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# THE FULL BINARY TREE CANNOT BE INTERPRETED IN A CHAIN

ALEXANDER RABINOVICH

**Abstract.** We show that for no chain $C$ there is a monadic-second order interpretation of the full binary tree in $C$.

**§1. Introduction.** Two fundamental results of classical automata theory are the decidability of the monadic second-order theory of $\omega$ by Büchi [Bu62] and the decidability of the monadic second-order theory of the full binary tree by Rabin [Rab69].

Büchi and Rabin's proofs use methods and results of finite automata theory. Rabin's original proof [Rab69] is highly nontrivial. Even though it deals with facts concerning finite automata, transfinite induction over ordinals up to the first uncountable ordinal $\omega_1$ is used in an essential way. Much effort has been invested over the last 40 years to simplify his proof. The modern proofs of Rabin's theorem are quite compact.

The full binary tree $T_2$ is a structure over the signature $\{<, Left, Right\}$, where $<$ is a binary relation and $Left, Right$ are unary relations. The domain of $T_2$ is the set of all finite strings over $\{0, 1\}$; the symbol $<$ is interpreted as the prefix relation; $Left(s)$ holds on $s$ iff the last letter of $s$ is 0; $Right(s)$ holds on $s$ iff the last letter of $s$ is 1.

The interpretation method was used by Rabin [Rab69] to recover most of the then-known decidability results as well as many new results. In particular he provided an interpretation of the rationals in $T_2$ and proved that the monadic theory of the rationals is decidable.

Büchi generalized the concept of an automaton to allow automata to "work" on words of any countable length (ordinal) and proved that the monadic theory of any ordinal $< \omega_2$ is decidable [BS73].

In the breakthrough paper [Sh75], Shelah developed a model-theoretical composition method and proved all then-known decidability results about the monadic theory of linear orders in a uniform way and much more. He also demonstrated limits on extending these decidability results, showing in particular that the monadic theory of the reals is undecidable. However, it is still open whether the decidability of $T_2$ can be obtained using the composition method.

Gurevich, Magidor, and Shelah [GMS83] proved that the decidability of the monadic theory of $\omega_2$ is independent of ZFC.

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We address a question of the interpretability power of the monadic theory of order, which is a stronger criterion for difficulty [LS99]. A structure $B$ is (semantically) interpreted in a structure $A$ if $B$ can be isomorphically reproduced in $A$ by relations definable in $A$.

A chain is a linear order expanded by monadic predicates. We prove that for no chain $C$ there is a monadic second-order interpretation of the full binary tree in $C$.

Hence, there is something inherently more complicated in the full binary tree than in any linear order.

Our paper is organized as follows. In the next section we recall the notion of (semantical) interpretation, we state two main technical lemmas and derive our main results. In Section 3 we recall elements of the composition method which we need for our proof of main lemmas. These lemmas are proved in Sections 4 and 5. In Section 6 related results are discussed.

**§2. Non-interpretability theorem.** Let $\Sigma$ and $\Delta$ be relational signatures, $A$ a $\Delta$-structure and $B$ a $\Sigma$-structure. A (one-dimensional) semantic interpretation $\Gamma$ of $B$ in $A$ is consists of three items

1. a formula $\partial_\Gamma(x_1)$ of signature $\Delta$,
2. for each $m$-ary symbol $R$ of $\Sigma$, a formula $R_\Gamma(x_1, \dots, x_m)$ of signature $\Delta$,
3. a bijective map $f_\Gamma$ from $\partial_\Gamma(A)$ the set of elements that satisfy $\partial_\Gamma$ into the domain of $B$

such that for every $m$-ary symbol $R$ of $\Sigma$ and all $a_1, \dots, a_m \in \partial_\Gamma(A)$

$$B \models R(f_\Gamma(a_1), \dots, f_\Gamma(a_m)) \Leftrightarrow A \models R_\Gamma(a_1, \dots, a_m).$$

The formula $\partial_\Gamma(x_1)$ is the domain formula of $\Gamma$. The set of elements that satisfy $\partial_\Gamma$ is the domain of $\Gamma$. We assume that $\partial_\Gamma(x_1)$ and $R_\Gamma(x_1, \dots, x_m)$ are formulas of monadic second-order logic.

Monadic (second-order) logic is the fragment of full second-order logic allowing quantification only over elements and monadic predicates. One way to define the monadic second-order language for a signature $\Delta$ is to augment the first-order language for $\Delta$ by quantifiable set variables (monadic predicate variables) and by new atomic formulas $x \in Y$, where $x$ is a first-order variable and $Y$ is a set variable. The intended interpretation here is that $\in$ is the membership relation and the set variables range over all subsets of a structure for $\Delta$.

We will use lower case letters $x, y, z$ for first-order variables and upper case letters $X, Y, Z$ for set variables.

We say that the quantifier depth of $\Gamma$ is $n$ if the quantifier depth of the formulas that appear in (1)-(2) is at most $n$.

For an interpretation $\Gamma$ in a chain $A$ we consider the domain of $\Gamma$ as a substructure of $A$.

