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Temporal logics with incommensurable distances are undecidable

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Abstract

Temporal logic based on the two modalities "Since" and "Until" (TL) is the most popular logic for the specification of reactive systems. It is often called the linear time temporal logic. However, metric properties of real time cannot be expressed in this logic. The simplest modalities with metric properties are "X will happen within δ units of time". The extension of TL by all these modalities with rational δ is decidable. We show that the extension of the linear time temporal logic by two modalities "X will happen within one unit of time", "X will happen within τ unit of time" is undecidable, whenever τ is irrational.

1. Introduction

Temporal logic based on the two modalities "Since" and "Until" (TL) is the most popular framework for reasoning about the evolving of a system in time. By Kamp's theorem [9] this logic has the same expressive power as the monadic first order predicate logic. Therefore, the choice between monadic logic and temporal logic is merely a matter of personal preference.

Temporal logic and first-order monadic logic of order are equivalent whether the system evolves in discrete time or in continuous time. For continuous time neither logic is strong enough to express

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properties like: “X will happen within one unit of time”; hence, to specify metric properties we need a more expressive version of these logics.

R. Koymans [12] extended this logic by modality “X will happen at distance one from the current point”. Unfortunately, the satisfiability problem for this logic is undecidable. Following the works of T. Henzinger and others [12,3,2,10,4,14,1,5] and more, we introduced in [6,8] the logic *QTL* (quantitative temporal logic), which has besides the modalities *Until* and *Since* two metric modalities: $\diamond_1(X)$ and $\overleftarrow{\diamond}_1(X)$. The first one says that X will happen (at least once) within the next unit of time, and the second says that X happened within the last unit of time. We proved that:

(1) The validity and satisfiability problem for this logic is decidable, whether we are interested in systems with *finite variability*, or in all systems evolving in time (a system has finite variability if it changes only at finitely many points, in any finite interval of time).
(2) This logic subsumes the different decidable metric temporal logics that we found in the literature, like *MITL* [2,1,5]. In particular, for a natural n, it is easy to express in this logic “X will happen in the next interval of length n” and “X happened within the last interval of length n” (see survey [7]).

Result (1) above implies that:

(3) The validity and satisfiability problem for the temporal logic with modalities {until, since} $\cup$ $\{\diamond_q, \overleftarrow{\diamond}_q : q$ ranges over the non-negative rationals$\}$ is decidable, whether we are interested in systems with finite variability, or in all systems evolving in time. Here, modality $\diamond_q(X)$ expresses “X will happen in the next interval of length q” and $\overleftarrow{\diamond}_q(X)$—“$X$ happened within the last interval of length q”.

A natural question is whether an extension of this logic by modalities $\diamond_\tau(X)$ and $\overleftarrow{\diamond}_\tau(X)$ for an irrational τ is decidable.

We show that

Theorem 1 (Main). *For every irrational τ, the temporal logic with four modalities Until, Since, $\diamond_1(X)$ and $\diamond_\tau(X)$ is undecidable over the reals both for finite variability and for all interpretations.*

This theorem and its proof can be straightforwardly generalized to the case when 1 and τ are replaced by any numbers c, d such that c/d is irrational.

The undecidability proof is obtained by reduction from the reachability problem for two-counter machines.

In the next section, we recall definitions and facts about temporal logics. The proof of Theorem 1 is given in Section 3. In Section 4 some further results are proved. In particular, it is shown that the logics of Theorem 1 are undecidable over the rational line.

2. Temporal logic

Temporal logics evolved in philosophical logic and were enthusiastically embraced by a large body of computer scientists. A temporal logic uses logical constructs called “modalities” to create a language that is free from quantifiers. The temporal logic with a set M of modalities is denoted by $TL(M)$. Here, we consider $TL(\mathbf{U}, \mathbf{S}, \Diamond_1, \Diamond_\tau, \overleftarrow{\Diamond}_1, \overleftarrow{\Diamond}_\tau)$. Its syntax is defined as follows.

