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THE CHURCH PROBLEM FOR COUNTABLE ORDINALS

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ABSTRACT. A fundamental theorem of Büchi and Landweber shows that the Church synthesis problem is computable. Büchi and Landweber reduced the Church Problem to problems about ω-games and used the determinacy of such games as one of the main tools to show its computability. We consider a natural generalization of the Church problem to countable ordinals and investigate games of arbitrary countable length. We prove that determinacy and decidability parts of the Büchi and Landweber theorem hold for all countable ordinals and that its full extension holds for all ordinals $< \omega^\omega$.

1. INTRODUCTION

Two fundamental results of classical automata theory are decidability of the monadic second-order logic of order (MLO) over $\omega = (\mathbb{N}, <)$ and computability of the Church synthesis problem. These results have provided the underlying mathematical framework for the development of formalisms for the description of interactive systems and their desired properties, the algorithmic verification and the automatic synthesis of correct implementations from logical specifications, and advanced algorithmic techniques that are now embodied in industrial tools for verification and validation.

In order to prove decidability of the monadic theory of ω, Büchi introduced finite automata over ω-words. He provided a computable reduction from formulas to finite automata.

Büchi generalized the concept of an automaton to allow automata to "work" on words of any countable length (ordinal) and used this to show that the MLO-theory of any countable ordinal is decidable (see [BS73]).

What is known as the "Church synthesis problem" was first posed by A. Church in [Ch63] for the case of $(\omega, <)$. The Church problem is much more complex than the decidability problem for MLO. Church uses the language of automata theory. It was McNaughton (see [Mc65]) who first observed that the Church problem can be equivalently phrased in game-theoretic language.

Let $\alpha > 0$ be an ordinal and let $\varphi(X_1, X_2)$ be a formula, where X_1 and X_2 are set (monadic predicate) variables. The *McNaughton game* $\mathcal{G}^\alpha_\varphi$ is defined as follows.

(1) The game is played by two players, called Player I and Player II.
(2) A *play* of the game has α rounds.

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(3) At round $\beta < \alpha$: first, Player I chooses $\pi_{X_1}(\beta) \in \{0,1\}$; then, Player II chooses $\pi_{X_2}(\beta) \in \{0,1\}$.
(4) By the end of the play $\pi_{X_1}, \pi_{X_2} : \alpha \to \{0,1\}$ have been constructed. Set

$$P_\pi := \pi_{X_1}^{-1}(1), Q_\pi := \pi_{X_2}^{-1}(1).$$

(5) Then, Player I wins the play if $(\alpha, <) \models \varphi(P_\pi, Q_\pi)$; otherwise, Player II wins the play.

What we want to know is: Does either one of the players have a *winning strategy* in $\mathcal{G}^\alpha_\varphi$? If so, which one? That is, can Player I choose his moves so that, whatever way Player II responds we have $\varphi(P_\pi, Q_\pi)$? Or can Player II respond to Player I's moves in a way that ensures the opposite?

Since at round $\beta < \alpha$, Player I has access only to $Q_\pi \cap [0, \beta)$ and Player II has access only to $P_\pi \cap [0, \beta]$, it seems that the following formalizes well the notion of a strategy in this game:

Definition 1.1 (Causal operator)**.** Let α be an ordinal, $f : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ maps the subsets of α into the subsets of α. We call f *causal* (resp. *strongly causal*) iff for all $P, P' \subseteq \alpha$ and $\beta < \alpha$, if

$$P \cap [0, \beta] = P' \cap [0, \beta] \quad (\text{resp. } P \cap [0, \beta) = P' \cap [0, \beta)) \quad \text{implies} \quad f(P) \cap [0, \beta] = f(P') \cap [0, \beta] \, .$$

That is, if P and P' agree *up to and including* (resp. *up to*) β, then so do $f(P)$ and $f(P')$.

So a winning strategy for Player I is a strongly causal $f : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ such that for every $P \subseteq \alpha$, $(\alpha, <) \models \varphi(f(P), P)$; a winning strategy for Player II is a causal $f : \mathbb{P}(\alpha) \to \mathbb{P}(\alpha)$ such that for every $P \subseteq \alpha$, $(\alpha, <) \models \neg\varphi(P, f(P))$.

It is clear that if Player I has a winning strategy in $\mathcal{G}^\alpha_\varphi$, then $\alpha \models \forall X_2 \exists X_1 \varphi$. It is also easy to see that $\alpha \models \forall X_2 \exists X_1 \varphi$ does not imply that Player I has a winning strategy.

This leads to

Definition 1.2 (Game version of the Church problem)**.** Let α be an ordinal. Given a *MLO*-formula $\varphi(X_1, X_2)$, decide whether Player I has a winning strategy in $\mathcal{G}^\alpha_\varphi$.

From now on, we will use "formula" for "*MLO*-formula" unless stated otherwise.

To simplify notations, games and the Church problem were previously defined for formulas with two free variables X_1 and X_2. It is easy to generalize all definitions and results to formulas $\psi(X_1, \ldots, X_m, Y_1, \ldots Y_n)$ with many variables. In this generalization at round β, Player I chooses values for $X_1(\beta), \ldots, X_m(\beta)$, then Player II replies by choosing values for $Y_1(\beta), \ldots, Y_n(\beta)$. Note that, strictly speaking, the input to the Church problem is not only a formula, but a formula plus a partition of its free variables to Player I's variables and Player II's variables.

In [BL69], Büchi and Landweber prove the computability of the Church problem in $\omega = (\mathbb{N}, <)$. Even more importantly, they show that in the case of ω we can restrict ourselves to *definable strategies*, that is causal (or strongly causal) operators computable by finite state automata or, equivalently, definable in *MLO*. (Recall that $F : \mathbb{P}(\mathbb{N}) \to \mathbb{P}(\mathbb{N})$ is definable in *MLO* if there is an *MLO* formula $\psi(X, Y)$ such that for every $P, Q \subseteq \mathbb{N}$ we have $P = F(Q)$ iff $\omega \models \psi(Q, P)$.)

Theorem 1.3 (Büchi-Landweber, 1969)**.** *Let $\varphi(\bar{X}, \bar{Y})$ be a formula, where $\bar{X}$ and $\bar{Y}$ are disjoint lists of variables. Then:*

Determinacy: *One of the players has a winning strategy in the game $\mathcal{G}^\omega_\varphi$.*

Decidability: *It is decidable* which *of the players has a winning strategy.*

Definable strategy: *The player who has a winning strategy, also has a* definable *winning strategy.*

Synthesis algorithm: *We can compute a formula* $\psi(\bar{X}, \bar{Y})$ *that defines (in* ω*) a winning strategy for the winning player in* $\mathcal{G}^{\omega}_{\varphi}$*.*

After stating their main theorem, Büchi and Landweber write:

> "We hope to present elsewhere a corresponding extension of [our main theorem] from ω to any countable ordinal."

However, despite the fundamental role of the Church problem, no such extension is even mentioned in a later book by Büchi and Siefkes [BS73], which summarizes the theory of finite automata over words of countable ordinal length.

In [RS08], we provided a counter-example to a full extension of the Büchi-Landweber theorem to $\alpha \geq \omega^{\omega}$. Let $\varphi(Y)$, $\psi(Y)$ be formulas and $\mathcal{M}$ be a structures. We say that ψ *selects* (or, is a *selector* for) φ *in* $\mathcal{M}$ iff:

(1) $\mathcal{M} \models \exists^{\leq 1} Y \psi(Y)$ (i.e., there is at most one Y that satisfies ψ),
(2) $\mathcal{M} \models \forall Y(\psi(Y) \rightarrow \varphi(Y))$, and
(3) $\mathcal{M} \models \exists Y \varphi(Y) \rightarrow \exists Y \psi(Y)$.

In [RS08], we proved that for every ordinal $\alpha \geq \omega^{\omega}$ there is an *MLO* formula $\psi_{\alpha}(Y)$ such that $\alpha \models \exists Y \psi_{\alpha}$; however, there is no *MLO* formula that selects ψ_{α} in α.

Now, consider McNaughton game of length α for $\varphi(X_1, X_2)$ defined as $\psi_{\alpha}(X_1) \wedge X_2 = X_2$ (so, we ignore X_2 and this can be considered as a one-player game). Player I cannot have a definable strategy in $\mathcal{G}^{\alpha}_{\varphi}$. Indeed, if χ could define such a strategy, then $\exists X_2(X_2 = \emptyset \wedge \chi)$ would select $\psi_{\alpha}(X_1)$ over α. On the other hand, Player I does win this game: she simply plays a *fixed* X_1 that satisfies $\psi_{\alpha}(X_1)$ over α, ignoring Player II moves. Hence, the definability and the synthesis parts of the Büchi-Landweber theorem fail for every $\alpha \geq \omega^{\omega}$. Shomrat [Sho07] proved

Theorem 1.4. *The Büchi-Landweber theorem holds for an ordinal* α *if and only if* $\alpha < \omega^{\omega}$*.*

The proof in [Sho07] is very long and requires the development of an extensive game-theoretic apparatus. Moreover, the technique of [Sho07] cannot be extended to games of length $\geq \omega^{\omega}$. In this article, we provide a simple proof of Theorem 1.4.