Usually in the literature it is only required that $f_\Gamma$ be surjective. However, if there is a surjective interpretation of a structure $B$ in a chain $C$ then there is a bijective interpretation of $B$ in an expansion of $C$ by a monadic predicate that contains exactly one element from each set $\delta_b := \{c \in \partial_\Gamma(C) \mid f_\Gamma(c) = b\}$, where $b \in B$. Sometimes, interpretations with parameters are considered. In such an interpretation with parameters the domain formula and the formulas defining relations can contain parameters. However, if there is an interpretation with parameters of a structure $B$ in a chain $C$ then there is an interpretation without parameters of $B$ in an expansion of $C$ by monadic predicates.

**Terminology.** Up to Section 6 "interpretation" stands for "one-dimensional monadic second-order semantic interpretation."

Recall that a linear order $A$ is scattered if there is no order-preserving mapping from the rationals into $A$.

LEMMA 2.1. *Assume that G is an infinite connected graph such that every node has finite degree. If there is an interpretation of G in a chain A, then the domain of the interpretation is a scattered order.*

LEMMA 2.2. *If there is an interpretation of the rationals in a chain A, then the domain of the interpretation is not a scattered order.*

These lemmas are proved in Sections 4 and 5. As a corollary we obtain our main result:

THEOREM 2.3 (non-interpretability theorem). *There is no interpretation of the full binary tree in a chain.*

PROOF. Assume that $\Gamma$ is an interpretation of $T_2$ in a chain $A$. Let $\overrightarrow{G}_2$ be the directed graph of $T_2$; its nodes are the finite strings over $\{0, 1\}$. A node $s$ is connected to nodes $s0$ and $s1$. Let $G_2$ be the corresponding undirected graph. There is a simple interpretation of $G_2$ in $T_2$ with the domain formula *True*. Hence, by Lemma 2.1, the domain $D$ of $\Gamma$ is a scattered subchain of $A$. By [Rab69], there is an interpretation $\Gamma'$ of the rationals in $T_2$. Taking the composition of the interpretations $\Gamma$ and $\Gamma'$ we obtain an interpretation $\overline{\Gamma}$ of the rationals in $A$ such that the domain $\overline{D}$ of $\overline{\Gamma}$ is a subset of $D$. By Lemma 2.2, $\overline{D}$ is not scattered. This is a contradiction, because all subchains of a scattered chain are scattered. ⊣

## §3. Elements of the composition method.

Our proofs of Lemmas 2.1 and 2.2 make use of the technique known as the composition method developed by Feferman-Vaught and Shelah [FV59, Sh75]. To fix notations and to aid the reader unfamiliar with this technique, we briefly review the required definitions and results.

**3.1. Hintikka formulas and $n$-types.** Fix a finite relational signature $\Sigma$. Let $n, l \in \mathbb{N}$. We denote by $\mathfrak{Form}_l^n$ the set of monadic second-order formulas (in $\Sigma$) with free variables among $x_1, \dots, x_l$ and of quantifier depth $\leq n$.

DEFINITION 3.1. Let $n, l \in \mathbb{N}$ and let $\mathcal{M}, \mathcal{N}$ be structures and $a_1, \dots, a_l \in \mathcal{M}$, $b_1, \dots, b_l \in \mathcal{N}$. The *n-theory* of $(\mathcal{M}, a_1, \dots a_l)$ is

$$Th^n(\mathcal{M}, a_1, \dots, a_l) := \{\varphi \in \mathfrak{Form}_l^n \mid \mathcal{M}, a_1, \dots, a_l \models \varphi\}.$$

If $Th^n(\mathcal{M}, a_1, \dots, a_l) = Th^n(\mathcal{N}, b_1, \dots, b_l)$, we say that $(\mathcal{M}, a_1, \dots, a_l)$ and $(\mathcal{N}, b_1, \dots, b_l)$ are *n-equivalent* and write $\mathcal{M}, a_1, \dots, a_l \equiv^n \mathcal{N}, b_1, \dots, b_l$. If $l = 0$ we write $\mathcal{M} \equiv^n \mathcal{N}$.

Clearly, $\equiv^n$ is an equivalence relation. For any $l, n \in \mathbb{N}$, the set $\mathfrak{Form}_l^n$ is infinite. However, it contains only finitely many semantically distinct formulas.

LEMMA 3.2 (Hintikka Lemma). *For $n, l \in \mathbb{N}$, we can compute a* finite *set $Hin_l^n \subseteq \mathfrak{Form}_l^n$ such that*:

1. *For every $\equiv^n$-equivalence class $E$ there is a* unique *$\tau \in Hin_l^n$ such that for every structure $\mathcal{M}$ and $a_1, \dots, a_l \in \mathcal{M}$: $(\mathcal{M}, a_1, \dots, a_l) \in E$ if and only if $\mathcal{M}, a_1, \dots, a_l \models \tau$.*

2. *Every formula* $\varphi(x_1, \dots, x_l)$ *with* $\mathrm{qd}(\varphi) \leq n$ *is equivalent to a (finite) disjunction of formulas from* $Hin_l^n$. *There is an algorithm which for every formula* $\varphi(x_1, \dots, x_l)$ *computes a finite set* $G_\varphi \subseteq Hin_l^{\mathrm{qd}(\varphi)}$ *of formulas, such that* $\varphi$ *is equivalent to the disjunction of all the formulas in* $G_\varphi$. *Moreover,* $\tau \in G_\varphi$ *iff* $\tau \rightarrow \varphi$.