It has *monadic predicate names* $P_1, P_2, \ldots$ and *modality names* $\mathbf{U}, \mathbf{S}, \Diamond_1, \Diamond_\tau, \overleftarrow{\Diamond}_1, \overleftarrow{\Diamond}_\tau$. The formulas of this temporal logic are given by the grammar:

$$\varphi ::= true \mid P \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi\, \mathbf{U}\varphi \mid \varphi\, \mathbf{S}\varphi \mid \Diamond_1\varphi \mid \Diamond_\tau\varphi \mid \overleftarrow{\Diamond}_1\varphi \mid \overleftarrow{\Diamond}_\tau\varphi$$

We consider the interpretations of this logic over the set $\mathbb{R}^{\geqslant 0}$ of non-negative reals. A structure for this logic is $A = \langle \mathbb{R}^{\geqslant 0}, <, P_1, P_2, \ldots, P_n \rangle$, where the monadic predicates $P_i \subseteq \mathbb{R}^{\geqslant 0}$ are the interpretations of corresponding predicate names mentioned in the formulas of the logic. For $a \in \mathbb{R}^{\geqslant 0}$ the relation φ holds at a in a model A (notations $A, a \models \varphi$) is defined by structural induction:

- $A, a \models true$
- For atomic formulas: $A, a \models P$ iff $a \in P$.
- For Boolean combinations, the definition is the usual one.
- $A, a \models \varphi_1\, \mathbf{U}\varphi_2$ iff there is $b > a$ such that $A, b \models \varphi_2$ and $A, c \models \varphi_1$ for all c in the open interval (a, b).
- $A, a \models \varphi_1\, \mathbf{S}\varphi_2$ iff there is $b < a$ such that $A, b \models \varphi_2$ and $A, c \models \varphi_1$ for all c in the open interval (b, a).
- Let $\delta \in \mathbb{R}^{\geqslant 0}$. Then $A, a \models \Diamond_\delta\varphi$ iff there is b in the interval $(a, a+\delta)$ such that $A, b \models \varphi$. Similarly, $A, a \models \overleftarrow{\Diamond}_\delta\varphi$ iff there is b in the interval $(a-\delta, a)$ such that $A, b \models \varphi$.

We use standard abbreviations (derived modalities):

$$\begin{array}{ll} \Diamond X \triangleq true\ \mathbf{U}X & \overleftarrow{\Diamond} X \triangleq true\ \mathbf{S}X \\ \Box X \triangleq \neg\Diamond\neg X & \overleftarrow{\Box} X \triangleq \neg\overleftarrow{\Diamond}\neg X \\ \Box_1 X \triangleq \neg\Diamond_1\neg X & \overleftarrow{\Box}_1 X \triangleq \neg\overleftarrow{\Diamond}_1\neg X \end{array}$$

Note that the semantics of all modalities above is a "strict" one, for example, the formula $\neg X \wedge \Box X$ is satisfiable.

We proved in [6] and [8] that if our logic contains $\Diamond_\delta\varphi$ and $\overleftarrow{\Diamond}_\delta\varphi$, then it can express more liberal bounds in time like: "X will happen in the future, within the period that starts in $m\delta$ units of time, and ends in $n\delta$ units of time" ($m < n$ are naturals). We may also include or exclude one or both of the endpoints of the period.

Below are some useful equivalences:

$$\Diamond_a(\Diamond_b(X)) \leftrightarrow \Diamond_{a+b}(X)$$

$$\overleftarrow{\Diamond}_a(\overleftarrow{\Diamond}_b(X)) \leftrightarrow \overleftarrow{\Diamond}_{a+b}(X)$$

Observe that

$(\neg X)\, \mathbf{U}X$ holds at b iff there is $c > b$ such that $c \in X$ and no point from the interval (b, c) is in X.

$\neg(\neg X\, \mathbf{U} true)$ holds at b iff for every $c > b$ a point from the interval (b, c) is in X.

Hence,

$(\neg\Diamond_a(X)) \wedge ((\neg\Diamond_a(X))\ \mathbf{U}\Diamond_a(X)) \wedge (\neg(\neg\Diamond_a(X)\ \mathbf{U} true))$ holds at b iff the next occurrence of X is at distance a from b, i.e., $\neg X$ holds on $(b, b+a)$ and X holds at $b+a$.