Our main results show that the determinacy and the decidability parts of the Büchi-Landweber Theorem hold for every countable ordinal.

Theorem 1.5 (Main)**.** *Let* α *be a countable ordinal and* $\varphi(\bar{X}, \bar{Y})$ *be a formula. Then:*

Determinacy: *One of the players has a winning strategy in the game* $\mathcal{G}^{\alpha}_{\varphi}$*.*

Decidability: *It is decidable* which *of the players has a winning strategy in* $\mathcal{G}^{\alpha}_{\varphi}$*.*

Our proof uses both game theoretical techniques and the "composition method" developed by Feferman-Vaught, Shelah and others (see, e.g. [Sh75]).

The article is organized as follows. The next section recalls standard definitions about monadic logic of order, summarizes elements of the composition method and reviews known facts about the monadic theory of countable ordinals. In Section 3, we introduce game-types, define games on game types and show that these game are a special case of McNaughton games. Section 4 contains the main technical lemmas of the paper and shows that the role of the game types is similar to the role of monadic types in the composition method. In

Section 5, we prove that the Büchi-Landweber theorem holds in its entirety for all ordinals $\alpha < \omega^\omega$. In Sections 6 and 7, it is proved that the determinacy and decidability parts of the Büchi-Landweber theorem hold for all countable ordinals. Section 8 provides an MLO-characterisation of the winner and shows that for every formula $\varphi(X,Y)$ there exists an *MLO* sentence ψ such that for every countable ordinal α: Player I wins $\mathcal{G}^\alpha_\varphi$ iff $\alpha \models \psi$. Section 9 addresses the problem whether the winner has a definable winning strategy. We were unable to show that this problem is decidable for ordinals $\geq \omega^\omega$; however, we reduced the decidability of this problem for countable ordinals $\geq \omega^\omega$ to the case of games of length ω^ω. Section 10 contains a conclusion and states some open problems.

2. Preliminaries and Background

2.1. **Notations and terminology.** We use n, k, l, m, p, q for natural numbers and $\alpha, \beta, \gamma, \delta$ for ordinals. We use $\mathbb{N}$ for the set of natural numbers and ω for the first infinite ordinal. We write $\alpha + \beta$, $\alpha\beta$, α^β for the sum, multiplication and exponentiation, respectively, of ordinals α and β. We use the expressions "*chain*" and "*linear order*" interchangeably.

We use $\mathbb{P}(A)$ for the set of subsets of A.

2.2. The Monadic Logic of Order (*MLO*).

2.2.1. *Syntax.* The syntax of the monadic second-order logic of order - *MLO* has in its vocabulary *individual* (first order) variables $t_1, t_2 \ldots$, monadic *second-order* variables $X_1, X_2 \ldots$ and one binary relation $<$ (the order).

Atomic formulas are of the form $X(t)$ and $t_1 < t_2$. Well formed formulas of the monadic logic *MLO* are obtained from atomic formulas using Boolean connectives $\neg, \vee, \wedge, \rightarrow$ and the first-order quantifiers $\exists t$ and $\forall t$, and the second-order quantifiers $\exists X$ and $\forall X$. The quantifier depth of a formula φ is denoted by $\mathrm{qd}(\varphi)$.

We use upper case letters X, Y, Z,... to denote second-order variables; with an overline, $\bar{X}$, $\bar{Y}$, etc., to denote finite tuples of variables.

2.2.2. *Semantics.* A *structure* is a tuple $\mathcal{M} := (A, <^{\mathcal{M}}, \bar{P}^{\mathcal{M}})$ where: A is a non-empty set, $<^{\mathcal{M}}$ is a binary relation on A, and $\bar{P}^{\mathcal{M}} := \left(P_1^{\mathcal{M}}, \ldots, P_l^{\mathcal{M}}\right)$ is a *finite* tuple of sub*sets* of A.

If $\bar{P}^{\mathcal{M}}$ is a tuple of l sets, we call $\mathcal{M}$ an *l-structure*. If $<^{\mathcal{M}}$ linearly orders A, we call $\mathcal{M}$ an *l-chain*.

Suppose $\mathcal{M}$ is an l-structure and φ a formula with free-variables among $X_1, \ldots, X_l$. We define the relation $\mathcal{M} \models \varphi$ (read: $\mathcal{M}$ *satisfies* φ) as usual, understanding that the second-order quantifiers range over *subsets* of A.

Let $\mathcal{M}$ be an l-structure. The *monadic theory* of $\mathcal{M}$, $MTh(\mathcal{M})$, is the set of all formulas with free-variables among $X_1, \ldots, X_l$ satisfied by $\mathcal{M}$.

From now on, we omit the superscript in '$<^{\mathcal{M}}$' and '$\bar{P}^{\mathcal{M}}$'. We often write $(A, <) \models \varphi(\bar{P})$ meaning $(A, <, \bar{P}) \models \varphi$.

Definition 2.1 (*MLO* Definable Function)**.** Let $\mathcal{M} := (A, <)$ be a chain. A function $F : \mathbb{P}(A)^n \to \mathbb{P}(A)^m$ is *MLO-definable* in $\mathcal{M}$ if there is an *MLO* formula $\varphi(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ such that $\mathcal{M} \models \varphi(P_1, \ldots, P_n, Q_1, \ldots Q_m)$ iff $(Q_1, \ldots, Q_m) = F(P_1, \ldots, P_n)$.

2.3. **Elements of the composition method.** Our proofs make use of the technique known as the composition method developed by Feferman-Vaught and Shelah [FV59, Sh75]. To fix notations and to aid the reader unfamiliar with this technique, we briefly review the definitions and results that we require. A more detailed presentation can be found in [Th97] or [Gu85].

2.3.1. *Hintikka formulas and n-types.* Let $n, l \in \mathbb{N}$. We denote by $\mathfrak{Form}_l^n$ the set of formulas with free variables among $X_1, \ldots, X_l$ and of quantifier depth $\leq n$.

Definition 2.2. Let $n, l \in \mathbb{N}$ and let $\mathcal{M}, \mathcal{N}$ be l-structures. The *n-theory* of $\mathcal{M}$ is

$$Th^n(\mathcal{M}) := \{\varphi \in \mathfrak{Form}_l^n \mid \mathcal{M} \models \varphi\}.$$

If $Th^n(\mathcal{M}) = Th^n(\mathcal{N})$, we say that $\mathcal{M}$ and $\mathcal{N}$ are *n-equivalent* and write $\mathcal{M} \equiv^n \mathcal{N}$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and $l > 0$, the set $\mathfrak{Form}_l^n$ is infinite. However, it contains only finitely many semantically distinct formulas. So, there are finitely many $\equiv^n$-equivalence classes of l-structures. In fact, we can compute characteristic sentences for the $\equiv^n$-classes:

Lemma 2.3 (Hintikka Lemma)**.** *For $n, l \in \mathbb{N}$, we can compute a* finite *set $Char_l^n \subseteq \mathfrak{Form}_l^n$ such that:*

- *For every $\equiv^n$-equivalence class A there is a* unique *$\tau \in Char_l^n$ such that for every l-structure $\mathcal{M}$: $\mathcal{M} \in A$ iff $\mathcal{M} \models \tau$.*
- *Every MLO formula $\varphi(X_1, \ldots X_l)$ with $\mathrm{qd}(\varphi) \leq k$ is equivalent to a (finite) disjunction of characteristic formulas from $Char_l^k$. Moreover, there is an algorithm which for every formula $\varphi(X_1, \ldots X_l)$ computes a finite set $G \subseteq Char_l^{\mathrm{qd}(\varphi)}$ of characteristic formulas, such that φ is equivalent to the disjunction of all the formulas in G.*

*Any member of $Char_l^k$ we call a (k,l)-*Hintikka formula *or (k,l)-characteristic formula. We use τ, τ_i, τ^j to range over the characteristic formulas and G, G_i, G' to range over sets of characteristic formulas.*

Definition 2.4 (n-Type)**.** For $n, l \in \mathbb{N}$ and an l-structure $\mathcal{M}$, we denote by $type_n(\mathcal{M})$ the unique member of $Char_l^n$ satisfied by $\mathcal{M}$ and call it the *n-type* of $\mathcal{M}$.

Thus, $type_n(\mathcal{M})$ determines $Th^n(\mathcal{M})$ and, indeed, $Th^n(\mathcal{M})$ is computable from $type_n(\mathcal{M})$.

2.3.2. *The ordered sum of chains and of n-types.*

Definition 2.5.

