GAMES - BASIC NOTIONS

Alexander Rabinovich

Department of Computer Science
Tel-Aviv University

Plan

1. Topological Games.
2. Determinacy for Open games
3. Martin Theorem.
4. Games on Graphs
5. Fundamental algorithmic questions.

Topological GAMES

A - alphabet (might be infinite).
Game $G(X)$ is given by $X \subseteq A^{\omega}$ - a set of ω strings over A.
There are two Players which play ω rounds:
Round i:

- Player I chooses $a_{2i} \in$ A.
- Player II chooses $a_{2i+1} \in$ A.

A play - $x = a_0 a_1 a_2 \ldots$.
Winning conditions: Player I wins a play if $x \in X$, otherwise Player II wins x.

Strategy

A position is a finite word $u \in A^*$.

If $|u|$ is even then u is a position of Player I

If $|u|$ is odd then u is a position of Player II

A strategy for Player I is a function $f : (A^2)^* \to A$.
Player I follows a strategy f in a play
$x = a_0 a_1 \ldots a_{2i} a_{2i+1} \ldots$ iff

$$a_{2i} = f(a_0, \ldots a_{2i-1})$$

f is a winning strategy for Player I in the game $G(X)$ iff every play x which follows f is in X.

Strategy

A position is a finite word $u \in A^*$.

If $|u|$ is even then u is a position of Player I

If $|u|$ is odd then u is a position of Player II

A strategy for Player I is a function $f : (A^2)^* \to A$.
Player I follows a strategy f in a play
$x = a_0 a_1 \ldots a_{2i} a_{2i+1} \ldots$ iff

$$a_{2i} = f(a_0, \ldots a_{2i-1})$$

f is a winning strategy for Player I in the game $G(X)$ iff every play x which follows f is in X.

Strategies and winning strategies for Player II are defined similarly.

EXAMPLE

$A = \{a, b\}$ and X is the set of ω-strings with infinitely many b.
A strategy f_1 for Player I: always choose b.
f_1 is a winning strategy.

EXAMPLE

$A = \{a, b\}$ and X is the set of ω-strings with infinitely many b.
A strategy f_1 for Player I: always choose b.
f_1 is a winning strategy.

A strategy f_2 for Player I: if in the last round Player II choice was b, then choose a; otherwise choose b.

f_2 is also a winning strategy for Player I.

Determinacy

Lemma In $G(X)$ at most one of the Players has a winning strategy.

Proof. If f and g are strategies of Player I and Player II then there is a unique play x which follows these strategies. Hence, it is impossible that both f and g are winning.

Determinacy

Lemma In $G(X)$ at most one of the Players has a winning strategy.
Proof. If f and g are strategies of Player I and Player II then there is a unique play x which follows these strategies. Hence, it is impossible that both f and g are winning.

Def. $G(X)$ is determinate if one of the players has a winning strategy.

Determinacy

Lemma In $G(X)$ at most one of the Players has a winning strategy.
Proof. If f and g are strategies of Player I and Player II then there is a unique play x which follows these strategies. Hence, it is impossible that both f and g are winning.

Def. $G(X)$ is determinate if one of the players has a winning strategy.

Central issue in descriptive set theory: Characterise determined games by topological properties of the winning conditions.

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.

Open Games are determinate

X is open if $X = UA^{\omega}$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game. Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u \ : \ $ Player I does not have a winning strategy in $G_u(X)\}$.

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game. Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u : \text{ Player I does not have a winning strategy in } G_u(X)\}$.
We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.

Open Games are determinate

X is open if $X = UA^{\omega}$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game. Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u\ :\ $ Player I does not have a winning strategy in $G_u(X)\}$.
We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.
(1) $A \subset P$ and (2) $\forall u \in P \exists a \forall b (uab \in P)$.

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game.
Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u\ :\ \text{Player I does not have a winning strategy in } G_u(X)\}$.
We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.
(1) $A \subset P$ and (2) $\forall u \in P \exists a \forall b (uab \in P)$.
Player II strategy $f_2 :\ P \to A$ - choose a as in (2).

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).

Theorem If X is open then $G(X)$ is determinate.

Proof I.

For a finite string of odd length $G_u(X)$ is a residual game. Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.

Let $P = \{u \ : \ $ Player I does not have a winning strategy in $G_u(X)\}$.

We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.

(1) $A \subset P$ and (2) $\forall u \in P \exists a \forall b (uab \in P)$.

Player II strategy $f_2 : P \to A$ - choose a as in (2).

We claim that f_2 is a winning strategy.