These equivalences and observations imply the following Proposition.

Proposition 2. *Let TL be a temporal logic which includes modalities* $\mathbf{U}$ *and* $\mathbf{S}$.

(1) *If* $\Diamond_{\delta_1}\varphi$ *and* $\Diamond_{\delta_2}\varphi$ *are expressible in TL, then* $\Diamond_{\delta_1+\delta_2}\varphi$ *is also expressible in TL. In particular, if* $\Diamond_{\delta}\varphi$ *is expressible in TL, then* $\Diamond_{n\delta}\varphi$ *is also expressible in TL.*
(2) *If* $\overleftarrow{\Diamond}_{\delta_1}\varphi$ *and* $\overleftarrow{\Diamond}_{\delta_2}\varphi$ *are expressible in TL, then* $\overleftarrow{\Diamond}_{\delta_1+\delta_2}\varphi$ *is also expressible in TL. In particular, if* $\overleftarrow{\Diamond}_{\delta}\varphi$ *is expressible in TL, then* $\overleftarrow{\Diamond}_{n\delta}\varphi$ *is also expressible in TL.*
(3) *If* $\Diamond_{\delta}\varphi$ *is expressible in TL, then we can express in TL that the next occurrence of* φ *is at the distance exactly* δ *from the current point, i.e., there is a formula next-occ*$_\delta(\varphi)$ *such that* $A, a \models$ *next-occ*$_\delta(\varphi)$ *iff* $A, a+\delta \models \varphi$ *and* $A, b \models \neg\varphi$ *for every* $b \in (a, a+\delta)$.
(4) *Similarly, if* $\overleftarrow{\Diamond}_{\delta}\varphi$ *is expressible in TL, then there is a TL formula prev-occ*$_\delta(\varphi)$ *such that* $A, a \models$ *prev-occ*$_\delta(\varphi)$ *iff* $A, a-\delta \models \varphi$ *and* $A, b \models \neg\varphi$ *for every* $b \in (a-\delta, a)$.

Note that unlike the property "the next occurrence of X will happen at distance exactly δ", the property "there is an occurrence of X which will happen at distance exactly δ" is not expressible from $\Diamond_\delta$.

3. Proof of Theorem 1

For the proof of Theorem 1, without restriction of generality, we can assume that $\tau < 1$. Indeed, assume that $\tau > 1$. Let n be a natural number greater than τ. By Proposition 2, $\Diamond_n$ is expressible from $\Diamond_1$. Therefore, the undecidability of the temporal logic with modalities $\{\mathbf{U}, \mathbf{S}, \Diamond_n, \Diamond_\tau\}$ will imply the undecidability of the temporal logic with modalities $\{\mathbf{U}, \mathbf{S}, \Diamond_1, \Diamond_\tau\}$.

This theorem can be straightforwardly generalized to the case when 1 and τ are replaced by any numbers c and d such that c/d is irrational. Indeed, let φ be a formula over modalities $\{\mathbf{U}, \mathbf{S}, \Diamond_1, \Diamond_\tau\}$. Let ψ be obtained from φ by replacing all occurrences of $\Diamond_1$ (respectively, of $\Diamond_\tau$) by $\Diamond_c$ (respectively, by $\Diamond_{c\tau}$). Then, φ is satisfiable iff ψ is satisfiable.

The undecidability proof is obtained by reduction from the reachability problem for two-counter machines. Similar reductions were used by [13,11] to prove undecidability results for timed automata with incommensurable constants.

A two-counter machine M consists of a finite set of states and two unbounded nonnegative integer variables called counters. Initially both counter values are 0. Three types of instructions are used: branching based upon whether a specific counter has the value 0, incrementing a counter, and decrementing a counter (which leaves unchanged a counter value of 0). We assume that from every state exactly one instruction can be executed. It is well known that the problem whether from state 1 a specified state k is reachable is undecidable.