(1) Let $l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a chain and $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ a sequence of l-chains. Write $\mathcal{M}_\alpha := (A_\alpha, <^\alpha, P_1{}^\alpha, \ldots, P_l{}^\alpha)$ and assume that $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$ are in I. The *ordered sum* of $\mathfrak{S}$ is the l-chain

$$\sum_{\mathcal{I}} \mathfrak{S} := (\bigcup_{\alpha \in I} A_\alpha, <^{\mathcal{I},\mathfrak{S}}, \bigcup\nolimits_{\alpha \in I} P_1{}^\alpha, \ldots, \bigcup_{\alpha \in I} P_l{}^\alpha),$$

where:

if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathcal{I},\mathfrak{S}} a$ iff $\beta <^{\mathcal{I}} \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathcal{M}_\alpha$'s are not disjoint, replace them with isomorphic l-chains that have disjoint domains, and proceed as before.

(2) If for all $\alpha \in I$, $\mathcal{M}_\alpha$ is isomorphic to $\mathcal{M}$ for some *fixed* $\mathcal{M}$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M} \times \mathcal{I}$.
(3) If $\mathcal{I} = (\{0,1\},<)$ and $\mathfrak{S} = (\mathcal{M}_0, \mathcal{M}_1)$, we denote $\sum_{\mathcal{I}} \mathfrak{S}$ by $\mathcal{M}_0 + \mathcal{M}_1$.

The next proposition says that taking ordered sums preserves $\equiv_n$-equivalence.

Proposition 2.6. *Let $n, l \in \mathbb{N}$. Assume:*
(1) *$(I, <^{\mathcal{I}})$ is a linear order,*
(2) *$\big(\mathcal{M}^0_\alpha \mid \alpha \in I\big)$ and $\big(\mathcal{M}^1_\alpha \mid \alpha \in I\big)$ are sequences of l-chains, and*
(3) *for every $\alpha \in I$, $\mathcal{M}^0_\alpha \equiv^n \mathcal{M}^1_\alpha$.*
Then, $\sum_{\alpha \in I} \mathcal{M}^0_\alpha \equiv^n \sum_{\alpha \in I} \mathcal{M}^1_\alpha$.

This allows us to define the sum of formulas in $Char^n_l$ with respect to any linear order.

Definition 2.7.
(1) Let $n, l \in \mathbb{N}$, $\mathcal{I} := (I, <^{\mathcal{I}})$ a chain, $\mathfrak{H} := (\tau_\alpha \mid \alpha \in I)$ a sequence of (n,l)-Hintikka formulas. The *ordered sum* of $\mathfrak{H}$, (notations $\sum_{\mathcal{I}} \mathfrak{H}$ or $\sum_{\alpha \in \mathcal{I}} \tau_\alpha$), is *an* element τ of $Char^n_l$ such that:
if $\mathfrak{S} := (\mathcal{M}_\alpha \mid \alpha \in I)$ is a sequence of l-chains and $type_n(\mathcal{M}_\alpha) = \tau_\alpha$ for $\alpha \in I$, then
$$type_n(\sum_{\mathcal{I}} \mathfrak{S}) = \tau.$$
(2) If for all $\alpha \in I$, $\tau_\alpha = \tau$ for some *fixed* $\tau \in Char^n_l$, we denote $\sum_{\alpha \in \mathcal{I}} \tau_\alpha$ by $\tau \times \mathcal{I}$.
(3) If $\mathcal{I} = (\{0,1\},<)$ and $\mathfrak{H} = (\tau_0, \tau_1)$, we denote $\sum_{\alpha \in \mathcal{I}} \tau_\alpha$ by $\tau_0 + \tau_1$.

The following fundamental result of Shelah can be found in [Sh75]:

Theorem 2.8 (Composition Theorem)**.** *Let $\varphi(X_1, \dots, X_l)$ be a formula, let $n = \mathrm{qd}(\varphi)$ and let $\{\tau_1, \dots, \tau_m\} = Char^n_l$. Then, there is a formula $\psi(Y_1, \dots, Y_m)$ such that for every chain $\mathcal{I} = (I, <^{\mathcal{I}})$ and a sequence $(\mathcal{M}_\alpha \mid \alpha \in I)$ of l-chains the following holds:*
$$\sum_{\alpha \in I} \mathcal{M}_\alpha \models \varphi \textit{ iff } \mathcal{I} \models \psi(Q_1, \dots Q_m), \textit{ where } Q_j = \{\alpha \in I \ : \ M_\alpha \models \tau_j\} \ .$$
Moreover, ψ is computable from φ.

We are usually interested in cases (2) and (3) of the Definition 2.7. The following Theorems are important consequences of the Composition Theorem:

Theorem 2.9 (Addition Theorem)**.** *The function which maps the pairs of characteristic formulas to their sum is a recursive function. Formally, the function $\lambda n, l \in \mathbb{N}.\lambda\tau_0, \tau_1 \in Char^n_l.\tau_0 + \tau_1$ is recursive.*

Theorem 2.10 (Multiplication Theorem)**.** *Let $\mathcal{I}$ be a chain. The function $\lambda n, l \in \mathbb{N}.\lambda\tau \in Char^n_l.\tau \times \mathcal{I}$ is recursive in the monadic theory of $\mathcal{I}$.*

2.4. **The monadic theory of countable ordinals.** Büchi (see, e.g., [BS73]) has shown that there is a *finite* amount of data concerning any countable ordinal that determines its monadic theory.

Definition 2.11 (Code of an ordinal)**.** Let $\alpha > 0$ be a countable ordinal. Write $\alpha = \omega^\omega \beta + \zeta$ where $\zeta < \omega^\omega$ (this can be done in a unique way). If $\zeta \neq 0$, write

$$\zeta = \sum_{i \leq n} \omega^{n-i} \cdot a_{n-i} \text{ , where } a_i \in \mathbb{N} \text{ for } i \leq n \text{ and } a_n \neq 0$$

(this, too, can be done in a unique way).

Define Code(α) as

$$\mathrm{Code}(\alpha) := \begin{cases} (0, a_n, \dots, a_0) & \text{if } \gamma = 0 \\ (1, a_n, \dots, a_0) & \text{if } \gamma \neq 0 \text{ and } \zeta \neq 0 \\ \langle 1 \rangle & \text{otherwise, i.e., if } \gamma \neq 0 \text{ and } \zeta = 0 \end{cases} .$$

The following is implicit in [BS73]:

Theorem 2.12 (Code Theorem)**.** *There is an algorithm that, given a sentence φ and the* code *of an ordinal α, determines whether $(\alpha, <) \models \varphi$.*

3. Game types

Recall that $Char_2^n$ is the set of characteristic formulas of the quantifier depth n with free variables among $\{X_1, X_2\}$.

For $G \subseteq Char_2^n$ we denote by $\mathcal{G}_G^\alpha$, the McNaughton game $\mathcal{G}_\varphi^\alpha$, where φ is the disjunction of all formulas in G.

By Lemma 2.3, for every formula $\varphi(X_1, X_2)$ of quantifier depth n there is $G \subseteq Char_2^n$ such that φ is equivalent to the disjunction of all formulas from G. Moreover, G is computable from φ. Hence, in order to show that every McNaughton game of length α is determinate, it is enough to show that for every n and $G \subseteq Char_2^n$, the game $\mathcal{G}_G^\alpha$ is determinate. Moreover, if it is decidable who wins the games of the form $\mathcal{G}_G^\alpha$, then it is decidable who wins $\mathcal{G}_\varphi^\alpha$ games.

Definition 3.1 (Game Types)**.** Let $n \in \mathbb{N}$.

Game type of ordinal: For an ordinal α: game-type$_n(\alpha)$ is defined as

$$\{G \subseteq Char_2^n \mid \text{ Player I wins } \mathcal{G}_G^\alpha\} .$$

Formal game-type: A formal n-game-type is an element[1] of $\mathbb{P}(\mathbb{P}(Char_2^n))$.

Let α be an ordinal, C be a formal n-game-type and $G \subseteq Char_2^n$. We consider the following α-game Game$_\alpha(C, G)$.

Game$_\alpha(C, G)$: The game has α rounds and it is defined as follows:

Round i:

- Player I chooses $G_i \in C$.
- Player II chooses $\tau_i \in G_i$.

Winning conditions: Let τ_i $(i \in \alpha)$ be the sequence that appears in the play. I wins the play if $\Sigma_{i \in \alpha} \tau_i \in G$.

The following lemma is immediate:

Lemma 3.2. *If $C_1 \subseteq C_2$, $G_1 \subseteq G_2$ and I wins Game$_\alpha(C_1, G_1)$, then I wins Game$_\alpha(C_2, G_2)$.*

[1] recall that $\mathbb{P}(A)$ stands for the set of subsets of A

As a consequence of Theorem 2.8 and Theorem 1.3 we obtain the following lemma which will play a prominent role in our proofs:

Lemma 3.3.