Any member of $Hin_l^n$ we call a $(n, l)$-*Hintikka formula*. We use $\tau$, $\tau_i$, $\tau'$ to range over the Hintikka formulas.

Definition 3.3 ($n$-Type). For $n, l \in \mathbb{N}$ and $a_1, \dots, a_l \in \mathscr{M}$, we denote by $type^n(\mathscr{M}; a_1, \dots, a_l)$ the unique member of $Hin_l^n$ satisfied by $a_1, \dots, a_l$ in $\mathscr{M}$.

Thus,

Lemma 3.4. 1. $type^n(\mathscr{M}; a_1, \dots, a_l)$ *determines* $Th^n(\mathscr{M}; a_1, \dots, a_l)$ *and, indeed,* $Th^n(\mathscr{M}; a_1, \dots, a_l)$ *is computable from* $type^n(\mathscr{M}; a_1, \dots, a_l)$.

2. $type^{n+1}(\mathscr{M})$ *determines* $\{type^n(\mathscr{M}; a_1) \mid a_1 \in M\}$ *and, indeed,* $\{type^n(\mathscr{M}; a_1) \mid a_1 \in M\}$ *is computable from* $type^{n+1}(\mathscr{M})$.

### 3.2. The ordered sum of chains and of $n$-types.

Definition 3.5 (sum of chains). Let $l \in \mathbb{N}$, $\mathscr{I} := (I, <^{\mathscr{I}})$ a linear order and $\mathfrak{S} := (\mathscr{M}_\alpha \mid \alpha \in I)$ a sequence of chains. Write $\mathscr{M}_\alpha := (A_\alpha, <^\alpha, {P_1}^\alpha, \dots, {P_l}^\alpha)$ and assume that $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$ are in $I$. The *ordered sum* of $\mathfrak{S}$ is the chain

$$\sum_{\mathscr{I}} \mathfrak{S} := (\bigcup_{\alpha \in I} A_\alpha, <^{\mathscr{I}, \mathfrak{S}}, \bigcup_{\alpha \in I} {P_1}^\alpha, \dots, \bigcup_{\alpha \in I} {P_l}^\alpha),$$

where:

if $\alpha, \beta \in I$, $a \in A_\alpha$, $b \in A_\beta$, then $b <^{\mathscr{I}, \mathfrak{S}} a$ iff $\beta <^{\mathscr{I}} \alpha$ or $\beta = \alpha$ and $b <^\alpha a$.

If the domains of the $\mathscr{M}_\alpha$'s are not disjoint, replace them with isomorphic chains that have disjoint domains, and proceed as before.

If $\mathscr{I} = (\{0, 1\}, <)$ and $\mathfrak{S} = (\mathscr{M}_0, \mathscr{M}_1)$, we denote $\sum_{\mathscr{I}} \mathfrak{S}$ by $\mathscr{M}_0 + \mathscr{M}_1$.

The next well-known lemma states that taking ordered sums preserves $\equiv^n$-equivalence.

Lemma 3.6. *Let* $n \in \mathbb{N}$. *Assume*:

1. $(I, <^{\mathscr{I}})$ *is a linear order,*
2. $(\mathscr{M}_\alpha^0 \mid \alpha \in I)$ *and* $(\mathscr{M}_\alpha^1 \mid \alpha \in I)$ *are sequences of chains (in the same signature), and*
3. *for every* $\alpha \in I$, $\mathscr{M}_\alpha^0 \equiv^n \mathscr{M}_\alpha^1$.

*Then,* $\sum_{\alpha \in I} \mathscr{M}_\alpha^0 \equiv^n \sum_{\alpha \in I} \mathscr{M}_\alpha^1$.

This allows us to define the sum of formulas in $Hin_0^n$ with respect to any linear order. We will use only finite sums of Hintikka's formulas. The finite sums can be defined over the Hintikka formulas with free variables.