Let M be a two-counter machine with a set of states $\{1, \ldots, k\}$. We are going to construct a formula $\varphi_M(Q_1, \ldots, Q_k, C_1, C_2, Z, N, R_0, R_1, R_2)$ which "encodes" the computation of M. Let us first explain how computations can be encoded on reals. An ω-sequence $\langle n_0, m_0, q_0\rangle\langle n_1, m_1, q_1\rangle \dots \langle n_i, m_i, q_i\rangle \dots$ of M configurations can be encoded as a structure with predicates $Q_1, \dots Q_k$, C_1 and C_2. Predicates Q_i will encode states in the ω-sequence, predicate C_1 (respectively, C_2) will encode the sequence n_i (respectively, m_i) of the values of the first (respectively, second) counter. The ith configuration $\langle n_i, m_i, q_i\rangle$ will be encoded on the interval $(i-1, i]$ as follows: Q_{q_i} will be true at i and false at other points of this interval; Q_j for $j \neq q_i$ will be false at all points in the interval; C_1 (respectively, C_2) will hold only at the point $i - \{n_i\tau\}$ (respectively, $i - \{m_i\tau\}$), where $\{a\}$ denotes the fractional part of a, i.e., a number $r \in [0, 1)$ such that $a - r$ is an integer. The fact that τ is irrational implies that for all $n, m \in \mathbb{N}$ if $\{m\tau\} = \{n\tau\}$ then $n = m$. Hence, the encoding is injective.

Now we are going to construct a formula $\varphi_M(Q_1, \dots, Q_k, C_1, C_2, Z, N, R_0, R_1, R_2)$ which "encodes" the computation of M and has the following properties:

Properties of φ_M: $\mathbb{R}^{\geqslant 0}, 0 \models \varphi_M$ if and only if the following conditions hold:

(1) Z is interpreted as $\{0\}$.
(2) N is interpreted as the set of naturals.
(3) R_j is interpreted as $\bigcup_{n\in\mathbb{N}}[3n + j, 3n + j + 1)$ for $j = 0, 1, 2$.
(4) $\bigcup_{i=1}^{k} Q_i = \mathbb{N}$ and at every $n \in \mathbb{N}$ exactly one of Q_i holds.
(5) In every interval $(n, n + 1]$ there is one occurrence of C_j for $n \in \mathbb{N}$ and $j = 1, 2$.
(6) $\langle n_0, m_0, q_0\rangle\langle n_1, m_1, q_1\rangle \dots \langle n_i, m_i, q_i\rangle \dots$ is the (unique) computation of M iff $\langle Q_1, \dots, Q_k, C_1, C_2\rangle$ encodes this computation, i.e.,
  (a) Q_{q_i} holds at i.
  (b) C_1 holds at $i - \{n_i\tau\}$ and C_2 holds at $i - \{m_i\tau\}$.

The first five conditions are independent of a counter machine and are expressible by a formula ψ_{1-5} which is the conjunction of the following formulas:

(1) $\square(Z \leftrightarrow \neg \overleftarrow{\Diamond}\, true)$.
(2) $\square\big((Z \to N) \wedge (N \to \text{next-occ}_1(N))\big)$.

Note that Z and N satisfy the above two formulas iff Z is interpreted as $\{0\}$ and N as the set $\mathbb{N} \subseteq \mathbb{R}^{\geqslant 0}$ of natural numbers. Properties (3)–(5) can be defined from Z and N by "non-metrical" modalities **U** and **S**.

(3) R_j are disjoin and for $j = 0, 1, 2$:

$$\square\Big((Z \to R_0) \bigwedge ((R_j \wedge N) \to (R_j \wedge \neg N)\, \mathbf{U} (N \wedge (R_{j+1 mod 3}))\Big)$$

(4) $\square\big((\vee Q_i \leftrightarrow N) \bigwedge (\wedge_{i\neq j}(Q_i \to \neg Q_j))\big)$.
(5) $\square\big((N \to ((\neg N)\, \mathbf{U} C_j) \wedge (C_j \to ((\neg C_j)\, \mathbf{U} N)\big)$ for $j =, 1, 2$.

To express the last condition, we are going first to describe formulas *unchanged*, *incr*, *decr* which have the following properties.

