathcal{G}^\alpha_\varphi$, then (say) $\exists X(X = \varnothing \wedge \psi(X, Y))$ would select φ in $(\alpha, <)$, which is impossible. On the other hand, Y does win this game: she simply plays some fixed $P \subseteq \alpha$ which satisfies φ in $(\alpha, <)$ (ignoring X's moves).

The Büchi-Landweber Theorem in its entirety generalizes to ordinals smaller than ω^ω. Its determinacy and decidability clauses generalize to all countable ordinals. Thus,

Theorem 29. *Let α be a* countable *ordinal, $\varphi(X, Y)$ a formula.*

- Determinacy*: One of the players has a winning strategy in the game $\mathcal{G}^\alpha_\varphi$.*
- Decidability*: It is decidable* which *of the players has a winning strategy.*
- Definable strategy*: If $\alpha < \omega^\omega$, then the player who has a winning strategy, also has a* definable *(in $(\alpha, <)$) winning strategy. For every $\alpha \geq \omega^\omega$, there is a formula for which this fails.*
- Synthesis algorithm*: If $\alpha < \omega^\omega$, we can compute a formula $\psi(X, Y)$ that defines a winning strategy for the winning player in $\mathcal{G}^\alpha_\varphi$.*

A proof of this theorem can be found in [15]. It uses the composition method to reduce games of every countable length to games of length ω.

Finally, for uncountable ordinals the situation changes radically. Let $\varphi_{spl}(X, Y)$ say: "X is stationary, $Y \subseteq X$ and both Y and $X \setminus Y$ are stationary" (recall Definition 12). Then it follows immediately from [10] that each of the following statements is consistent with ZFC:

1. None of the players has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.
2. Y has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.
3. X has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.

In other words, ZFC can hardly tell us anything concerning this game. On the other hand, S. Shelah (private communication) tells us he believes it should be possible to prove:

Conjecture 30. It is consistent with ZFC that $\mathcal{G}^{\omega_1}_\varphi$ is determined for *every* formula φ.

10 The Bounded Uniformization Problem

As mentioned above, the uniformization problem in $(\omega^\omega, <)$ has not so far been solved (or shown to be undecidable). The task of constructing a uniformizer is intuitively harder than that of constructing a selector in that a uniformizer must *respond* to a given subset substituted for the domain variable X with an appropriate subset to be substituted for the image variable Y; it must (uniformly) answer a variety of challenges. In selection X simply does not appear in the formula. Put more abstractly, its *variability* has been reduced to zero. A natural move therefore, when X does appear in the formula, is to place various restrictions on the subsets of the domain substituted for it. One restriction which comes to mind is to consider formulas $\varphi(t, Y)$ where the t is an *first-order* variable, i.e. ranges over elements of the domain. Once we show the solvability of the uniformization problem for such formulas, our next step may be to allow X to range only over *finite* subsets of the domain, or perhaps over sets of order-type ω, etc. These examples are generalized by the following proposition.

Proposition 31 (Solvability of δ-bounded uniformization). *There is an algorithm that, given ordinals $\alpha \in [\omega^\omega, \omega_1]$, $\delta < \omega^\omega$ and a formula $\varphi(X, Y)$, decides whether there is a ψ which uniformizes φ in $(\alpha, <)$,* when X is restricted to range over subsets of order-type $< \delta$. *If such a ψ exists, the algorithm constructs it.*

Roughly speaking, the proof proceeds by a (non-trivial) reduction of this problem to uniformization over the class of ordinals smaller than δ and to selection in $(\omega^\omega, <)$ (or in $(\omega_1, <)$ when $\alpha = \omega_1$). Proposition 15 tells us the former is solvable, while Proposition 16 handles the latter.

11 Open Problems

We end by presenting several questions and conjectures, whose investigation, we believe, represents the next natural step in the exploration of definable and Church uniformization in the monadic theory of ordinals. First, a question already mentioned.

Question 32. Is the uniformization problem in $(\omega^\omega, <)$ solvable?