(1) *The game* $Game_\omega(C, G)$ *is determinate.*
(2) *It is decidable which of the players wins* $Game_\omega(C, G)$.

Proof. We provide a reduction from $\text{Game}_\alpha(C, G)$ to a McNaughton game.

Let $\{\tau_1, \dots, \tau_m\} = Char_2^n$. For every $G' \subseteq Char_2^n$:

- Let $J(G') := \{j \mid \tau_j \in G'\}$.
- Let $\varphi_{G'}(X_1, X_2)$ be $\bigvee_{\tau \in G'} \tau$ - the disjunction of all formulas from G'.
- Let $\psi_{G'}(Y_1, \dots, Y_m)$ be constructed from $\varphi_{G'}$ as in the Composition Theorem (Theorem 2.8).

Let $C = \{G_1, \dots, G_k\}$. Define formula $\varphi_{C,G}(X_1, \dots, X_k, Y_1, \dots, Y_m)$ as the disjunction of

(1) For all t exactly one of $X_i(t)$ $(i = 1, \dots, k)$ holds and $\psi_G(Y_1, \dots Y_m)$.
(2) There is t such that not exactly one of $Y_j(t)$ holds.
(3) There is t and $i \in \{1, \dots, k\}$ such that $X_i(t)$ and $\neg \bigvee_{j \in J(G_i)} Y_j(t)$.

Consider the McNaughton game $\mathcal{G}^\alpha_{\varphi_{C,G}}$. The second disjunct forces Player II at each round to assign the value 1 exactly to one of Y_j, and the third disjunct forces Player II to reply to the choice of X_i of Player I by choosing Y_j such that $\tau_j \in G_i$. It is clear that Player I (respectively, Player II) has a winning strategy in $\text{Game}_\alpha(C, G)$ iff Player I (respectively, Player II) has a winning strategy in $\mathcal{G}^\alpha_{\varphi_{C,G}}$.

Therefore, by the Büchi-Landweber theorem, $\text{Game}_\omega(C, G)$ is determinate, and it is decidable whether I wins $\text{Game}_\omega(C, G)$. □

4. Addition and Multiplication Lemmas for Game Types

This section contains the main technical lemmas of this paper. In particular, the role of game types in Lemma 4.1 and Lemma 4.4 is similar to the role of the monadic types in the Addition and Multiplication theorems.

We say that an ordinal α is *determinate* if for every *MLO* formula φ, one of the player has a winning strategy in $\mathcal{G}^\alpha_\varphi$.

In the next lemma and throughout this section we often use "α-game with a winning condition G" for the McNaughton game $\mathcal{G}^\alpha_\varphi$, where $\varphi = \vee_{\tau \in G} \tau$; recall that this game is also denoted by $\mathcal{G}^\alpha_G$ (see beginning of Section 3).

Lemma 4.1. *Let* β *be an ordinal,* $C_\beta = \textit{game-type}_n(\beta)$ *and* G *be a formal n-game-type. For every* $G_i \in C_\beta$, *let* $K(G_i, G) = \{\tau \in Char_2^n \mid \forall \tau' \in G_i(\tau + \tau' \in G)\}$. *Let* $K(C_\beta, G) = \cup_{G_i \in C_\beta} K(G_i, G)$. *If* α *and* β *are determinate, then:*

A: *If Player I wins the* α*-game with winning condition* $K(C_\beta, G)$, *then Player I wins the* $\alpha + \beta$*-game with winning condition* G.

B: *If Player I cannot win the* α*-game with winning condition* $K(C_\beta, G)$, *then Player II wins the* $\alpha + \beta$*-game with winning condition* G.

Proof. (A) Assume that Player I has a winning strategy F_α for the α-game with winning condition $K(C_\beta, G)$. He plays the first α rounds according to the strategy F_α. Assume that after α rounds the play satisfies $\tau \in K(C_\beta, G)$. Then there is $G_i \in C_\beta$ such that

$\tau \in K(G_i, G)$. Player I plays the next β rounds according to a winning strategy F_{β,G_i} for the β game with winning condition G_i. (Such a winning strategy exists, by definition of C_β.) Hence during these β rounds $\tau_i \in G_i$ is realized. The type of the play will be $\tau + \tau_i \in G$. Hence, Player I wins the $\alpha + \beta$-game with winning condition G.

(B) Now assume that Player I cannot win the α-game with winning condition $K(C_\beta, G)$. Then, by determinacy of α games, Player II has a winning strategy for this game. Let him play the first α rounds according to this winning strategy. Let $\tau \notin K(C_\beta, G)$ be the type reached after α rounds.

Let $G_\tau = \{\tau' \,:\, \tau + \tau' \in G\}$. We claim that Player II wins the β-game for G_τ. Indeed, if he cannot win this game, then, by determinacy of β games, Player I wins it, i.e., $G_\tau \in C_\beta$, and hence, $\tau \in K(C_\beta, G)$. Contradiction

The next β rounds Player II can play according to his winning strategy for G_τ. This will ensure that the type τ' of this β-play is not in G_τ. Hence, the type of the entire play is $\tau + \tau' \notin G$.

Hence, Player II wins the $\alpha + \beta$ game for G.

This completes the proof of (A) and (B). □

As an immediate consequence, we obtain the following Theorem:

Theorem 4.2. *If α and β are determinate, then*

(1) *$\alpha + \beta$ is determinate.*

(2) *game-type$_n(\alpha + \beta)$ is computable from game-type$_n(\alpha)$ and game-type$_n(\beta)$.*

Proof.

(1) is an immediate consequence of Lemma 4.1.

(2) Assume C_α = game-type$_n(\alpha)$ and C_β = game-type$_n(\beta)$. To check whether $G \in$ game-type$_n(\alpha + \beta)$, first compute $K(C_\beta, G)$ and then check whether $K(C_\beta, G) \in C_\alpha$. Lemma 4.1 implies that $G \in$ game-type$_n(\alpha + \beta)$ iff $K(C_\beta, G) \in C_\alpha$. □

The above theorem allows us to define the addition of n-game types for determinate ordinals. We will use "+" for the addition of game types.

Recall that an ordinal α is definable if there is a sentence θ_α such that for every chain $M = (A, <)$: $M \models \theta_\alpha$ iff M is isomorphic to α.

From the proof of Lemma 4.1 we deduce the following variant of Theorem 4.2 for *definable strategies*:

Lemma 4.3. *Assume that α is definable and in every game of length α or β one of the players has a* definable *winning strategy. Then in every $\alpha + \beta$ game one of the players has a* definable *winning strategy. Moreover, if there are algorithms which for every φ compute a definable winning strategy for $\mathcal{G}^\alpha_\varphi$ and $\mathcal{G}^\beta_\varphi$, then there is an algorithm that computes a definable winning strategy for $\mathcal{G}^{\alpha+\beta}_\varphi$.*

Proof. Let $G \subseteq Char^n_2$. We will construct a definable winning strategy for $\mathcal{G}^{\alpha+\beta}_G$. We use here the notations of Lemma 4.1. As shown there, if Player I has a winning strategy in $\mathcal{G}^{\alpha+\beta}_G$, then the following strategy F is winning for Player I:

(1) F plays the first α rounds according to a winning strategy in $\mathcal{G}^\alpha_{K(C_\beta,G)}$.

(2) If after α rounds the play satisfies $\tau \in K(G_i, G)$, then the last β steps F plays according to a winning strategy in $\mathcal{G}^\beta_{G_i}$.

For $\tau \in K(C_\beta, G)$, we denote by i_τ the minimal i such that $\tau \in G_i$. Now assume that $\psi(X_1, X_2)$ defines a winning strategy for $\mathcal{G}^{\alpha}_{K(C_\beta,G)}$ and $\chi_i(X_1, X_2)$ defines a winning strategy for $\mathcal{G}^{\beta}_{G_i}$, and θ_α defines the ordinal α. Then the formula

$$\exists t\big(\theta_\alpha^{<t} \wedge \psi^{<t} \wedge \bigwedge_{\tau \in K(C_\beta,G)} (\tau^{<t} \rightarrow \chi_{i_\tau}^{\geq t})\big)$$

defines a winning strategy described above. (Here, for a variable t which does not occur free in φ, we denote by $\varphi^{<t}$ the formula obtained from φ by relativizing all first order quantifiers to $< t$, i,e. by replacing "$\forall u(\dots)$" by "$\forall u(u < t \rightarrow \dots)$". The formula $\varphi^{\geq t}$ is defined similarly.)