If N, R_0, R_1, R_2, C_1 and C_2 satisfy the conditions (1)–(5), then for $C \in \{C_1, C_2\}$ and every $n \in N$:

$\mathbb{R}^{\geqslant 0}, n \models \mathit{unchanged}(C)$ if the value of the counter C was unchanged between interval $(n-1,n]$ and $(n,n+1]$. That is, if $n-v=n+1-u$, where v (respectively, u) is the unique point in the interval $(n-1,n]$ (respectively, in the interval $(n,n+1]$ which is in C.

$\mathbb{R}^{\geqslant 0}, n \models \mathit{decr}(C)$ if the value of the counter C was decremented between interval $(n-1,n]$ and $(n,n+1]$. That is, if $\{n-v-\tau\}=n+1-u$, where v (respectively, u) is the unique point in the interval $(n-1,n]$ (respectively, in the interval $(n,n+1]$ which is in C.

$\mathbb{R}^{\geqslant 0}, n \models \mathit{incr}(C)$ if the value of the counter C was incremented between interval $(n-1,n]$ and $(n,n+1]$. That is, if $\{n-v+\tau\}=n+1-u$, where v (respectively, u) is the unique point in the interval $(n-1,n]$ (respectively, in the interval $(n,n+1]$ which is in C.

unchanged(C) can be defined as

$$\begin{array}{c}(\neg C)\rightarrow\big((\neg N)\ \mathbf{S}(C\wedge \text{next-occ}_1(C))\big)\\ \bigwedge\\ C\rightarrow\big((\neg N)\ \mathbf{U}(C\wedge N)\big).\end{array}$$

decr(C) can be defined as follows. If C holds at n, then C will hold at $n+1$ (recall that by our agreement the decrement of zero counter is zero). In other cases, if the distance from n to the preceding C point v is strictly less than τ, then the distance from v to the next C point is τ; otherwise, the distance from v to the next C point is $1+\tau$. This is formalized by the formula

$$\begin{array}{c}C\rightarrow(\neg N\ \mathbf{U}(N\wedge C))\\ \bigwedge\\ (\neg N)\ \mathbf{S}(C\wedge\Box_\tau\neg N)\rightarrow(\neg N)\ \mathbf{S}(C\wedge\Diamond_{1+\tau}C)\\ \bigwedge\\ \Big(\neg\big((\neg N)\ \mathbf{S}(C\wedge\Box_\tau\neg N)\big)\Big)\rightarrow\Big((\neg N)\ \mathbf{S}(C\wedge\Diamond_\tau C)\Big)\end{array}$$

Finally, *incr*(C) can be defined by the following cases

Case $v-(n-1)>\tau$. In this case u should be at distance one from the point $v-\tau$. In our logic, we cannot say that P occurs at exactly distance one; however, we can express that the first occurrence of P is at distance one. Unfortunately, u is a second occurrence of P at distance one from $v-\tau$. This is also inexpressible in our logic. To overcome this difficulty, we will use predicates R_j $(j=0,1,2)$. Note that if R_j holds at v, then u will be the first occurrence of $P\wedge R_{j+1 mod 3}$ at distance one from $v-\tau$. All this can be formalized by the formula ψ_1 defined as

$$\wedge_j\Big(R_{j+1mod3}\rightarrow(\neg N)\ \mathbf{S}(R_j\wedge\text{next-occ}_\tau(C)\wedge\text{next-occ}_{1+\tau}(C\wedge R_{j+1mod3}))\Big)$$

Case $v-(n-1)<\tau$. In this case u should be at distance two from the point $v-\tau$. And if v was in R_j then $v-\tau$ is in $R_{j-1mod3}$ and u is in $R_{j+1mod3}$. This can be formalized by the formula ψ_2 defined as

$$\wedge_j\Big(R_{j+1mod3}\rightarrow(\neg N)\ \mathbf{S}\big(N\wedge(\neg N)\ \mathbf{S}(\text{next-occ}_\tau(C)\wedge\text{next-occ}_{2+\tau}(C\wedge R_{j+1mod3}))\big)\Big)$$

