Expressiveness of Metric Modalities for Continuous Time

Yoram Hirshfeld and Alexander Rabinovich

Sackler Faculty of Exact Sciences, Tel Aviv Univ., Israel 69978 {joram, rabinoa}@post.tau.ac.il

Abstract. We prove a conjecture by A. Pnueli and strengthen it showing a sequence of "counting modalities" none of which is expressible in the temporal logic generated by the previous modalities, over the real line, or over the positive reals. We use this sequence to prove that over the real line there is no finite temporal logic that can express all the natural properties that arise when dealing with systems that evolve in continuous time.

1 Introduction

Temporal Logic based on the two modalities "Since" and "Until" (TL) is a most popular framework for reasoning about the evolving of a system in time. By Kamp's theorem [12] this logic has the same expressive power as the monadic first order predicate logic. Therefore the choice between monadic logic and temporal logic is merely a matter of personal preference.

Temporal logic and the monadic logic are equivalent whether the system evolves in discrete steps or in continuous time, But for continuous time both logics are not strong enough to express properties like: "X will happen within 1 unit of time", and we need a better version of the logics.

Following the work R. Koymans, T. Henzinger and others, [14, 3, 2, 13, 5, 17, 1, 7], and more, we introduced in [9, 11] the logic QTL (Quantitative Temporal Logic), which has besides the modalities Until and Since two metric modalities: $\diamondsuit_1(X)$ and $\overleftrightarrow_1(X)$. The first one says that X will happen (at least once) within the next unit of time, and the second says that X happened within the last unit of time. We proved:

- 1. This logic consumes the different metric temporal logics that we found in the literature, like MITL [2, 1, 7].
- 2. The validity and satisfiability problem for this logic is decidable, whether we are interested in systems with *finite variability*, or in all systems evolving in time (a system has finite variability if it changes only at finitely many points, in any finite interval of time).

An important question was not answered: is this logic expressive enough to express all the important properties about evolving systems? If not, which modalities should we add? A. Pnueli suggested the modality $P_2(X, Y)$: "X and then Y will both occur in the next unit of time". $P_2(X, Y)$ was probably thought of as a natural strengthening of the metric logics that were presented before. It can serve as a first in a sequence of extensions of the logic, where for each natural number n, we add the modality $P_n(X_1, \ldots, X_n)$. $P_n(X_1, \ldots, X_n)$ says that there is an increasing sequence of points t_1, \ldots, t_n in the coming unit interval such that $X_i(t_i)$ holds for $i = 1, \ldots, n$. It probably seemed pointless to define new modalities when you cannot prove that they can express something new. Pnueli conjectured that the modality $P_2(X, Y)$ cannot be expressed in the different (equivalent) metric logic that we defined above, but he left it at that (we were unable to locate where this conjecture was first published. It is attributed to Pnueli in later papers like [2] and [17]).

Here we prove Pnueli's conjecture: We denote by $C_2(X)$ the modality "X will be true at least twice in the next unit of time". This is a special case of $P_2(X,Y)$ where Y = X. We prove:

- $-C_2(X)$ cannot be expressed in QTL (and the equivalent languages). Moreover:
- For every *n* let us define the modality $C_n(X)$ that says that *X* will hold at least at *n* points of the next unit interval. Then the modality $C_{n+1}(X)$ cannot be expressed in the logic $QTL(C_1, \ldots, C_n)$, which is generated by QTL, and the extra *n* modalities $QTL(C_1(X), \ldots, C_n(X))$

Therefore there is a proper hierarchy of temporal logics, and it is important to investigate how to extend the logic QTL to a full strength, yet decidable logic. Counting modalities like $C_n(X)$ are not a natural choice of a modality and it maybe suspected that a better chosen finite set of modalities together with QTL is as strong as, or even stronger than QTL with all the modalities C_n . Not so! We were able to prove:

- No finite temporal logic can express all the statements $C_n(X)$.

The last claim needs to be made exact: No finite temporal logic, whose modalities are defined in a natural monadic predicate logic, can express all the counting modalities over continuous time, extended in both directions; i.e, over the full real line. We believe that the same is true also when we consider continuous time with a first point 0, i.e, positive time line, but the proof will be more difficult.

When stated formally the result seems even stronger: Let L be second order monadic logic of order, together with the predicate B(t,s) which says that s = t + 1. The modalities $C_n(X)$ are expressible in this logic, but no temporal logic with a finite or infinite family of modalities which are defined by formulas with bounded quantifier depth can express all the modalities $C_n(X)$.

