



Composition Theorems for Generalized Sum and Recursively Defined Types

Alexander Rabinovich

*School of Computer Science
Tel Aviv University
Tel Aviv 69978, Israel
e-mail: rabinooa@post.tau.ac.il*

Composition theorems are tools which reduce reasoning about compound data structures to reasoning about their parts. For example, the truth value of a sentence about the Cartesian product of two structures can be reduced to the truth values of sentences on the components of the product.

A seminal example of a compositional theorem is the Feferman-Vaught Theorem [2]. Feferman and Vaught introduced a generalized product construct which encompasses many algebraic constructions on mathematical structures. Their main theorem reduces the first-order theory of generalized products to the first order theory of its factors and the monadic second-order theory of its index structure.

Shelah [24] defined the notion of a generalized sum and provided the composition theorem which reduces the monadic second-order theory of the generalized sum to the monadic second-order theories of the summands and of its index structure. An important example of generalized sums is the ordinal sum of linearly ordered sets. In [24] the composition theorem for linear orders was one of the main tools for obtaining remarkable decidability results for the monadic theory of linear orders.

In [6] several composition theorems for monadic-second order logic over trees were given.

Two important applications of the compositional methods to algebra and logics are related to

Decidability. Here the method serves as a very powerful tool to obtain new results as well as to simplify other decidability proofs [2,24,3,6,8,1,12].

Expressibility. Analyze the expressive power of the first-order and monadic second-order languages. The compositional method is useful both for the proofs of the limitation of the expressive power [7,8,9,15,16] and for the proofs of positive results on the expressive power [10,14,15,16,5,18,19].

The composition theorem for linear orders was described in survey expositions by Gurevich [4] and Thomas [26]; it was illustrated there for the decidability of monadic logic of order, as an alternative to the automata-theoretic approach advanced by Büchi. The composition method, despite of its success, is still largely ignored by the theoretical computer science community. W. Thomas surveys in [26] Shelah’s composition theorem for linear orders and writes: “Although the subject was exposed in Gurevich’s concise survey [4], it did not attract much attention among theoretical computer scientists. Preference was (and is still) given to automata theoretic method . . . because it does not involve frightening logical technicalities as one finds them in [24]. Thus, there is a tendency that the merits of model theoretic approach are overlooked. This is unfortunate, because it excludes some interesting applications.”

In this talk in addition to surveying classical compositional results, my main technical contributions will be as follows:

- (i) A composition theorem for first-order logic over the generalized sum [20]
- (ii) A composition theorem for recursively defined types [21].

References

- [1] J.P. Burges, Y. Gurevich. The Decision Problem for Temporal Logic. *Notre Dame J. Formal Logic*, 26 (1985) 115-128.
- [2] S. Feferman and R.L. Vaught (1959). The first-order properties of products of algebraic systems. *Fundamenta Mathematica* 47:57–103.
- [3] Y. Gurevich (1979). Modest theory of short chains I. *Journal of Symbolic Logic* 44:481–490.
- [4] Y. Gurevich (1985). Monadic second-order theories. In *Model-Theoretic Logics*, (J. Barwise and S. Feferman, eds.), 479-506, Springer-Verlag.
- [5] Y. Gurevich and A. Rabinovich. Definability and Undefinability with Real Order at the Background. *Journal of Symbolic Logic* 65(2) (2000), 946-958.
- [6] Y. Gurevich and S. Shelah (1979). Modest theory of short chains II. *Journal of Symbolic Logic* 44:491–502.
- [7] Y. Gurevich and S. Shelah (1979). Rabin’s uniformization problem. *Journal of Symbolic Logic* 48:1105–1119.
- [8] Y. Gurevich and S. Shelah (1985). The decision problem for branching time logic. *Journal of Symbolic Logic* 50(3):668–681.

- [9] Y. Gurevich and S. Shelah. On the strength of the interpretation method. *The Journal of Symbolic Logic*, **54**:305–323, 1989.
- [10] T. Hafer and W. Thomas (1987). Computation tree logic CTL* and path quantifiers in the monadic theory of the binary tree. In *Proceedings of ICALP'87: International Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science* **267**:269–279, Springer-Verlag.
- [11] Y. Hirshfeld and A. Rabinovich. A Framework for Decidable Metrical Logics. In ICALP99, LNCS vol. 1644, pp 422-432, Springer Verlag 1999.
- [12] W. Hodges (1993). *Model Theory*. Cambridge University Press.
- [13] N. Immerman (1998). *Descriptive Complexity*, Springer Verlag, 1998.
- [14] S. Lifsches and S. Shelah. Peano Arithmetic may not be interpretable in the monadic theory of orders. *Journal of Symbolic Logic*, **62**(3):848–872, 1997.
- [15] S. Lifsches and S. Shelah. Uniformization, choice functions and well orders in the class of trees. *Journal of Symbolic Logic*, **61**:1206–1227, 1996.
- [16] S. Lifsches and S. Shelah. Uniformization and Skolem Functions in the Class of Trees. *Journal of Symbolic Logic*, **63**:103–127, 1998.
- [17] J.A. Makowsky Algorithmic aspects of the Feferman-Vaught Theorem. 2003.
- [18] F. Moller and A. Rabinovich. On the expressive power of CTL*. LICS 1999, 360-369.
- [19] F. Moller, A. Rabinovich. Counting on CTL*: On the Expressive power of Monadic Path Logic. *Information and Computation* 184 (2003) 147-159..
- [20] A. Rabinovich. Composition Theorems for Generalized Sum. Technical Report, Tel Aviv University www.cs.tau.ac.il/~rabinoa, 1998.
- [21] A. Rabinovich. Composition Theorems for Recursively Defined Structures. Technical Report, 1999.
- [22] A. Rabinovich. On compositionality and its limitations. Technical Report, Edinburgh University, 2001.
- [23] E. Ravve. Decomposition of Databases with translation schemes. Ph.D. Thesis, Technion, Israel Institute of Technology, 1998.
- [24] S. Shelah (1975). The monadic theory of order. *Annals of Mathematics* **102**:379–419.
- [25] S. Shelah (1996). On the very weak 0 – 1 law for random graphs with orders. *Journal of Logic and Computation* **6**:137–159.
- [26] W. Thomas (1997). Ehrenfeucht games, the composition method, and the monadic theory of ordinal words. In *Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht, Lecture Notes in Computer Science* **1261**:118-143, Springer-Verlag.
- [27] S. Zeitman. The Compositional Method. Ph.D. Dissertation, Wayne State University, 1994.