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game.
Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u : \text{ Player I does not have a winning strategy in } G_u(X)\}$.
We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.
(1) $A \subset P$ and (2) $\forall u \in P \exists a \forall b (uab \in P)$.
Player II strategy $f_2 : P \to A$ - choose a as in (2).
We claim that f_2 is a winning strategy.
If x is consistent with f_2 we are always in P.

Open Games are determinate

X is open if $X = UA^\omega$ (U a set of finite strings).
Theorem If X is open then $G(X)$ is determinate.
Proof I.
For a finite string of odd length $G_u(X)$ is a residual game.
Player II moves fist,then Player I, etc.; a play x wining for Player I in $G_u(X)$ if $ux \in X$.
Let $P = \{u$: Player I does not have a winning strategy in $G_u(X)\}$.
We assume that I does not have a winning strategy in $G(X)$ and show that II has a winning strategy.
(1) $A \subset P$ and (2) $\forall u \in P \exists a \forall b (uab \in P)$.
Player II strategy $f_2 : P \to A$ - choose a as in (2).
We claim that f_2 is a winning strategy.
If x is consistent with f_2 we are always in P.
x has no prefix in U.

Second Proof

$X = UA^{\omega}$.

Second Proof

$X = UA^\omega$.

Define $W_0 := U$.

$$W_{i+1} := W_i \cup \{u \in (A^2)^* \ : \ ua \in W_i \text{ for some } a\}$$
$$\cup\{u \in A(A^2)^* : \ ua \in W_i \text{ for all } a\}$$

Second Proof

$X = UA^\omega$.

Define $W_0 := U$.

$$W_{i+1} := W_i \cup \{u \in (A^2)^* \ : \ ua \in W_i \text{ for some a}\}$$
$$\cup \{u \in A(A^2)^* : \ ua \in W_i \text{ for all a}\}$$

Define rank : $A^* \to \mathrm{Nat} \cup \{\infty\}$ as:
rank$(u) := \min\{i \ : \ u \in W_i\}$.

Second Proof

$X = UA^\omega$.

Define $W_0 := U$.

$$W_{i+1} := W_i \cup \{u \in (A^2)^* \ : \ ua \in W_i \text{ for some a}\}$$
$$\cup \{u \in A(A^2)^* : \ ua \in W_i \text{ for all a}\}$$

Define rank : $A^* \to \mathrm{Nat} \cup \{\infty\}$ as:
$\text{rank}(u) := \min\{i \ : \ u \in W_i\}$.

Player I has a winning strategy in $G_u(X)$ iff $\text{rank}(u)$ is finite.

The strategy - decrease the rank.

Borel Sets

F and G are the family of open and closed sets respectively.

Borel Sets

$\mathcal{F}$ and $\mathcal{G}$ are the family of open and closed sets respectively.
$\mathcal{F}_\sigma$ - the countable unions of closed sets.
$\mathcal{G}_\delta$ - the complements of sets from $\mathcal{F}_\sigma$ - the countable intersection of open sets.
$\mathcal{F}_{\sigma\delta}$ the countable intersections of $\mathcal{F}_\sigma$
$\mathcal{G}_{\delta\sigma}$ the complements of sets from $\mathcal{F}_{\sigma\delta}$.

Borel Sets

$\mathcal{F}$ and $\mathcal{G}$ are the family of open and closed sets respectively.
$\mathcal{F}_\sigma$ - the countable unions of closed sets.
$\mathcal{G}_\delta$ - the complements of sets from $\mathcal{F}_\sigma$ - the countable intersection of open sets.
$\mathcal{F}_{\sigma\delta}$ the countable intersections of $\mathcal{F}_\sigma$
$\mathcal{G}_{\delta\sigma}$ the complements of sets from $\mathcal{F}_{\sigma\delta}$.

Etc., for every $s \in (\sigma\delta)^*$, we define $\mathcal{F}_s$ and their complements - Borel sets of finite ranks.

Borel Sets

F and G are the family of open and closed sets respectively.
F_σ - the countable unions of closed sets.
G_δ - the complements of sets from F_σ - the countable intersection of open sets.
$F_{\sigma\delta}$ the countable intersections of F_σ
$G_{\delta\sigma}$ the complements of sets from $F_{\sigma\delta}$.

Etc., for every $s \in (\sigma\delta)^*$, we define F_s and their complements - Borel sets of finite ranks.

Def. The class of Borel sets is the smallest class of sets containing open sets and closed under countable unions and countable intersections.

Almost all sets in WORKING Math. are Borel.