In predicate logic the expressive power grows with the increasing of the quantifier depth. In temporal logic this is achieved by increasing the nesting depth of the modalities. Kamp showed that for the simplest logic of order iterating the modal operations can replace the complex use of quantifier. Our result, together with previous evidence (see [15]) suggests that this was a lucky peculiarity of the first-order monadic logic of linear order, and it cannot be expected to hold for strong logics.

These results leave open and emphasize even more the main question: Is the logic QTL enriched by all the modalities $P_n(X_1, \ldots, X_n)$ the appropriate maximal decidable temporal logic? If not, what is its strength, and what is the appropriate continuous metric temporal logic?

The paper is divided as follows: In section 2 we recall the definitions and the previous results concerning the continuous time logics. In section 3 we prove Pnueli's conjecture and its generalization, that the modalities C_i create a strictly increasing family of logics. In section 4 we discuss the more general and abstract result: that no temporal logic based on modalities with finite quantifier depth can express all the modalities C_n .

2 Monadic Logic and Quantitative Temporal Logic

2.1 MLO - Monadic Logic of Order

The natural way to discuss systems that evolve in time is classical predicate logic. The language has a name for the order relation of the time line, and a supply of unary predicate names to denote a properties that the system may or may not have at any point in time. Hence:

The syntax of the monadic predicate logic of order - MLO has in its vocabulary *individual* (first order) variables t_0, t_1, \ldots , monadic *predicate* variables X_0, X_1, \ldots , and one binary relation < (the order). Atomic formulas are of the form $X(t), t_1 = t_2$ and $t_1 < t_2$. Well formed formulas of the monadic logic MLO are obtained from atomic formulas using Boolean connectives $\neg, \lor, \land, \rightarrow$ and the (first order) quantifiers $\exists t$ and $\forall t$ and the (second-order) quantifiers $\exists X$ and $\forall X$. The formulas which do not use $\exists X$ and $\forall X$ are called first-order MLO formulas (FOMLO). Note that FOMLO formulas may contain free monadic predicate variables, and they will be assigned to particular predicates in a structure.

A structure for MLO is a tuple $M = \langle A, \langle, P_1, \ldots, P_n \rangle$, where A is a set linearly ordered by the relation \langle , and P_1, \cdots, P_n , are one-place predicates (sets) that correspond to the predicate names in the logic. We shall use the simple notation $\langle A, \langle \rangle$ when the particular predicates are not essential to the discussion.

The main models are: the **continuous canonical model** $\langle R^+, \langle \rangle$, the non-negative real line, and the **discrete canonical model** $\langle N, \langle \rangle$, the naturals.

As is common we will use the assigned formal names to refer to objects in the meta discussion. Thus we will write:

$$M \models \varphi[t_1, \dots, t_k; X_1, \dots, X_m]$$

where M is a structure, φ a formula, t_1, \dots, t_k elements of M and X_1, \dots, X_m predicates in M, instead of the correct but tedious form:

$$M, \tau_1, \ldots, \tau_k; P_1, \ldots, P_m \models_{MLO} \varphi(t_1, \ldots, t_k; X_1, \ldots, X_m),$$

where τ_1, \ldots, τ_k and $P_1 \cdots, P_m$ are names in the metalanguage for elements and predicates in M.

2.2**Temporal Logics**

Temporal logics evolved in philosophical logic and were enthusiastically embraced by a large body of computer scientists. It uses logical constructs called "modalities" to create a language that is free from variables and quantifiers. Here is the general logical framework to define temporal logics:

The syntax of the Temporal Logic $TL(O_1^{(k_1)}, \ldots, O_n^{(k_n)}, \ldots)$ has in its vo-cabulary monadic predicate names P_1, P_2, \ldots and a sequence of modality names with prescribed arity, $O_1^{(k_1)}, \ldots, O_n^{(k_n)}, \ldots$ (the arity notation is usually omitted). The formulas of this temporal logic are given by the grammar:

$$\varphi ::= True | P | \neg \varphi | \varphi \land \varphi | O^{(k)}(\varphi_1, \cdots, \varphi_k)$$

A temporal logic with a finite set of modalities is called a finite (base) temporal logic.