Next, we saw that for every countable $\alpha \geq \omega^\omega$ there are McNaughton games, where the winner does not have a definable winning strategy. This leads to (compare Questions ($2''$) and ($3''$) of Sect. 1):

Conjecture 33. There is an algorithm that, given $\alpha \in [\omega^\omega, \omega_1)$ and a formula $\varphi(X, Y)$, decides whether there is a *definable* winning strategy in $\mathcal{G}^\alpha_\varphi$, and if so, returns a ψ defining one.

Rabinovich ([15]) shows that if the conjecture holds for $\alpha = \omega^\omega$, then it is true.

We have seen that the only stumbling block for selection in $(\omega^\omega, <)$ was selecting an unbounded ω-sequence (recall Corollary 24). We believe an analogous statement may be true concerning definability of a winning strategy for games of length ω^ω:

Conjecture 34. For every formula $\varphi(X, Y)$, there is a formula $\psi(W, X, Y)$ such that for every unbounded ω-sequence $S \subseteq \omega^\omega$, $\psi(S, X, Y)$ defines in $(\omega^\omega, <)$ a winning strategy for the winner of $\mathcal{G}^{\omega^\omega}_\varphi$.

It is possible to extend the definition of selection degrees to the case of uniformization. First, extend MLO by allowing also atomic formulas of the form $F(X, Y)$ where F is a new relation symbol. Call the resulting language MLO^F. Let $\mathcal{M}$ be a structure with domain A. For every $f : \mathcal{P}(A) \to \mathcal{P}(A)$, denote by $\mathcal{M}^f$ the expansion of $\mathcal{M}$ which interprets F as (the graph of) f.

Let $\varphi_0(X, Y)$, $\varphi_1(X, Y)$ be MLO-formulas (that is, where F does not appear). Say that φ_0 is *easier than* φ_1 *to uniformize in* $\mathcal{M}$ if and only if there exists an MLO^F-formula $\psi(X, Y)$ such that for every f which uniformizes (the relation defined by) φ_1 in $\mathcal{M}$, ψ uniformizes φ_0 in $\mathcal{M}^f$. Now continue as in the case of selection to define *uniformization degree*. It is easy to see that this definition generalizes the one given for selection. A natural question is then:

Question 35. What are the uniformization degrees in $(\omega^\omega, <)$?

Of course, there is no reason to limit ourselves to *countable* ordinals.

Question 36. What are the selection/uniformization degrees in $(\omega_1, <)$?

Recall that it was only for notational convenience that we stated our results for formulas $\varphi(Y)$ having only a single free-variable Y. Our discussion carries through to formulas $\varphi(\bar{Y})$ with finitely many free-variables. In particular, so does the definition of selection degrees. Thus, $\varphi_0(Y_0, \ldots, Y_{l-1})$ is easier than $\varphi_1(X_0, \ldots, X_{m-1})$ to select in $\mathcal{M}$ iff there is $\psi(X_0, \ldots, X_{m-1}, Y_0, \ldots, Y_{l-1})$ such that for every m-tuple $\bar{P}$ satisfying φ_1 in $\mathcal{M}$, $\psi(\bar{P}, \bar{Y})$ selects φ_0 in $\mathcal{M}$. This is important to remember when discussing selection degrees in $(\omega_1, <)$. Indeed, for each $n \in \omega \setminus 1$, let $\varphi_n(X_0, \ldots, X_{n-1})$ say "for all $i < j < n$, X_i is a stationary subset of ω_1 and $X_i \cap X_j = \varnothing$." Then it can be shown that every formula $\varphi(\bar{Y})$ is easier than φ_n for some $n \in \omega \setminus 1$. We suspect also (but this is yet to be proven) that the φ_n represent distinct selection degrees in $(\omega_1, <)$ and that, more generally, φ_{n+1} never shares a degree with a formula having only n free-variables. If this is indeed so, then unlike what held true for countable ordinals, not all interesting phenomena having to do with selection in $(\omega_1, <)$ are exhibited by formulas having a single free-variable.

Further open questions in the context of uniformization and selection are suggested in [19].

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