Lemma 3.7. *Let* $n, l, k \in \mathbb{N}$ *and let* $\mathscr{M}^i, \mathscr{N}^i$ *for* $i = 0, 1$ *be chains in the same signature and* $a_1^i, \dots, a_l^i \in \mathscr{M}^i$, $b_1^i, \dots, b_k^i \in \mathscr{N}^i$. *If* $type^n(\mathscr{M}^0; a_1^0, \dots, a_l^0) = type^n(\mathscr{M}^1; a_1^1, \dots, a_l^1)$ *and* $type^n(\mathscr{N}^0; b_1^0, \dots, b_k^0) = type^n(\mathscr{N}^1; b_1^1, \dots, b_k^1)$, *then*

$$type^n(\mathscr{M}^0 + \mathscr{N}^0; a_1^0, \dots, a_l^0, b_1^0, \dots, b_k^0) = type^n(\mathscr{M}^1 + \mathscr{N}^1; a_1^1, \dots, a_l^1, b_1^1, \dots, b_k^1).$$

Definition 3.8 (sum of types). Let $n, l, k \in \mathbb{N}$, and $\tau_0 \in Hin_l^n$ and $\tau_1 \in Hin_k^n$ be Hintikka formulas. The *ordered sum* $\tau_0 + \tau_1$ of $\tau_0$ and $\tau_1$ is *an* element $\tau$ of $Hin_{l+k}^n$ such that:

if $\mathcal{M}, \mathcal{N}$ are chains, $a_1, \dots, a_l \in \mathcal{M}$ and $b_1, \dots, b_k \in \mathcal{N}$,
$type^n(\mathcal{M}; a_1, \dots, a_l) = \tau_0$ and $type^n(\mathcal{N}; b_1, \dots, b_k) = \tau_1$, then
$type^n(\mathcal{M} + \mathcal{N}; a_1, \dots, a_l, b_1, \dots, b_k) = \tau$.

### 3.3. Ramsey theorem for additive colourings.

Definition 3.9. 1. A colouring of a chain $C$ is a function $col : [C]^2 \to T$ where $[C]^2$ is the set of unordered pairs of distinct elements of $C$ and $T$ is a finite set (the set of colours).

2. The colouring $f$ is additive if for every $x_1 < y_1 < z_1$ and $x_2 < y_2 < z_2$ in $C$, it hold that $col(x_1, y_1) = col(x_2, y_2)$ and $col(y_1, z_1) = col(y_2, z_2)$ implies $col(x_1, z_1) = col(x_2, z_2)$. In this case a partial operation $+$ is well defined on $T$: $t_1 + t_2 = t$ iff there are $x < y < z$ such that $col(x, y) = t_1$, $col(y, z) = t_2$ and $col(x, z) = t$.

3. A sub-chain $D \subseteq C$ is homogeneous (for $col$) if there exists $t_0 \in T$ such that for every $x, y \in D$, $col(x, y) = t_0$.

Shelah [Sh75] proved the following remarkable theorem.

Theorem 3.10 (Ramsey theorem for additive colourings). *Let $col : [C]^2 \to T$ be an additive coloring where $C$ is a chain and $T$ is finite.*

1. *If $C$ is isomorphic to an infinite ordinal, then there is $H \subseteq C$, cofinal and homogeneous for $col$.*
2. *If $C$ is a dense chain, then there is an interval $J$ which has a dense homogeneous subset.*

**§4. Proof of Lemma 2.1.** Let $\Gamma$ be an interpretation of $G$ in a chain $A$. Let $D \subseteq A$ be the domain of $\Gamma$. Define an equivalence relation $\sim$ on $A$ as follows: for $a \le b$, $a \sim b$ if $[a, b] \cap D$ is a scattered subchain of $A$. It is clear that every $\sim$-equivalence class is an interval in $A$. An $\sim$-equivalence class is a $D$-class if $[a, b] \cap D \ne \emptyset$; otherwise it is a $\neg D$-class.

The set of $\sim$-equivalence classes inherit the order from $A$; namely, if $I_1, I_2$ are $\sim$-equivalence classes, then $I_1 \le I_2$ iff there is $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1 \le a_2$.

Between any two $\sim$-equivalence classes there is a $D$-class. Therefore, either there is one $\sim$-equivalence class or the chain $D_\sim$ of $D$-equivalence classes is dense.

In the first case $D$ is scattered and we are done.

We assume that the chain $D_\sim$ of $D$-equivalence classes is a dense order and derive a contradiction.

Let $n$ be an upper bound on the quantifier-depth of the formulas that appear in $\Gamma$. Define a coloring on the chain $D_\sim$ of $D$-classes as follows: if $I_1 < I_2 \in D_\sim$, then $col(I_1, I_2) := \big(type^{n+1}(I_1), type^n(A \restriction (I_1, I_2))\big)$, where $A \restriction (I_1, I_2)$ is the subchain of $A$ over the set $\{a \mid \forall a_1 \in I_1 \forall a_2 \in I_2 (a_1 < a < a_2)\}$. This is an additive coloring of a dense chain, therefore by Theorem 3.10, there is an interval $(J_1, J_2)$ of $D_\sim$ and a homogeneous subset $H \subseteq D_\sim$ everywhere dense in the interval $(J_1, J_2)$ of $D_\sim$. Let $\big(\tau_1^H, \tau_2^H\big)$ be the color of $H$.

For an $\sim$-equivalence class $I$ we denote by $A \restriction< I$ the subchain of $A$ over the set $\{a \mid \forall b \in I (a < b)\}$; for $\sim$-equivalence classes $I_1 < I_2$ we denote by $A \restriction (I_1, I_2]$ the subchain of $A$ over the set $\{a \mid \forall a_1 \in I_1 \forall a_2 \in I_2 (a_1 < a < a_2)\} \cup I_2$; the chains $A \restriction\le I$, $A \restriction> I$ and $A \restriction (I_1, I_2)$ are defined similarly.