Martin Theorem

Martin Determinacy Theorem If X is Borel then $G(X)$ is determinate.

Martin Theorem

Martin Determinacy Theorem If X is Borel then $G(X)$ is determinate.

Theorem There is X such that $G(X)$ is not determinate. Proof relies on Axiom of Choice.

Games on Graphs

Games on Graphs - Slightly more general than Top. games. But they are “reducible" to Topological games.

Games on Graphs

Games on Graphs - Slightly more general than Top. games. But they are “reducible" to Topological games.

Def. Arena - A directed graph G=(V,E) and a partition of V into V_1 and V_2 with technical requirements:

1. Bipartite graph - All edges between V_1 and V_2.
2. Out degree of every node is at least one.

Games on Graphs

Games on Graphs - Slightly more general than Top. games. But they are “reducible" to Topological games.

Def. Arena - A directed graph G=(V,E) and a partition of V into V_1 and V_2 with technical requirements:

1. Bipartite graph - All edges between V_1 and V_2.
2. Out degree of every node is at least one.

A graph game is an arena and a set X of infinite paths.

Games on Graphs

Games on Graphs - Slightly more general than Top. games. But they are “reducible" to Topological games.

Def. Arena - A directed graph G=(V,E) and a partition of V into V_1 and V_2 with technical requirements:

1. Bipartite graph - All edges between V_1 and V_2.
2. Out degree of every node is at least one.

A graph game is an arena and a set X of infinite paths.
A play starting from a node v.
Round 0: the owner of the vertex $v = v_0$ chooses an adjacent node v_1. Then the other player chooses a node v_2 adjacent to v_1.

Games on Graphs

Games on Graphs - Slightly more general than Top. games. But they are “reducible" to Topological games.

Def. Arena - A directed graph G=(V,E) and a partition of V into V_1 and V_2 with technical requirements:

1. Bipartite graph - All edges between V_1 and V_2.
2. Out degree of every node is at least one.

A graph game is an arena and a set X of infinite paths.
A play starting from a node v.
Round 0: the owner of the vertex $v = v_0$ chooses an adjacent node v_1. Then the other player chooses a node v_2 adjacent to v_1.
Round i: the owner of the vertex $v = v_{2i}$ chooses an adjacent node v_{2i+1}. Then the other player chooses a node $v_{2(i+1)}$ adjacent to v_{21+1}.

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I $\ldots$

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I $\ldots$
A winning strategy for player I - $\ldots$.

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I . . .
A winning strategy for player I - . . . .

Reduction between game graphs and topological games.

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I ...
A winning strategy for player I - ....

Reduction between game graphs and topological games.
Examples Arena The compete bipartite graph on

$\{v_1, v_3\}$, $\{v_2\}$
Winning set X - the path passing v_3 infinitely often

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \dots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I $\dots$
A winning strategy for player I - $\dots$.

Reduction between game graphs and topological games.
Examples Arena The compete bipartite graph on

$\{v_1, v_3\}, \ \{v_2\}$
Winning set X - the path passing v_3 infinitely often

Who has a winning strategy?

Games on Graphs- Cont.

Winning conditions Player I wins a play $x = v_0 v_1 \ldots$ if $x \in X$; otherwise Player II wins.

A strategy for Player I $\ldots$
A winning strategy for player I - $\ldots$.

Reduction between game graphs and topological games.
Examples Arena The compete bipartite graph on

$\{v_1, v_3\}, \ \{v_2\}$
Winning set X - the path passing v_3 infinitely often

Who has a winning strategy?

X paths which pass infinitely often in v_1 and in v_3 Who has a winning strategy?

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Computability Question: Compute a description of a winning strategy.

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Computability Question: Compute a description of a winning strategy.

a finite representation.e.g., the arena is finite graph and X is given by a Muller condition.

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Computability Question: Compute a description of a winning strategy.

a finite representation.e.g., the arena is finite graph and X is given by a Muller condition.

a finite description of a winning strategy. Recursive function? Finite memory function, memoryless strategy.

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Computability Question: Compute a description of a winning strategy.

a finite representation.e.g., the arena is finite graph and X is given by a Muller condition.

a finite description of a winning strategy. Recursive function? Finite memory function, memoryless strategy.

Fundamental Algorithmic Questions

Input: a finite representation of a game.

Decidability Question: Who has a winning strategy?

Computability Question: Compute a description of a winning strategy.

a finite representation.e.g., the arena is finite graph and X is given by a Muller condition.

a finite description of a winning strategy. Recursive function? Finite memory function, memoryless strategy.