Structures for TL are again linear orders with monadic predicates M = $\langle A, \langle P_1, P_2, \ldots, P_n \rangle$, where the predicate P_i are those which are mentioned in the formulas of the logic. Every modality $O^{(k)}$ is interpreted in every structure M as an operator $O_M^{(k)} : [\mathbb{P}(A)]^k \to \mathbb{P}(A)$ which assigns "the set of points where $O^{(k)}[S_1,\ldots,S_k]$ holds" to the k-tuple $\langle S_1,\ldots,S_k \rangle \in \mathbb{P}(A)^k$. (Here \mathbb{P} is the power set notation, and $\mathbb{P}(A)$ denotes the set of all subsets of A.) Once every modality corresponds to an operator the semantics is defined by structural induction:

- $\begin{array}{ll} \mbox{ for atomic formulas: } \langle M,t\rangle \models_{_{TL}} P & \mbox{iff} \quad t \in P. \\ \mbox{ for Boolean combinations the definition is the usual one.} \end{array}$
- for $O^{(k)}(\varphi_1, \cdots, \varphi_k)$

$$\langle M,t\rangle \models_{_{TL}} O^{(k)}(\varphi_1,\cdots,\varphi_k) \quad \text{iff} \quad t \in O^{(k)}_M(A_{\varphi_1},\cdots,A_{\varphi_k})$$

where $A_{\varphi} = \{ \tau : \langle M, \tau \rangle \models_{T_L} \varphi \}$ (we suppressed predicate parameters that may occur in the formulas).

We are interested in a more restricted case; for the modality to be of interest the operator $O^{(k)}$ should reflect some intended connection between the sets A_{φ_i} of points satisfying φ_i and the set of points $O[A_{\varphi_1}, \ldots, A_{\varphi_k}]$. The intended meaning is usually given by a formula in an appropriate predicate logic:

Truth Tables: A formula $\overline{O}(t_0, X_1, \dots, X_k)$ in the predicate logic L is a Truth Table for the modality $O^{(k)}$ if for every structure M

$$O_M(A_1,\ldots,A_k) = \{\tau : M \models_{MLO} \overline{O}[\tau,A_1,\ldots,A_k]\}.$$

The modalities *until* and *since* are most commonly used in temporal logic for computer science. They are defined through the following truth tables:

- The modality $X \mathbf{U} Y$, "X until Y", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 < t_1 \land Y(t_1) \land \forall t(t_0 < t < t_1 \to X(t)))$$

- The modality $X \mathbf{S} Y$, "X since Y", is defined by

$$\psi(t_0, X, Y) \equiv \exists t_1(t_0 > t_1 \land Y(t_1) \land \forall t(t_1 < t < t_0 \to X(t))).$$

If the modalities of a temporal logic have truth tables in a predicate logic then the temporal logic is equivalent to a fragment of the predicate logic. Formally:

Proposition 1. If every modality in the temporal logic TL has a truth table in the logic MLO then to every formula $\varphi(X_1, \ldots, X_n)$ of TL there corresponds effectively (and naturally) a formula $\overline{\varphi}(t_0, X_1, \ldots, X_n)$ of MLO such that for every $M, \tau \in M$ and predicates P_1, \ldots, P_n

$$\langle M, \tau, P_1, \dots, P_n \rangle \models_{TL} \varphi \quad \text{iff} \quad \langle M, \tau, P_1, \dots, P_n \rangle \models_{MLO} \overline{\varphi}$$

In particular the temporal logic $TL(\mathbf{U}, \mathbf{S})$ with the modalities "until" and "since" corresponds to a fragment of first-order MLO (FOMLO).

The two modalities \mathbf{U} and \mathbf{S} are also enough to express all the formulas of first-order *MLO* with one free variable:

Theorem 2. ([12, 6]) The temporal logic $TL(\mathbf{U}, \mathbf{S})$ is expressively complete for FOMLO over the two canonical structures: For every formula of FOMLO with at most one free variable, there is a formula of $TL(\mathbf{U}, \mathbf{S})$, such that the two formulas are equivalent to each other, over the positive integers (discrete time) and over the positive real line (continuous time).