(1) For $I \in H$

$$type^n(A \restriction< I) = type^n(A \restriction\leq J_1) + type^n(A \restriction (J_1, I)) = type^n(A \restriction\leq J_1) + \tau_2^H$$

and this is independent of the choice of $I \in H$.

Similarly,

(2) For $b \in A$ and $I, I' \in H$, if all elements in $I, I'$ are less than $b$, and $a \in I$, $a' \in I'$ and $type^n(I; a) = type^n(I'; a')$, then $type^n(A; a, b) = type^n(A; a', b)$.

Proof of 2.

$$\begin{aligned} type^n(A \restriction\leq I; a) &= type^n(A \restriction< I) + type^n(I; a) && \text{by Def. 3.8} \\ &= type^n(A \restriction< I') + type^n(I; a) && \text{by (1)} \\ &= type^n(A \restriction< I') + type^n(I'; a') && \text{by the assumption} \\ &= type^n(A \restriction\leq I'; a') && \text{by Def. 3.8.} \end{aligned}$$

Take any $I'' \in H$ between $I \cup I'$ and $b$. Since the color of $H$ is $(\tau_1^H, \tau_2^H)$ we obtain

$$\begin{aligned} type^n(A \restriction> I; b) &= type^n(A \restriction (I, I'')) + type^n(A \restriction\geq I''; b) \\ &= \tau_2^H + type^n(A \restriction\geq I''; b) \\ &= type^n(A \restriction (I', I'')) + type^n(A \restriction\geq I''; b) \\ &= type^n(A \restriction> I'; b). \end{aligned}$$

Since $type^n(A; a, b) = type^n(A \restriction\leq I; a) + type^n(A \restriction> I; b)$ and $type^n(A; a', b) = type^n(A \restriction\leq I'; a') + type^n(A \restriction> I'; b)$, we obtain $type^n(A; a, b) = type^n(A; a', b)$. ⊣

Let $I \in H$ and let $a \in I \cap D$ we are going to show that if $b \in D$ and there is an edge in $G$ between $f_\Gamma(a)$ and $f_\Gamma(b)$, then $b \in I$. Since $G$ is connected this implies that $D \subseteq I$ and this contradicts the assumption that there is more than one $\sim$-equivalence class.

There is an edge in $G$ between $f_\Gamma(a)$ and $f_\Gamma(b)$ iff $type^n(A, a, b) \rightarrow \varphi(x_1, x_2)$, where $\varphi$ is the formula that defines the edge relation in $\Gamma$. Without loss of generality we assume that $b > a$.

If $b$ is not in the $\sim$-equivalence class $I$ of $a$, then $H$ contains infinitely many classes between $a$ and $b$. Let $I'$ be such a class. Since $I, I' \in H$, we have that $type^{n+1}(I) = type^{n+1}(I') = \tau_1^H$. Therefore, by Lemma 3.4(2), there is $a' \in I'$ such that $type^n(I'; a') = type^n(I; a)$. Hence, by (2), $type^n(A; a, b) = type^n(A; a', b)$ and therefore there is an edge in $G$ between $f_\Gamma(a')$ and $f_\Gamma(b)$. Hence, the degree of $f_\Gamma(b)$ is infinite – contradiction.

**§5. Proof of Lemma 2.2.** Recall that a chain is scattered if there is no order preserving embedding from $Q$ into the chain. To every scattered chain we assign a pair of ordinals - the rank of the chain. The set of such pairs is ordered lexicographically. The rank of a chain refines the Hausdorff degree of the chain and is the minimal function which satisfies the following conditions:

1. $Rank(A) = (0, 0)$ if $A$ is a finite chain.
2. $Rank(A) \leq (\alpha, \beta)$ if $A = \Sigma_{i \in I} A_i$ where $I$ is isomorphic to the ordinal $\beta$ or the inverse of $\beta$ and for every $i \in I$: $Rank(A_i) \leq (\alpha_i, \beta_i)$ where $\alpha_i < \alpha$.

If $Rank(A) = (\alpha, \beta)$, then the Hausdorff degree of $A$ is $\alpha$. By a result of Hausdorff [Ro82], $A$ is a scattered chain if and only if $Rank(A)$ is well defined (i.e., there is one and only one pair of ordinals such that $Rank(A) = (\alpha, \beta)$).

Lifsches and Shelah [LS98] proved that there is no interpretation of an ordinal $\alpha \times \omega$ in a chain over $\alpha$. However, Lemma 2.2 does not follow from the Lifsches-Shelah theorem or its proof.