Case $v-(n-1)=\tau$ In this case u should be $n+1$. This can be formalized by the formula

$$\psi_3 \triangleq (\neg N)\ \mathbf{U}(C \wedge N)$$

Therefore, $incr(C)$ can be defined as

$$\begin{gathered}\big((\neg N)\ \mathbf{S}(N \wedge \Box_\tau(\neg C))\big) \to \psi_1 \\ \wedge \\ \big((\neg N)\ \mathbf{S}(N \wedge \Diamond_\tau(C))\big) \to \psi_2 \\ \wedge \\ \big((\neg N)\ \mathbf{S}(N \wedge \text{next-occ}_\tau(C))\big) \to \psi_3\end{gathered}$$

Now we are ready to express property (6). Let M be a two-counter machine. For every state i of M we introduce a formula $step_i$. Recall that we have assumed that in every state exactly one instruction is enabled.

Branching test. Assume that the instruction for a state $i \in \{0,\ldots,k\}$ is:
if c_j is zero *then change to state* i_0 *else to state* i_1.
Then the corresponding formula is

$$\Box(Q_i \to (unchanged(C_1) \wedge unchanged(C_2)) \wedge (C_j \to ((\neg N)\ \mathbf{U}Q_{i_0}) \wedge ((\neg C_j) \to (\neg N)\ \mathbf{U}Q_{i_1})$$

Increment a counter. Assume that the instruction for a state $i \in \{0,\ldots,k\}$ is:
increment c_j *and change to state* i_0.
Then the corresponding formula is

$$\Box(Q_i \to (incr(C_j) \wedge unchanged(C_{3-j}) \wedge ((\neg N)\ \mathbf{U}Q_{i_0})$$

Decrement a counter. Assume that the instruction for a state $i \in \{0,\ldots,k\}$ is:
decrement c_j *and change to state* i_0.
Then the corresponding formula is

$$\Box(Q_i \to (incr(C_j) \wedge unchanged(C_{3-j}) \wedge ((\neg N)\ \mathbf{U}Q_{i_0})$$

Finally, assume that 1 is the initial state of a counter machine M. The formula φ_M which encodes the computation of M can be defined as

$$\psi_{1-5} \wedge (Z \to (Q_1 \wedge C_1 \wedge C_2)) \wedge \bigwedge_i step_i$$

If k is a state of M then $\mathbb{R}^{\geqslant 0}, 0 \models \varphi_M \wedge \Diamond Q_k$ if and only if M reaches a state k. Since, the reachability problem for two-counter machines is undecidable, we obtain that the satisfiability problem is undecidable for any temporal logic which contains the modalities $\mathbf{U}$, $\mathbf{S}$, $\Diamond_1(X)$ and $\Diamond_\tau(X)$.

4. Further results

Our proof can be easily modified to show the following Corollary:

Corollary 3. *Let TL be a temporal logic which contains modalities* $\mathbf{U}$ *and* $\mathbf{S}$. *Assume that* $c, b \in \mathbb{R}^{\geqslant 0}$ *are such that* c/b *is irrational and that at least one of the modalities from the set* $\{\Diamond_c, \overleftarrow{\Diamond}_c\}$ *and one of the modalities from the set* $\{\Diamond_b, \overleftarrow{\Diamond}_b\}$ *is expressible in TL. Then the satisfiability problem for TL over the reals is undecidable.*

Our choice of modalities $\Diamond_b$ corresponds to an open interval $(0, b)$. One can consider modalities $\Diamond_{(0,b]}$:

$\Diamond_{(0,b]}X$ holds at a point u if there is v such that $v \in X$ and $v \in (u, u + b]$.

Our proof can be adopted to show that if b/c is irrational, then the temporal logic with modalities $\{\mathbf{U}, \mathbf{S}, \Diamond_{(0,b]}, \Diamond_{(0,c]}\}$ and the temporal logic with modalities $\{\mathbf{U}, \mathbf{S}, \Diamond_{(0,b]}, \Diamond_c\}$ are undecidable.