The case when Player II has a winning strategy is treated similarly. □

Lemma 4.4. *Let $n \in \mathbb{N}$, let $(\alpha_i : i \in \omega)$ be an ω-sequence of ordinals and let $C \subseteq \mathbb{P}(Char_2^n)$ be a formal n-game type. Assume that for every i:*
(1) *For every $G \subseteq Char_2^n$, the α_i game for G is determinate.*
(2) *$C = game\text{-}type_n(\alpha_i)$.*
Then

A: *If Player I wins $Game_\omega(C, G)$, then Player I wins the $\sum \alpha_i$-game for G.*

B: *If Player I cannot win $Game_\omega(C, G)$, then Player II wins the $\sum \alpha_i$-game for G.*

C: *For every $G \subseteq Char_2^n$, the $\sum \alpha_i$ game for G is determinate.*

D: *$G \in game\text{-}type_n(\sum \alpha_i)$ iff Player I wins $Game_\omega(C, G)$.*

Proof. (A) Let F be a winning strategy for Player I in $\mathrm{Game}_\omega(C, G)$. Consider the following strategy for Player I

First α_0 Rounds: Let $G_0 \in C$ be the first move of Player I according to F. Player I will play the first α_0 rounds according to the his winning strategy in the α_0-game for G_0.

Let π_0 be a play according to this strategy and let $\tau_0 = type_n(\pi_0)$. Note that $\tau_0 \in G_0$ and a (partial) play $G_0\tau_0$ is consistent with F.

Next α_{i+1} Rounds: Let $G_{i+1} \in C$ be the move of Player I according to F after $G_0\tau_0 \dots G_i\tau_i$. Player I will play the next α_{i+1} rounds according to his winning strategy in the α_{i+1}-game for G_{i+1}.

Let π_{i+1} be a play according to this strategy during these α_{i+1} rounds.

Let $\tau_{i+1} = type_n(\pi_{i+1})$. Note that $\tau_{i+1} \in G_{i+1}$ and the play $G_0\tau_0, \dots, G_{i+1}\tau_{i+1}$ is consistent with F.

$type_n(\pi_0 \dots \pi_i \dots) = \Sigma\tau_i$ is in G, because the play $G_0\tau_0, \dots, G_i\tau_i, \dots$ is consistent with the winning strategy F of Player I in $\mathrm{Game}_\omega(C, G)$. Hence, the described strategy is a winning strategy for Player I in the $\sum \alpha_i$ game for G.

Now let us prove (B). Assume that Player I has no winning strategy in $\mathrm{Game}_\omega(C, G)$. By Lemma 3.3 this game is determinate. Hence, Player II has a winning strategy F_2 for $\mathrm{Game}_\omega(C, G)$.

We show that the following strategy of Player II is winning in the $\sum \alpha_i$ game for G

First α_0 Rounds: For every $D \in C$ let τ_D be the response of Player II according to F_2 to the first move D of Player I in $\mathrm{Game}_\omega(C, G)$.

Let $H_0 = \{\tau_D : D \in C\}$. We claim that Player II has a winning strategy in the α_0-game for $\neg H_0$ - the complement of H_0, i.e., for $Char_2^n \setminus H_0$. Indeed, if he has no such strategy, then, by determinacy of α_0, Player I has a winning strategy for $\neg H_0$. Therefore, $\neg H_0 \in C$. Let $\tau = F_2(\neg H_0)$. Then $\tau \in \neg H_0$ and $\tau \in H_0$. Contradiction.

The first α_0 rounds Player II will play according to his winning strategy for $\neg H_0$.

Let π_0 be a play according to this strategy and let $\tau_0 = type_n(\pi_0)$. Note that $\tau_0 \in H_0$. Let G_0 be such that $\tau_0 = F_2(G_0)$. The (partial) play $G_0\tau_0$ is consistent with F_2.

Next α_{i+1} Rounds: For every $D \in C$ let τ_D be the response of Player II according to F_2 to $G_0\tau_0 \dots G_i\tau_i D$. ($F_2$ is defined because $G_0\tau_0 \dots G_i\tau_i D$ is a play according to F_2 for every $D \in C$.) Let $H_{i+1} = \{\tau_D \mid D \in C\}$. We claim that Player II has a winning strategy in α_{i+1} game for $\neg H_{i+1}$. (The arguments are the same as the arguments which show that Player II has a winning strategy for $\neg H_0$.)

The next α_{i+1} rounds Player II will play according to his winning strategy for $\neg H_{i+1}$.

Let π_{i+1} be a play according to this strategy and let $\tau_{i+1} = type_n(\pi_{i+1})$. Note that $\tau_{i+1} \in H_{i+1}$. Let G_{i+1} be such that $\tau_{i+1} = F_2(G_0\tau_0 \dots G_i\tau_i G_{i+1})$. The play $G_0\tau_0 \dots G_{i+1}\tau_{i+1}$ is consistent with F_2.

$type_n(\pi_0 \dots \pi_i \dots) = \Sigma\tau_i$ is not in G, because the play $G_0\tau_0 \dots G_i\tau_i \dots$ is consistent with the winning strategy F_2 of Player II in $\text{Game}_\omega(C, G)$. Hence, the described strategy is a winning strategy for Player II in $\sum \alpha_i$ game for G.

(C) and (D) are immediate consequences of (A) and (B). □

As a consequence of Lemma 4.4 and Lemma 3.3 we obtain the following Theorem:

Theorem 4.5. *Assume α is determinate. Then*
(1) *$\alpha \times \omega$ is determinate.*
(2) *game-type$_n(\alpha \times \omega)$ is computable from game-type$_n(\alpha)$.*

From the proof of Lemma 4.4, by arguments similar to those in the proof of Lemma 4.3, we deduce the following variant of Theorem 4.5 for *definable strategies*:

Lemma 4.6. *Assume that α is definable and in every game of length α one of the players has a definable winning strategy. Then in every $\alpha \times \omega$ game one of the players has a definable winning strategy. Moreover, if there is an algorithm which for every φ computes a definable winning strategy for $\mathcal{G}^{\alpha}_{\varphi}$, then there is an algorithm that computes a definable winning strategy for $\mathcal{G}^{\alpha\times\omega}_{\varphi}$.*

Note that Theorem 4.5 allows to define the multiplication by ω of n-game types for determinate ordinals. The following Lemma allows to define ω-sums of n-game types of determinate ordinals. It is an analog of Proposition 2.6 for game types.

Lemma 4.7. *Let $n \in \mathbb{N}$. Assume that*
(1) *$\bar{\alpha} = (\alpha_i \ : \ i \in \omega)$ and $\bar{\beta} = (\beta_i \ : \ i \in \omega)$ are ω-sequences of determinate ordinals and*
(2) *game-type$_n(\alpha_i)$ = game-type$_n(\beta_i)$ for every $i \in \mathbb{N}$.*
Then game-type$_n(\sum \alpha_i)$ = game-type$_n(\sum \beta_i)$ and the ordinals $\sum \alpha_i$ and $\sum \beta_i$ are determinate.

The proof of Lemma 4.7 can be derived from Lemmas 4.1, 4.4 and the Ramsey theorem. Its proof is omitted, since we won't use this lemma in the sequel.

5. Büchi-Landweber theorem holds for $\alpha < \omega^\omega$

In this section we provide a simple proof of Shomrat's theorem [Sho07] which extends the Büchi-Landweber theorem to all ordinals $\alpha < \omega^\omega$.

Theorem 5.1. *Let $\alpha < \omega^\omega$ and $\varphi(X, Y)$ be a formula. Then:*

Determinacy: *One of the players has a winning strategy in the game $\mathcal{G}^\alpha_\varphi$.*

Decidability: *It is decidable* which *of the players has a winning strategy.*

Definable strategy: *The player who has a winning strategy also has a* definable *winning strategy.*

Synthesis algorithm: *We can compute a formula $\psi(X, Y)$ that defines (in $(\alpha, <)$) a winning strategy for the winning player in $\mathcal{G}^\alpha_\varphi$.*

Proof. Note that every ordinal $\alpha < \omega^\omega$ is definable. Games of length one have definable winning strategies.

First, prove by the induction on $n \in \mathbb{N}$ that the theorem holds for $\alpha = \omega^n$. The base $\alpha = 1$ is trivial. For the inductive step use Lemma 4.6.

Next, by Lemma 4.3, we obtain that the theorem is true for every α of the form $\omega^m n_m + \omega^{m-1} n_{m-1} + \ldots + +\omega^0 n_0$, where $m, n_i \in \mathbb{N}$.

Finally, note that every $\alpha < \omega^\omega$ is equal to $\omega^m n_m + \omega^{m-1} n_{m-1} + \ldots + +\omega^0 n_0$, for some $k, n_i \in \mathbb{N}$. □

In [RS08] we proved that for every $\alpha \geq \omega^\omega$ there is a formula ψ_α such that Player I wins $\mathcal{G}^\alpha_{\psi_\alpha}$; however, he has no definable winning strategy in this game, i.e., the definability part of the Büchi-Landweber theorem fails for every $\alpha \geq \omega^\omega$. Therefore, the Büchi-Landweber theorem holds for α iff $\alpha < \omega^\omega$.