2.3 QTL - Quantitative Temporal Logic

The logics MLO and $TL(\mathbf{U}, \mathbf{S})$ are not suitable to deal with quantitative statements like "X will occur within one unit of time". In [8, 9, 10] we introduced the *Quantitative Temporal Logic*, adding to TL the modalities $\diamondsuit_1 X$ (X will happen within the next unit of time) and $\bigtriangledown_1 X$ (X happened within the last unit of time):

Definition 3 (Quantitative Temporal Logic). QTL, quantitative temporal logic is the logic $TL(\mathbf{U}, \mathbf{S})$ enhanced by the two modalities: $\Diamond_1 X$ and $\overleftarrow{\Diamond}_1 X$. These modalities are defined by the tables with free variable t_0 :

- (3) $\diamondsuit_1 X : \exists t ((t_0 < t < t_0 + 1) \land X(t))$
- (4) $\overleftarrow{\bigtriangledown}_1 X : \quad \exists t ((t < t_0 < t + 1) \land X(t)) \; .$

QTL was the latest in a list of metric logics for continuous time, developed over approximately 15 years. When interpreted carefully all these logics are equivalent. For completeness we list the two main modalities that were suggested before QTL together with their natural truth table: 1. The logic MITL [2] has as modalities X $until_{(m,n)}Y$ with natural numbers m < n, which holds at t_0 iff

$$\exists t_1 [(t_0 + m < t_1 < t_0 + n) \land Y(t_1) \land \forall t(t_0 < t < t_1 \to X(t))].$$

Other modalities with closed and half closed intervals as indices, and dual modalities with "since" replacing "until" are defined similarly.

2. Manna and Pnueli [13] base their logic on modalities $[\Gamma(X) > n]$ which holds at t_0 iff

$$\forall t(t_0 - n < t < t_0 \to X(t)).$$

The dual modality for the future is defined similarly. To these they add modalities $[\Gamma(X) = n]$ saying that X started exactly n units of time ago.

We proved in [9] and [11] that:

- 1. The logic QTL can express more liberal bounds in time like: "X will happen in the future, within the period that starts in m units of time, and ends in n units of time" (m < n). We may also include or exclude one of both of the endpoints of the period.
- 2. QTL consumes the different decidable metric temporal logics that we found in the literature, including MITL and the Manna-Pnueli logic described above.
- 3. There is a natural fragment QMLO (quantitative monadic logic of order), of the classical monadic logic of order with the +1 function, that equals in expressive power to QTL.
- 4. The validity and satisfiability problem for this logic is decidable, whether we are interested in systems with *finite variability*, or in all systems evolving in time (a system has finite variability if it changes only at finitely many points, in any finite interval of time).

The advantages of the logic QTL were the subject of [8, 9, 10, 11]. In particular, it is decidable. Here we investigate the limitations of its expressive power.

3 Modalities Which Are Not Expressible in QTL

We start the investigation of the limitations of the temporal logic proving Pnueli's conjecture:

Theorem 4. The modality $C_2(X)$ is not expressible in QTL.

Proof. Let M be the real non negative line with the predicate P(t) that is true exactly at the points $n \cdot \frac{2}{3}$ for all natural numbers n. Let us call the following four predicates: $P, \neg P, True, False$ the **trivial predicates**. We show by structural induction that for every statement φ of QTL there is a point t_{φ} such that from this point on φ is equivalent to one of the trivial predicates.

- This is trivially true for atomic statements.
- The collection of truth sets for the four trivial predicates is closed under Boolean combinations. Therefore the set of formulas satisfying our claim is closed under the Boolean connectors.
- Assume now that $\varphi = (\theta \ Untill \ \psi)$ and t_0 is a point beyond which both θ and ψ are equivalent to one of the trivial predicates. We check the different possibilities for the truth value of φ at a point t beyond t_0 . If θ is equivalent to P or to False then φ is false. If θ is equivalent to $\neg P$ or to True then φ is true if ψ is equivalent to either of P, $\neg P$ or True, and φ is false if ψ is equivalent to False. In every case φ is equivalent either to True or to False.
- For $\varphi = (\theta \ Since \ \psi)$ we need only a minor modification: Let t_1 be an even integer beyond t_0 (so that P is true at t_1). Then for points beyond $t_1 \ \varphi$ is true if $\theta \equiv True$ and ψ occurred at t_1 or earlier, or if $\theta \equiv \neg P$ and ψ is equivalent to any of the special predicates except *False* (the choice of t_1 ensures the case that $\psi \equiv P$) in all other cases $\varphi \equiv False$.
- Assume that $\varphi = \Diamond_1 \theta$ and from t_0 on θ is equivalent to one of the four trivial predicates. If θ is equivalent to *False* then φ is equivalent to *False* from t_0 on. In the other three cases φ is equivalent to *True* from t_0 on.
- A similar argument works when $\varphi = \overleftarrow{\Diamond}_1 \theta$.

On the other hand the statement $C_2(P)$ is false at any point in the interval (n, n + 1/3) if n is even and it is true at any point in the interval (n, n + 1/3) if n is odd. This shows that $C_2(P)$ is not equivalent to any QTL formula.