Another extension deals with temporal logics over the rationals $\mathbb{Q}^{\geqslant 0}$. Note that there are properties definable by first-order monadic logic which are undefinable in the temporal logic with modalities $\mathbf{U}$ and $\mathbf{S}$. Stavi introduced two modalities $\mathbf{U}_{Stavi}$ and $\mathbf{S}_{Stavi}$ and proved that the temporal logic with modalities $\mathbf{U}, \mathbf{S}, \mathbf{U}_{Stavi}, \mathbf{S}_{Stavi}$ has the same expressive power as the first-order monadic logic over the class of all linear orders, i.e., for every formula $\varphi(t_0, P_1, \ldots P_k)$ there is an equivalent (over the class of linear order) formula in $TL(\mathbf{U}, \mathbf{S}, \mathbf{U}_{Stavi}, \mathbf{S}_{Stavi})$ [4]. In our undecidability proof we do not use the Stavi modalities. Over the rationals, the modalities $\Diamond_a X$ and $\overleftarrow{\Diamond}_a X$ are defined exactly like over the reals. The formula which was used extensively in our proof expresses the property "there is a next occurrence of X after the current point at distance exactly δ" is false for all irrational δ.

Theorem 4. *For every irrational* τ, *the temporal logic with four modalities Until, Since,* $\Diamond_1(X)$ *and* $\Diamond_\tau(X)$ *is undecidable over the rationals.*

In the rest of this section a proof of Theorem 4 is outlined. Let M be a two-counter machine. The ith configuration $\langle n_i, m_i, q_i\rangle$ will be encoded on the interval $(i-1, i]$ as follows: Q_{q_i} will be true at i and false at other points of this interval; Q_j for $j \neq q_i$ will be false at all points in the interval; C_1 (respectively, C_2) will hold on the subinterval $(i - \{n_i\tau\}, i)$ (respectively, $(i - \{m_i\tau\}, i)$, where $\{a\}$ denotes the fractional part of a. Note that if τ is irrational, then for all $i \in \mathbb{N}$ and $0 < n \in \mathbb{N}$ the number $i - \{n_i\tau\}$ is not rational; however, the interval $(i - \{n_i\tau\}, i)$ is well defined.

We can write a formula which encodes the computation of M in a way similar to the encoding in the proof of Theorem 1. For this purpose one has to specify that one configuration follows after the other according to the table of M. First, we can specify predicates Z, N, R_0, R_1, R_2, R_3 with the same interpretation as in the proof of Theorem 1. This can be formalized by $TL(U, \Diamond_1)$ formulas. Next, we can express that for every $i \in \mathbb{N}$, in the interval $(i-1, i]$ each of the (counter) predicates C_1 and C_2 either holds only at i or at an open interval with a right end point i.

Then one can construct formulas *unchanged*, *decr* and *incr* such that for $C \in \{C_1, C_2\}$ under the interpretation which satisfy the above requirement

(1) $\mathbb{Q}^{\geqslant 0}, i \models unchanged(C)$ if the value of the counter (coded by) C is the same in the intervals $(i-1, i]$ and $(i, i+1]$.

(2) $\mathbb{Q}^{\geqslant 0}, i \models incr(C)$ if the value of the counter (coded by) C in $(i-1, i]$ plus one is same as its value in $(i, i+1]$.
(3) $\mathbb{Q}^{\geqslant 0}, i \models decr(C)$ if the value of the counter (coded by) C in $(i-1, i]$ minus one is same as its value in $(i, i+1]$.

For example, *unchanged*(C) can be defined as $\Box\alpha$ where α is

$$\begin{array}{c} (N \wedge C) \rightarrow \big((\neg C)\ \mathbf{U}(N \wedge C)\big) \\ \wedge \\ \bigwedge_j \\ (N \wedge \neg C \wedge R_j) \rightarrow \Big(\big[\big((\neg C) \rightarrow \Diamond_1(R_{j+1 mod 3} \wedge \neg C)\big) \wedge \big((C \rightarrow \Diamond_1(R_{j+1 mod 3} \wedge C)\big)\big]\ \mathbf{S}N)\Big) \end{array}$$

We leave to the reader to write down formulas *incr* and *decr*. Finally, a formula which is satisfiable in $\mathbb{Q}^{\geqslant 0}$ at 0 iff state k of M is reachable from state 1, can be constructed from the above formulas exactly like in the proof of Theorem 1.

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