Now we are ready to prove Lemma 2.2. Toward a contradiction assume that there is an interpretation $\Gamma$ of $Q$ in a chain $A$ such that the domain $D$ of $\Gamma$ is a scattered subchain of $A$. Define an equivalence $\sim_\Gamma$ on $A$ as follows: For $a, b \in A$

$$a \sim_\Gamma b \text{ iff } a = b \text{ or } [\min(a, b), \max(a, b)] \cap D = \emptyset$$

Every $\sim_\Gamma$-equivalence class is an interval; it is called a $D$-class if it contains an element from $D$ (in this case it is a singleton chain); otherwise, it is called a $\neg D$-class. The $\sim_\Gamma$-equivalence classes inherit the order from $A$; we denote by $a_{\sim_\Gamma}$ the $\sim_\Gamma$-equivalence class of $a$ and by $A_{\sim_\Gamma}$ the chain of the $\sim_\Gamma$-equivalence classes. We make the following observations:

1. Between any two $\neg D$ equivalence classes there is a $D$-class.
2. If $D$ is scattered, then $A_{\sim_\Gamma}$ is scattered.

By induction on the rank of $A_{\sim_\Gamma}$ we show that $\Gamma$ is not an interpretation of $Q$ in $A$. The *basis* is trivial since in this case the domain $D$ of the interpretation is finite.

Inductive step. We consider two cases.

Case 1. Assume that $Rank(A_{\sim_\Gamma}) = (\alpha, \beta)$ where $\beta = \gamma + 1$ is a successor.

Without loss of generality we can assume that $A_{\sim_\Gamma} = \Sigma_{\delta<\beta} C_\delta = (\Sigma_{\delta<\gamma} C_\delta) + C_\gamma$, where $Rank(C_\delta) < (\alpha, \beta)$ (the case that $A_{\sim_\Gamma}$ is the sum over the inverse of an ordinal is similar). Let

$$A_\delta := \{a \in A \mid a_\sim \in C_\delta\}$$
$$Q_0 := f_\Gamma(\Sigma_{\delta<\gamma} A_\delta \cap D)$$
$$Q_1 := f_\Gamma(A_\gamma \cap D)$$

Note that $(Q_0, Q_1)$ is a partition of $Q$. Therefore either $Q_0$ or $Q_1$ contains a subset $Q'$ of $Q$ isomorphic to $Q$. W.l.o.g. assume that $Q_0$ contains such a subset. Let $A'$ be the expansion of $\Sigma_{\delta<\gamma} A_\delta$ by a unary predicate $D' := \{a \in D \mid f_\Gamma(a) \in Q'\}$. We provide an interpretation $\Gamma'$ of $Q'$ in $A'$. Define

$$\partial_{\Gamma'} := D'(x_1).$$

Let $n$ be an upper bound on the quantifier-depth of the formulas that appear in $\Gamma$. Let $K \subseteq Hin^n_2$ be such that $\tau \in K$ iff $\tau + type^n(A_\gamma) \rightarrow <_\Gamma (x_1, x_2)$ (here $<_\Gamma (x_1, x_2)$ is the formula that defines the interpretation of $<$). Hence, by 3.2, 3.8 and 3.7, for a chain $B$ and $a, b \in B$:

$$B, a, b \models \bigvee\{\tau \mid \tau \in K\} \text{ iff } B + A_\gamma, a, b \models <_\Gamma (x_1, x_2).$$

Define $<_{\Gamma'} (x_1, x_2)$ by

$$<_{\Gamma'} (x_1, x_2) := D'(x_1) \wedge D'(x_2) \wedge \bigvee\{\tau \mid \tau \in K\}.$$

Clearly, $\Gamma'$ provides an interpretation of $Q'$ in $A'$. However, the rank of $A'_{\sim_{\Gamma'}}$ is less than the rank of $A_{\sim_\Gamma}$, which is a contradiction.

CASE 2. Assume that $Rank(A_{\sim_\Gamma}) = (\alpha, \beta)$ where $\beta$ is a limit ordinal. Let $A_\delta$ for $\delta < \beta$ be defined as in the previous case. Let $n$ be an upper bound on the quantifier-depth of the formulas that appear in $\Gamma$.

Define an additive coloring on $\beta$ as follows: if $\gamma_1 < \gamma_2 < \beta$, then $col(\gamma_1, \gamma_2) := type^{n+1}(\Sigma_{\gamma_1 \leq \delta < \gamma_2} A_\delta)$.

By Theorem 3.10, there is $H = \{h_0 < h_1 < \dots\} \subseteq \beta$, cofinal and homogeneous for $col$. Let $\tau_H := col(h_0, h_1)$ and note that $\tau_H \rightarrow \exists x_1 \partial_\Gamma(x_1)$. Let $m$ be the cardinality of the set of Hintikka formulas of quantifier depth $n$ with one free variable $x_1$. We can assume that $\Sigma_{\delta < h_0} A_\delta$ contains at least $m + 1$ elements from $D$ (otherwise, take $H := \{h_{m+1}, h_{m+2} \dots\}$). Hence, there are $a_0, a_1 \in D \cap \Sigma_{\delta < h_0} A_\delta$ such that $type^n(\Sigma_{\delta < h_0} A_\delta; a_0) = type^n(\Sigma_{\delta < h_0} A_\delta; a_1) = \tau$. Therefore, for any $\gamma \geq h_0$ and $b \in A_\gamma$:

$$type^n(A; a_0, b) = type^n(\Sigma_{\delta < h_0} A_\delta; a_0) + type^n(\Sigma_{\delta \geq h_0} A_\delta; b) =$$
$$= type^n(\Sigma_{\delta < h_0} A_\delta; a_1) + type^n(\Sigma_{\delta \geq h_0} A_\delta; b) = type^n(A; a_1, b)$$