The method of the proof can be modified to produce a hierarchy of temporal logics, each stronger than the previous.

Definition 5. The counting modalities are the modalities $C_n(X)$ for every n which state that X will be true at least at n points within the next unit of time.

Theorem 6. The modality $C_{n+1}(X)$ is not expressible in $QTL(C_1, \dots, C_n)$.

Proof. Let M be the real non negative line with the predicate P(t) that is true exactly at the points $k \cdot \frac{2}{n+1}$ for all natural numbers k. Call again the following four predicates: $P, \neg P, True, False$ the **trivial predicates**, and as before show that every formula of $QTL(C_1 \cdots, C_n)$ is equivalent from some point on to a trivial predicate. On the other hand $C_{n+1}(P)$ is always true on the interval $(k, k + \frac{1}{n+1})$ if k is even, and false on the interval if k is odd.

4 No Finite Temporal Logic Is Fully Expressive

The hierarchy

$$TL < QTL < QTL(C_2) < \cdots < QTL(C_1 \cdots, C_n) < \cdots$$

raises the suspicion that it will be difficult to find a finite temporal logic that includes all these logics. We showed that it is not difficult. It is impossible. To be precise: **Theorem 7.** Let L be the second order monadic logic of order, with an extra predicate B(t, s) that is interpreted on the real line as s = t + 1. Let L_1 be a temporal logic with possibly infinitely many modalities, for which there is a natural number m such that all the modalities have truth tables in L, with quantifier depth not larger than m. Then there is some n such that $C_n(X)$ is not equivalent over the real line to any L_1 formula.

The proof is quite technical, yet close in spirit to the proof of theorem 6: We define an infinite family of very uniform models, with P their only unary predicate. We define for each integer k > 0 the model M_k to be the full real line R with P(t) occurring at the points m_k^1 for every integer m (positive, negative or zero). We show that any pair of models in this class that can be distinguished by some formula in L_1 , can also be distinguished by one of finitely many simple formulas. It follows that there is an infinite subfamily of models that satisfy the same formulas of L_1 . On the other hand for large n < k the model M_n satisfies $C_n(P)$ and the model M_k does not. Hence the formula $C_n(X)$ is not definable in L_1 .

Discussion:

1. The theorem says both more and less than what the title of the section says. Less because we confined ourselves to temporal logics with truth tables in the second order monadic logic of order with the addition of the +1 function. Allowing more arithmetical operations would produce more modalities. Moreover, modalities need not have truth tables in any predicate logic, and the following is a natural question:

Is there a finite temporal logic that includes all the modalities $P_n(X_1, \ldots, X_n)$, if we do not require that the modalities are are defined by truth tables?

On the other hand we prove more than is claimed because we prove that no *infinite temporal logic* can express all the counting modalities $C_n(X)$ if the truth tables of the modalities are of bounded quantifier depth.

- 2. Second order monadic logic of order with the +1 function is a much stronger logic than is usually considered when temporal logics are defined. All the temporal logics that we saw in the literature are defined in a fragment of monadic logic, with a very restricted use of the +1 function. All the decidable temporal logics in the literature remain decidable when we add the counting modalities $C_n(X)$ [10]. On the other hand second order monadic logic is undecidable over the reals even without the +1 function [16]. When the +1 function is added even a very restricted fragment of *first order* monadic logic of order is undecidable over the positive reals.
- 3. A natural way to strengthen the predicate logic is by adding predicates $B_q(t,s)$ for every rational umber q, to express the relation s = t + q. We call this logic L_Q . The proof of the theorem will not apply if we replace L by L_Q , and even modalities with truth tables of quantifier depth 2 distinguish

any two models M_k and M_r in our class. On the other hand just as before no *finite* temporal logic defined in this logic can express all the counting modalities. This leaves open the question:

Is the theorem above true when the predicate logic L is replaced by L_Q ?

4. It is well known that to say in *predicate logic* "there are at least n elements with a given property" requires quantifier depth that increases with n. We emphasize again that the theorem is much more significant than that. Temporal logics do not have quantifiers, and the expressive power is achieved by deeper nesting of the modalities. Thus to say that P will not occur in the next n units of time requires formulas of predicate logic with quantifier depth that increases with n. On the other hand QTL itself suffices to claim it for any n (with increasing modality nesting), although all the modalities of QTLhave very simple truth tables with quantifier depth at most 2. Therefore it is far from obvious that no finite temporal logic expresses all the modalities C_n using unlimited modality nesting.

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