6. Determinacy

Theorems 4.2 and 4.5 imply that the set of determinate ordinals is closed under addition and multiplication by ω. In this section, we prove that every countable ordinal is determinate. First, let us show the following Lemma:

Lemma 6.1. *ω^α is determinate for every countable α.*

Proof. By Induction on α.

The basis: $\alpha = 0$ is immediate.

The case of successor follows from Lemma 4.5(1).

Assume that α is a countable limit ordinal. In this case $\alpha = \lim_{i \in \mathbb{N}} \beta_i$, where $\beta_i < \alpha$ is an increasing ω-sequence.

We are going to show that for every $n \in \mathbb{N}$ and every $G \subseteq Char^n_2$, the ω^α game for G is determinate.

For every n, the set $\mathbb{P}(Char^n_2)$ is finite. Therefore, there is C and an increasing ω-subsequence γ_i of β_i such that

$$C = \text{game-type}_n(\omega^{\gamma_i}) \text{ for every } i.$$

Let $\alpha_i = \omega^{\gamma_i}$. By the inductive assumption α_i are determinate. Therefore, by Lemma 4.4(C) and the equation above, for every $G \subseteq Char^n_2$, the $\sum \alpha_i$ game for G is determinate.

Note that $\sum \alpha_i = \sum \omega^{\gamma_i} = \omega^{\lim \gamma_i} = \omega^{\lim \beta_i} = \omega^\alpha$. Hence, for every $n \in \mathbb{N}$ and every $G \subseteq Char^n_2$, the ω^α game for G is determinate. Therefore, ω^α is determinate. □

Theorem 6.2. *Every countable ordinal is determinate.*

Proof. Every ordinal has a Cantor Normal Form representation $\omega^{\alpha_1}n_1+\omega^{\alpha_2}n_2+\cdots+\omega^{\alpha_k}n_k$, where $k, n_i \in \mathbb{N}$. Hence, by Lemmas 6.1 and 4.2, every countable ordinal is determinate. □

7. Decidability

Lemma 7.1. *For every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that game-type$_n(\omega^m)$ = game-type$_n(\omega^m \times \alpha)$ for every countable ordinal $\alpha > 0$. Moreover, m is computable from n.*

Proof. The proof is similar to the proof of Theorem 3.5(B) in [Sh75]. For every ordinal α, let us write $t(\alpha)$ for game-type$_n(\alpha)$. By Lemma 4.5 there is a multiplication by ω operation on the game-types of determinate ordinals. We denote it by $\times\omega$; for every determinate ordinal α if $C = t(\alpha)$, then $C \times \omega$ is $t(\alpha\omega)$). The multiplication by ω operation is even computable by Lemma 4.5. Moreover, If all α_i ($i \in \omega$) are determinate and $C = t(\alpha_i)$ for all $i \in \omega$ and *fixed* C, then $t(\sum \alpha_i) = C \times \omega$.

Let $m := |\mathbb{P}(\mathbb{P}(Char_2^n))|$, i.e., the number of possible formal n-game types. We are going to show that this m satisfies the Lemma.

We first show that there is a $p < m$ such that $t(\omega^p) = t(\omega^{p+1})$.

By the pigeon-hole principle, there are $q < r \leq m$ such that $t(\omega^q) = t(\omega^r)$. Moreover, by Theorem 4.5 we may compute such q and r. If $r = q+1$, our claim is proved for $p = q$, so assume $r \geq q+2$. We will show that $p = q+1$ works. By Theorem 4.5(2),

$$t(\omega^{q+2}) = t(\sum_{i\in\mathbb{N}}(\omega^{q+1}+\omega^q)) = \big(t(\omega^{q+1}+\omega^q)\big)\times\omega =$$
$$\big(t(\omega^{q+1})+t(\omega^q)\big)\times\omega = \big(t(\omega^{q+1})+t(\omega^r)\big)\times\omega =$$
$$\big(t(\omega^{q+1}+\omega^r)\big)\times\omega.$$

But, $q+1 < r$, so $\omega^{q+1}+\omega^r = \omega^r$. Thus, indeed,

$$t(\omega^{q+2}) = t(\omega^r)\times\omega = t(\omega^q)\times\omega = t(\omega^{q+1}).$$

Next, we show that any p, as above, will satisfy the claim of the lemma, i.e., $t(\omega^p) = t(\omega^p\alpha)$ for every countable $\alpha > 0$. First, note that

$$\begin{aligned} t(\omega^p)+t(\omega^p) = t(\omega^p)+t(\omega^{p+1}) = \\ t(\omega^p+\omega^{p+1}) = t(\omega^{p+1}) = t(\omega^p). \end{aligned} \qquad (\star\star)$$

Now, by induction on countable $\alpha > 1$, we prove $t(\omega^p\cdot\alpha) = t(\omega^p)$. Assume α is a successor, say, $\alpha = \beta+1$. If $\beta = 0$, there is nothing to prove. Assume $\beta > 0$.

$$t(\omega^p\alpha) = t(\omega^p\beta+\omega^p) = t(\omega^p\beta)+t(\omega^p) = t(\omega^p)+t(\omega^p) \overset{(\star\star)}{=} t(\omega^p).$$

Finally, assume that α is a limit ordinal. Then there is an increasing ω-sequence $(\alpha_i\ :\ i \in \omega)$ such that $\alpha = \lim \alpha_i$. Hence,

$$\omega^p\times\alpha = \lim(\omega^p\times\alpha_i) = \sum\omega^p(\alpha_{i+1}-\alpha_i). \qquad (7.1)$$

By the inductive assumption $t(\omega^p) = t(\omega^p\times(\alpha_{i+1}-\alpha_i))$. Hence, $t(\omega^{p+1}) = t(\omega^p)\times\omega = t(\sum_{i\in\omega}\omega^p\times(\alpha_{i+1}-\alpha_i)) = t(\omega^p\alpha)$, by Lemma 4.4. Since $t(\omega^p) = t(\omega^{p+1})$, we derive that $t(\omega^p) = t(\omega^p\times\alpha)$.

Recall that $m := |\mathbb{P}(\mathbb{P}(Char_2^n))| \geq q+1 = p$. Therefore, $t(\omega^m) = t(\omega^p\omega^{m-p}) = t(\omega^p\omega^{m-p+1}) = t(\omega^{m+1})$. Hence, m satisfies the lemma. □

Let $n \in \mathbb{N}$ and let $m = m(n) \in \mathbb{N}$ be computable from n as in Lemma 7.1. Every ordinal $\alpha > 0$ has a unique representation of the form $\alpha = \omega^m\gamma + \omega^{m-1}n_1 + \omega^{m-2}n_2 + \cdots + \omega^0 n_m$, where $n_i \in \mathbb{N}$. Define $\mathrm{Code}_n(\alpha)$ as

$$\mathrm{Code}_n(\alpha) := \begin{cases} (0, n_1, n_2, \dots, n_m) & \text{if } \gamma = 0 \\ (1, n_1, n_2, \dots, n_m) & \text{otherwise} \end{cases}.$$

Theorem 7.2 (Decidability)**.** *There is an algorithm that given a formula $\varphi(X_1, X_2)$ and the $Code_{\mathrm{qd}(\varphi)}(\alpha)$ of a countable ordinal $\alpha > 0$, determines which of the players has a winning strategy in $\mathcal{G}_\varphi^\alpha$.*

Proof. By Lemma 7.1, we have that $\text{game-type}_n(\omega^m) = \text{game-type}_n(\omega^m \times \gamma)$ for every $\gamma > 0$. Therefore, by Lemma 4.2, if $\mathrm{Code}_n(\alpha) = \mathrm{Code}_n(\beta)$, then $\text{game-type}_n(\alpha) = \text{game-type}_n(\beta)$.

Hence, for an input φ and $\mathrm{Code}_{\mathrm{qd}(\varphi)}(\alpha) = (n_0, n_1, n_2, \dots, n_m)$ it is enough to decide which of the players wins the $\mathcal{G}_\varphi^{\omega^m n_0 + \omega^{m-1}n_1 + \cdots + \omega^0 n_m}$. This is decidable by Theorem 5.1. □

8. MLO Characterisation of the Winner

In this section we show that for every formula $\varphi(X, Y)$ there exists an *MLO* sentence ψ such that for every countable ordinal α: Player I wins $\mathcal{G}_\varphi^\alpha$ iff $\alpha \models \psi$.

Our proof refines the proof of Theorem 7.2. For every $n \in \mathbb{N}$ we will define a mapping gcode_n which assigns to every countable ordinal a code (game code). Then, we show that (1) the $\mathrm{gcode}_n(\alpha)$ determines the $\text{game-type}_n(\alpha)$ (2) for every code c in the range of $gcode_n$, the set $A_c := \{\alpha \mid \mathrm{gcode}_n(\alpha) = c\}$ is definable by an *MLO* sentence and (3) the range of gcode_n is a finite set. From (1)-(3) it will be easy to show that the set of countable ordinals for which Player I has a winning strategy in $\mathcal{G}_\varphi^\alpha$ is *MLO* definable.