Therefore, for any $\gamma \geq h_0$ and $b \in A_\gamma \cap D$:

$$f_\Gamma(a_0) < f_\Gamma(b) \text{ iff } f_\Gamma(a_1) < f_\Gamma(b)$$

Hence, $f_\Gamma(\Sigma_{\delta < h_0} A_\delta \cap D)$ contains a set $Q' := \{q \mid \min(f_\Gamma(a_0), f_\Gamma(a_1)) < q < \max(f_\Gamma(a_0), f_\Gamma(a_1))\}$ isomorphic to $Q$. Now, exactly as in the case for a successor, we can construct an interpretation $\Gamma'$ of $Q$ in an expansion $A'$ of $\Sigma_{\delta < h_0} A_\delta$ by a unary predicate such that $Rank(A'_{\sim_{\Gamma'}}) < (\alpha, \beta)$. This leads to a contradiction.

**§6. Further Results.** We proved that there is no one-dimensional semantical interpretation of the full binary tree in any chain. In the literature other interpretations are also considered.

Let $L$ be either a first-order or monadic second-order language. $\Gamma$ is a (one-dimensional) interpretation of the $L$-theory of a structure $A$ in the monadic-theory of a structure $B$ iff there are $A_1$ and $B_1$ such that $A$ and $A_1$ have the same $L$ theory, $B$ and $B_1$ have the same monadic second-order theory, all formulas of $\Gamma$ are monadic second-order and $\Gamma$ is an interpretation of structure $A_1$ in structure $B_1$.

As a corollary of Theorem 2.3 we obtain:

THEOREM 6.1 (non-interpretability theorem). *There is no (one-dimensional) interpretation of the first-order theory of the full binary tree in the monadic second-order theory of a chain.*

PROOF. Toward a contradiction, assume that there is an interpretation of the first-order theory of the full binary tree in a chain $A$. Then there is $B$ and $A_1$ such that $B$ has the same first-order theory as the full binary tree, $A_1$ has the same monadic second-order theory as $A$, and $B$ is interpreted in $A_1$. Note that $A_1$ should be a chain. Since every element of the full binary tree is first-order definable, it follows that $B$ contains a substructure $B_2$ isomorphic to the full binary tree. Hence, from any interpretation $\Gamma$ of $B$ in $A_1$ we obtain an interpretation of the full binary tree $B_2$ in an expansion of $A_1$ by the pre-image of $B_2$ under $f_\Gamma$. This contradicts Theorem 2.3. ⊣

We have considered one-dimensional interpretations. A more general notion is that of set interpretation. A set interpretation of a structure $B$ in a structure $A$ is similar to a one-dimensional interpretation, but the first-order variables are replaced by set variables and the elements of $B$ are represented by subsets of $A$.

Let $\Sigma$ and $\Delta$ be relational signatures, $A$ a $\Delta$-structure and $B$ a $\Sigma$-structure. A semantical set interpretation $\Gamma$ of $B$ in $A$ consists of three items

1. a formula $\partial_\Gamma(X_1)$ in the monadic language for $\Delta$,
2. for each $m$-ary symbol $R$ of $\Sigma$, a formula $R_\Gamma(X_1, \dots, X_m)$ in the monadic language for $\Delta$,
3. a bijective map $f_\Gamma$ from $\partial_\Gamma(A) := \{S \subseteq A \mid A, S \models \partial_\Gamma(X_1)\}$, the set of subsets of $A$ that satisfy $\partial_\Gamma$, into the domain of $B$

such that for every $m$-ary symbol $R$ of $\Sigma$ and all $S_1, \dots, S_m \in \partial_\Gamma(A)$

$$B \models R(f_\Gamma(S_1), \dots, f_\Gamma(S_m)) \Leftrightarrow A \models R_\Gamma(S_1, \dots, S_m).$$

One-dimensional interpretations of $B$ in $A$ reduce the monadic second-order theory of $B$ to the monadic second-order theory of $A$. Set interpretations of $B$ in $A$ reduce the first-order theory of $B$ to the monadic second-order theory of $A$. Elgot and Rabin [ER66] showed a set interpretation of the full binary tree in $\omega$.

The notion of semantic interpretation is not uniform. In [LS97, LS99] several general definitions of the notion of interpretation are discussed. The next definition is probably the most appropriate for an interpretation of the monadic second-order theory of one structure in the monadic second-order theory of another structure.

Let $\Sigma$ and $\Delta$ be relational signatures, $A$ a $\Delta$-structure and $B$ a $\Sigma$-structure. A semantical many-dimensional set interpretation $\Gamma$ of $B$ in $A$ consists of the following items:

1. a positive integer $d$ (the dimension),
2. a formula $\partial_\Gamma(\overrightarrow{X}_1)$ in the monadic language for $\Delta$, where $\overrightarrow{X}_1$ is a $d$-tuple of distinct monadic variables,
3. a formula $\mathit{Subset}_\Gamma(\overrightarrow{X}_1, \overrightarrow{X}_2)$ (the subset formula) in the monadic language for $\Delta$, where $\overrightarrow{X}_1, \overrightarrow{X}_2$ are disjoint $d$-tuples of distinct monadic variables,
4. for each $m$-ary symbol $R$ of $\Sigma$, a formula $R_\Gamma(\overrightarrow{X}_1, \dots, \overrightarrow{X}_m)$ in the monadic language for $\Delta$, where $\overrightarrow{X}_1, \dots, \overrightarrow{X}_m$ are disjoint $d$-tuples of distinct monadic variables,
5. a surjective map $f_\Gamma$ from $\partial_\Gamma(A) := \{(S_1, \dots, S_d) \subseteq A^d \mid A, S_1, \dots, S_d \models \partial_\Gamma(\overrightarrow{X}_1)\}$, the set of $d$-tuples of subsets of $A$ that satisfy $\partial_\Gamma$, onto the set of subsets of the domain of $B$

such that for all $\overrightarrow{S}_1, \overrightarrow{S}_2 \in \partial_\Gamma(A)$

$$B \models f_\Gamma(\overrightarrow{S}_1) \subseteq f_\Gamma(\overrightarrow{S}_2) \Leftrightarrow A \models \mathit{Subset}_\Gamma(\overrightarrow{S}_1, \overrightarrow{S}_2),$$

and for every $m$-ary symbol $R$ of $\Sigma$ and all $\overrightarrow{S}_1, \dots, \overrightarrow{S}_m \in \partial_\Gamma(A)$

$A \models R_\Gamma(\overrightarrow{S}_1, \dots, \overrightarrow{S}_m)$ iff $f_\Gamma(\overrightarrow{S}_i) = \{a_i\}$ $(i = 1, \dots, m)$ are one element sets and $B \models R(a_1, \dots, a_m)$.

A many-dimensional set interpretation of $B$ in $A$ reduces the monadic second-order theory of $B$ to the monadic second-order theory of $A$. We believe that the following conjecture holds.

Conjecture 1. *There is no many-dimensional set interpretation of the full binary tree in a chain.*

Gurevich and Shelah [GS89] proved that if there is a many-dimensional set interpretation of a structure $B$ in a chain, then there is a one-dimensional set interpretation of $B$ in a linear order (no monadic predicates are needed).

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## REFERENCES

[Bu62] J. R. BÜCHI, *On a decision method in restricted second order arithmetic*, ***Logic, Methodology and Philosophy of Science***, Stanford University Press, 1962, pp. 1–11.

[BS73] J. R. BÜCHI and D. SIEFKES, ***The monadic second order theory of all countable ordinals***, Lecture Notes in Mathematics, vol. 328, Springer, 1973.

[ER66] C. ELGOT and M. O. RABIN, *Decidability and Undecidability of Extensions of Second (First) Order Theory of (Generalized) Successor*, this JOURNAL, vol. 31 (1966), no. 2, pp. 169–181.

[FV59] S. FEFERMAN and R. L. VAUGHT, *The first order properties of products of algebraic systems*, ***Fundamenta Mathematicae***, vol. 47 (1959), pp. 57–103.

[GMS83] Y. GUREVICH, M. MAGIDOR, and S. SHELAH, *The monadic theory of* $\omega_2$, this JOURNAL, vol. 48 (1983), pp. 387–398.

[GS89] Y. GUREVICH and S. SHELAH, *On the strength of the interpretation method*, this JOURNAL, vol. 54 (1989), pp. 305–323.

[LS97] S. LIFSCHES and S. SHELAH, *Peano arithmetic may not be interpretable in the monadic theory of linear orders*, this JOURNAL, vol. 62 (1997), pp. 848–872.

[LS98] ———, *Uniformization and Skolem functions in the class of trees*, this JOURNAL, vol. 63 (1998), pp. 103–127.

[LS99] ———, *Random graphs in the monadic theory of order*, ***Archive for Mathematical Logic***, vol. 38 (1999), pp. 273–312.

[Rab69] M. O. RABIN, *Decidability of second-order theories and automata on infinite trees.*, ***Transactions of the American Mathematical Society***, vol. 141 (1969), pp. 1–35.

[Ro82] J. G. ROSENSTEIN, ***Linear orderings***, Academic Press Inc., New York, 1982.

[Sh75] S. SHELAH, *The monadic theory of order*, ***Annals of Mathematics. Second Series***, vol. 102 (1975), pp. 379–419.

THE BLAVATNIK SCHOOL OF COMPUTER SCIENCE
TEL AVIV UNIVERSITY
TEL AVIV, ISRAEL 69978
*E-mail*: rabinoa@post.tau.ac.il