Let $\alpha = \omega^m n_0 + \omega^{m-1}n_1 + \omega^{m-2}n_2 + \cdots + \omega^0 n_m$. Theorem 7.2 shows that in order to determine $\text{game-type}_n(\alpha)$ we do not have to know the coefficients of the ω^i for every sufficiently large i. The next Lemma will imply that in order to determine $\text{game-type}_n(\alpha)$ we do not need to know the precise value of the coefficients n_i, even for small values of i. It is sufficient to know these coefficients modulo a number which depends on n.

Recall that αn denotes the multiplication of the ordinal α by n.

Lemma 8.1. *For every $n \in \mathbb{N}$ there is $p \in \mathbb{N}$ such that if $n_1 > n_2 \geq p$ and $n_1 = n_2$ mod p, then $\text{game-type}_n(\alpha n_1) = \text{game-type}_n(\alpha n_2)$ for every ordinal $\alpha > 0$. Moreover, p is computable from n.*

Proof. For every ordinal α, let us write $t(\alpha)$ for $\text{game-type}_n(\alpha)$.

Let $m := |\mathbb{P}(\mathbb{P}(Char_2^n))|$, i.e., the number of possible formal n-game types.

By the pigeon-hole principle, there are $q < r \leq m+1$ such that $t(\alpha q) = t(\alpha r)$. By Theorem 4.2, for every n:

$$t(\alpha(n+q)) = t(\alpha n) + t(\alpha q) = t(\alpha n) + t(\alpha r) = t(\alpha(n+r))$$

Therefore, if $n_1 > n_2 \geq q$ and n_1 is equal to n_2 modulo $r - q$, then $t(\alpha n_1) = t(\alpha n_2)$. Note that q and r depend on α. However, if we define $p := (m+1)!$, then this p satisfies the lemma. □

Now we are ready to define gcode_n. Let $n \in \mathbb{N}$ and let $p = p(n) \in \mathbb{N}$ be computable from n as in Lemma 8.1. Let $\mathrm{trun}_n : \mathbb{N} \to \mathbb{N}$ be defined as

$$trun_n(k) := \begin{cases} k & \text{if } k < p \\ p + (k \ mod \ p) & \text{otherwise} \end{cases} .$$

Let $m = m(n) \in \mathbb{N}$ be computable from n as in Lemma 7.1. Every ordinal $\alpha > 0$ has a unique representation of the form $\alpha = \omega^m\gamma + \omega^{m-1}n_1 + \omega^{m-2}n_2 + \cdots + \omega^0 n_m$, where $n_i \in \mathbb{N}$.

Define $\mathrm{gcode}_n(\alpha)$ as

$$\mathrm{gcode}_n(\alpha) := \begin{cases} (0, \mathrm{trun}_n(n_1), \mathrm{trun}_n(n_2), \ldots, \mathrm{trun}_n(n_m)) & \text{if } \gamma = 0 \\ (1, \mathrm{trun}_n(n_1), \mathrm{trun}_n(n_2), \ldots, \mathrm{trun}_n(n_m)) & \text{otherwise} \end{cases} .$$

(Hence, the tuple $\mathrm{gcode}_n(\alpha)$ is obtained from $\mathrm{Code}_n(\alpha)$ by applying $trunc_n$ pointwise.) By Theorem 4.2, Lemma 7.1 and Lemma 8.1 it follows that game-type$_n(\alpha)$ is determinate by $\mathrm{gcode}_n(\alpha)$. Moreover, by a proof similar to the proof of Theorem 7.2, we obtain the following result:

Lemma 8.2 (Game Code Lemma)**.** *There is an algorithm that given a formula $\varphi(X_1, X_2)$ and the $gcode_{\mathrm{qd}(\varphi)}(\alpha)$ of a countable ordinal $\alpha > 0$, determines which of the players has a winning strategy in $\mathcal{G}^\alpha_\varphi$.*

Lemma 8.3. *For every c in the range of $gcode_n$ there is a sentence ψ_c such that $\alpha \models \psi_c$ iff $gcode_n(\alpha) = c$.*

Proof. (Sketch) For every $i \in \mathbb{N}$ there is an *MLO* formula $Mult_{\omega^i}(X)$ which says that X is a subset of the order type $\omega^i\gamma$ for an ordinal $\gamma > 0$. For every $k < p \in \mathbb{N}$ there is an *MLO* formula $Mod_{p,k}(X, Y)$ which says that X contains $k \ mod \ p$ occurrences of Y. Using these formulas it is not difficult to formalize the definition of gcode_n and to write a desirable formula ψ_c. □

Now we are ready to provide an *MLO* characterization of the winner.

Theorem 8.4. *There is an algorithm that given a formula $\varphi(X_1, X_2)$ computes a sentence Win_φ such that for every countable ordinal α: Player I wins $\mathcal{G}^\alpha_\varphi$ if and only if $\alpha \models Win_\varphi$.*

Proof. Let n be the quantifier depth of φ. Let C_φ be defined as

$$C_\varphi := \{c \text{ is a } \mathrm{gcode}_n \mid \text{ Player I wins } \mathcal{G}^\alpha_\varphi \text{ if } \mathrm{gcode}_n(\alpha) = c\}$$

Since the range of gcode_n is finite, the set C_φ is finite and computable by Lemma 8.2. For every $c \in C_\varphi$ compute ψ_c as in Lemma 8.3. Hence, Win_φ defined as

$$Win_\varphi := \bigvee_{c \in C_\varphi} \psi_c$$

satisfies the lemma. □

9. Synthesis Problem

Let α be an ordinal. In this section we address the following synthesis problem:

Problem Synth(α):

Input: a formula $\varphi(X_1, X_2)$,

Task: decide whether one of the players has a definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$, and if so, construct ψ which defines his winning strategy.

The decidability version of the synthesis problem for α requires only to decide whether one of the players has a definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$ (but does not output it). We will denote this version by Dsynth(α).

By Theorem 5.1, these problems are computable for $\alpha < \omega^{\omega}$. Note Dsynth(α) is trivial for $\alpha < \omega^{\omega}$, because for these ordinals the winning player has a definable winning strategy.

As mentioned in the introduction, for every $\alpha \geq \omega^{\omega}$ there is a formula ψ_{α} such that Player I wins $\mathcal{G}^{\alpha}_{\psi_{\alpha}}$; however, he has no definable winning strategy. Therefore, Dsynth(α) is non-trivial for $\alpha \geq \omega^{\omega}$.

Unfortunately, we were unable to show that the synthesis problems are computable for every countable ordinal. However, we show here that a crucial ordinal is ω^{ω}.

First, note

Lemma 9.1. *There is an algorithm which for formulas $\psi(X_1, X_2)$ and $\varphi(X_1, X_2)$ constructs sentences $Win^{\varphi,\psi}_{I}$ and $Win^{\varphi,\psi}_{II}$ such that for every ordinal α:*

(1) $\alpha \models Win^{\varphi,\psi}_{I}$ *iff ψ defines a winning strategy for Player I in $\mathcal{G}^{\alpha}_{\varphi}$.*
(2) $\alpha \models Win^{\varphi,\psi}_{II}$ *iff ψ defines a winning strategy for Player II in $\mathcal{G}^{\alpha}_{\varphi}$.*

Proof. $Win^{\varphi,\psi}_{I}$ is the conjunction of the sentence that says that ψ defines a strongly causal operator and the sentence $\forall X_2 X_1 \psi(X_1, X_2) \rightarrow \varphi(X_1, X_2)$.

$Win^{\varphi,\psi}_{II}$ is defined similarly. □

Recall (see Section 2.4) that the monadic theory of a countable ordinal α is definable from Code(α). From Lemma 9.1 we deduce:

Lemma 9.2.

(1) *The problems Synth(α) and Dsynth(α) are recursive in each other.*
(2) *If Code(α_1)=Code(α_2), then the problems Synth(α_1) and Synth(α_2) are equivalent, and the problems Dsynth(α_1) and Dsynth(α_2) are equivalent.*

Proof.

(1) It is clear that Dsynth(α) is recursive in Synth(α). We will show that Synth(α) is recursive in Dsynth(α). Let $\psi_1, \dots \psi_i \dots$ be a recursive enumeration of all formulas. Let φ be an input for Synth(α). First, check Dsynth(α) on input φ. If the answer is "No", then neither of the players has a definable winning strategy. If the answer is "Yes", then for $i = 1, \dots$ use Lemma 9.1 and Theorem 2.12 to verify whether ψ_i is a definable winning strategy for one of the players. When such ψ_i is found, output it and terminate. The correctness of the reduction is immediate.
(2) If Code(α_1)=Code(α_2), then by Theorem 2.12, α_1 and α_2 satisfy the same monadic sentences. Hence, by Lemma 9.1, ψ is a definable winning strategy in $\mathcal{G}^{\alpha_1}_{\varphi}$ iff ψ is a definable winning strategy in $\mathcal{G}^{\alpha_2}_{\varphi}$. □

The following lemma provides a reduction from Dsynth(α) to Dsynth(ω^ω). Hence, Synth(α) is reducible to Synth(ω^ω).

Lemma 9.3. *There is an algorithm that given the Code(α) of a countable $\alpha > \omega^\omega$ and $\varphi_1(X_1, X_2)$ constructs $\varphi_2(X_1, X_2)$ such that a player has a definable winning strategy in $\mathcal{G}^{\omega^\omega}_{\varphi_2}$ iff he has a definable winning strategy in $\mathcal{G}^{\alpha}_{\varphi_1}$.*

Proof. Let $\alpha = \omega^\omega\alpha' + \beta$ with $\beta < \omega^\omega$ (this can be done in a unique way), and let $\alpha_1 = \omega^\omega + \beta$.

Code(α)=Code(α_1), therefore, by lemma 9.2, ψ defines a winning strategy in $\mathcal{G}^{\alpha}_{\varphi_1}$ iff it defines a winning strategy in $\mathcal{G}^{\alpha_1}_{\varphi_1}$.

Let n be the quantifier depth of φ_1. Compute $G \subseteq \mathit{Char}^n_2$ such that φ_1 is equivalent to the disjunction of formulas from G. Let C_β and $K(C_\beta, G)$ be defined as in Lemma 4.1. Let φ_2 be the disjunction of formulas from $K(C_\beta, G)$. Note that φ_2 is computable from code(α) and φ_1.

We claim that φ_2 satisfies the conclusion of the Lemma.

Indeed, from the proof of Lemma 4.1, it follows that if F is a winning strategy in $\mathcal{G}^{\omega^\omega+\beta}_{\varphi_1}$, then its first ω^ω rounds is a winning strategy $\mathcal{G}^{\omega^\omega}_{K(C_\beta,G)} = \mathcal{G}^{\omega^\omega}_{\varphi_2}$.

Assume that $\psi(X_1, X_2)$ defines F in $\omega^\omega + \beta$. We are going to construct ψ_2 which defines in ω^ω the strategy F_1 which plays F first ω^ω rounds.

Let $\Delta = \{(\tau, \tau') \in \mathit{Char}^n_2 \times \mathit{Char}^n_2 \;:\; \tau + \tau' \to \psi\}$. Define Δ_1 as $\Delta_1 := \{\tau \;:\; \exists \tau' \;:\; (\tau, \tau') \in \Delta$ and $\beta \models \exists X_1 X_2 \tau'\}$. Note that Δ and Δ_1 are computable.

From Theorem 2.9, it follows that the disjunction of formulas from Δ_1 defines F_1.

For the other direction, assume that there is a definable winning strategy F_1 for Player I in $\mathcal{G}^{\omega^\omega}_{K(C_\beta,G)}$. Note that for every $\tau \in K(C_\beta, G)$ there is a winning strategy for Player I in the β game for $H_\tau = \{\tau' \;:\; \tau + \tau' \in G\}$. Since $\beta < \omega^\omega$ Player I has a definable winning strategy F_τ in the β game for H_τ. The definable winning strategy for $\mathcal{G}^{\omega^\omega+\beta}_{G}$ is constructed from F_1 and the strategies F_τ as in Lemma 4.3. The only subtle point is that the ordinal ω^ω is not definable. However, there is a formula $\theta(t)$ such that $\alpha_1 \models \theta(\gamma)$ iff $\gamma = \omega^\omega$ (this formula expresses that γ is the minimal ordinal such that the interval $[\gamma, \alpha_1)$ is isomorphic to (a definable ordinal) β.

The case when there is a definable winning strategy for Player II in $\mathcal{G}^{\omega^\omega}_{K(C_\beta,G)}$ is similar. □

10. Conclusion and Open Problems

We considered a natural extension of the Church Problem to countable ordinals. We proved that the Büchi-Landweber theorem extends fully to all ordinals $< \omega^\omega$ and that its determinacy and decidability parts extend to all countable ordinals. We reduced the synthesis problem for countable ordinals $> \omega^\omega$ to the synthesis problem for ω^ω. However, the decidability of the synthesis problems for ω^ω remains open.

In preliminary version of this paper we asked whether the first uncountable ordinal ω_1 is determinate. For uncountable ordinals the situation changes radically. Let $\varphi_{spl}(X, Y)$ say: "X is stationary, $Y \subseteq X$ and both Y and $X \setminus Y$ are stationary" (recall that $S \subseteq \omega_1$ is called *stationary* iff for every closed unbounded $C \subseteq \omega_1$, $S \cap C \neq \emptyset$). P. B. Larson and S. Shelah pointed to us that it follows immediately from [LaS08] that each of the following statements is consistent with ZFC:

(1) None of the players has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.
(2) Mrs. Y has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.
(3) Mr. X has a winning strategy in $\mathcal{G}^{\omega_1}_{\varphi_{spl}}$.

In other words, ZFC can hardly tell us anything concerning this game. On the other hand, S. Shelah [She07] tells us he believes it should be possible to prove:

Conjecture 10.1. It is consistent with ZFC that $\mathcal{G}^{\omega_1}_{\varphi}$ is determined for *every* formula φ.

Let us discuss a question of uniform definability of the winning strategy. Recall that for every φ and $\alpha < \omega^\omega$ one of the players has a finite memory winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$. It is natural to ask the following uniform definability question: given $\varphi(X, Y)$ as above, is it possible to provide ψ such that, for each ordinal below ω^ω such that Player I wins $\mathcal{G}^{\alpha}_{\varphi}$, the formula ψ defines his winning strategy in $\mathcal{G}^{\alpha}_{\varphi}$?

The negative answer to the uniform definability question even for one-player games follows from our results in [RS07]. Consider the formula $\varphi_{\omega ub}(Y)$ expressing that "Y is an unbounded ω-sequence." It is clear that the moves of the Player I are unimportant in the games with the winning condition $\varphi_{\omega ub}(Y)$, and that Player II wins this game over every countable limit ordinal. However, it was shown in [RS07] that Player II has no definable winning strategy which uniformly works for all limit ordinals $< \omega^\omega$.

The negative answer to the uniform definability question leads to the following algorithmic problem:

Problem: (uniform definability of winning strategy) Given $\varphi(X, Y)$. Decide whether there is ψ which defines a winning strategy for Player I for each ordinal below ω^ω such that Player I wins the $\mathcal{G}^{\alpha}_{\varphi}$.

The decidability of uniform definability problem is an open question.

Next, we describe a uniformization problem, sometimes called the Rabin uniformization problem.

Let $\varphi(\bar{X}, \bar{Y})$, $\psi(\bar{X}, \bar{Y})$ be formulas and $\mathcal{M}$ be a structure. We say that ψ *uniformizes* (or, is a *uniformizer* for) ψ *in* $\mathcal{M}$ iff:

(1) $\mathcal{M} \models \forall \bar{X} \exists^{\leq 1} \bar{Y} \psi(\bar{X}, \bar{Y})$,
(2) $\mathcal{M} \models \forall \bar{X} \forall \bar{Y} (\psi(\bar{X}, \bar{Y}) \rightarrow \varphi(\bar{X}, \bar{Y}))$, and
(3) $\mathcal{M} \models \forall \bar{X} \big(\exists \bar{Y} \varphi(\bar{X}, \bar{Y}) \rightarrow \exists \bar{Y} \psi(\bar{X}, \bar{Y})\big)$.

$\mathcal{M}$ has the *uniformization property* iff every formula φ has a uniformizer ψ in $\mathcal{M}$.

In [LS98], Lifsches and Shelah show that an ordinal α has the uniformization property iff $\alpha < \omega^\omega$.

Uniformization, too, naturally leads to a decision problem:

Uniformization Problem for α:
Input: an MLO formula $\varphi(X, Y)$.
Task: determine whether φ has a uniformizer in α, and if so, construct it.

Note the similarities and dissimilarities between the Church synthesis problem (see Def. 1.2) and the uniformization problem. In uniformization, we are also given a formula $\varphi(X, Y)$ and to every P we try and "respond" with a Q, such that $\varphi(P, Q)$ holds. Only we do not restrict ourselves to causal responses. On the other hand, we do restrict ourselves to *definable* (in $(\alpha, <)$) responses. In the Church problem, we do *not* require that the strategy (=causal operator) is definable.

While we are not yet able to decide uniformization in $(\omega^\omega, <)$, we presented in [RS07] a restricted version of this problem, and proved that this version is decidable for every countable ordinal.

Our initial motivation to study games of length $> \omega$ was a hope to reduce ω-games with complex winning conditions [CDT02, Se06] to longer games with simple winning conditions. We plan to pursue this direction further.

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