On the Construction of Analytic Sequent Calculi for Sub-classical Logics

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On the Construction of Analytic Sequent Calculi for **Sub-classical Logics**

- A propositional logic is called sub-classical if:
  - Its language is contained in the language of classical logic.
  - It is weaker than classical logic.
- A classical rule is considered too strong, and is replaced by weaker rules.
- Examples:
  - Intuitionistic logic
  - Relevance logics
  - Many-valued logics
  - Paraconsistent logics
- Our goal: Construct effective proof systems for sub-classical logics.

On the Construction of Analytic **Sequent Calculi** for Sub-classical Logics

- Sequent calculi are a prominent proof-theoretic framework, suitable for a variety of logics.
- *Sequents* are objects of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas.

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m \quad \rightsquigarrow \quad A_1 \wedge \ldots \wedge A_n \supset B_1 \vee \ldots \vee B_m$$

- Special instance: $\Gamma \Rightarrow A$ (Δ has one element)

- *Pure sequent calculi* are propositional sequent calculi that include all usual structural rules, and a finite set of pure logical rules.
- *Pure logical rules* are logical rules that allow any context [Avron '91].

$$\frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \text{but not} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

On the Construction of Analytic **Sequent Calculi** for Sub-classical Logics

The Propositional Fragment of **LK** [Gentzen 1934]

Structural Rules:

$$(id)\quad \frac{}{\Gamma, A \Rightarrow A, \Delta} \qquad (cut)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta}$$

$$(W \Rightarrow)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\Rightarrow W)\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

Logical Rules:

$$(\neg \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad (\Rightarrow \neg)\quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$$

$$(\wedge \Rightarrow)\quad \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee \Rightarrow)\quad \frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\Rightarrow \vee)\quad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\quad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$$

On the Construction of **Analytic** Sequent Calculi for Sub-classical Logics

Definition

A calculus is *analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only subformulas of $\Gamma \cup \Delta$.

- If a pure calculus is analytic then it is decidable.
- Proof search can be focused on a finite space of proofs.
- **LK** is analytic (traditionally follows from *cut*-elimination).

- Sequent Calculi provide a natural way to define many sub-classical logics:
  - Begin with **LK**.
  - Discard some of its (logical) rules.
  - Add other (logical) rules, that are derivable in **LK**.

What general conditions guarantee the analyticity of the obtained calculus?

On the **Construction** of Analytic Sequent Calculi for Sub-classical Logics

- Consider the following applications of $\dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta}$:

$$\frac{A \Rightarrow A}{\Rightarrow A \supset A} \qquad \frac{A, A \wedge B \Rightarrow A, B}{A \Rightarrow (A \wedge B) \supset A, B} \qquad \frac{B \vee C, A \Rightarrow B}{B \vee C \Rightarrow A \supset B}$$

- These applications constitute new (weaker) rules:

$$\frac{\Gamma, A \Rightarrow A, \Delta}{\Gamma \Rightarrow A \supset A, \Delta} \qquad \frac{\Gamma, A, A \wedge B \Rightarrow A, B, \Delta}{\Gamma, A \Rightarrow (A \wedge B) \supset A, B, \Delta} \qquad \frac{\Gamma, B \vee C, A \Rightarrow B, \Delta}{\Gamma, B \vee C \Rightarrow A \supset B, \Delta}$$

Definition (Safe Application)

An application of an **LK** rule is *safe* if all its context formulas are subformulas of the principal formula.

Theorem

A calculus whose rules are all safe applications of **LK**-rules is analytic.

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Analytic-by-construction Calculi: Examples

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$$(\wedge \Rightarrow)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

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Every rule is a trivial safe application of itself.

Analytic-by-construction Calculi: Examples

The Atomic Paraconsistent Logic P_1 [Sette '73, Avron '14]

$$(\neg \Rightarrow) \; \cancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}}$$

$$\frac{\Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A \wedge B, \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A \vee B, \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A \supset B, \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta}$$

- Paraconsistency applies only in the atomic level.
- $\not\vdash_{P_1} p, \neg p \Rightarrow \varphi$.
- $\vdash_{P_1} \psi, \neg\psi \Rightarrow \varphi$ whenever ψ is compund.

Analytic-by-construction Calculi: Examples

Calculus for Primal Infon Logic [Gurevich,Neeman '09]

$$(\wedge \Rightarrow)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee \Rightarrow)\ \xcancel{\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\ \frac{\Gamma, B, A \Rightarrow B, \Delta}{\Gamma, B \Rightarrow A \supset B, \Delta}$$

- An extremely efficient propositional logic.
- One of the main logical engines behind DKAL (Distributed Knowledge Authorization Language).
- Provides a balance between expressivity and efficiency.

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Analytic-by-construction Calculi: Examples

Extended Primal Infon Logic

$$(\wedge \Rightarrow)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee \Rightarrow)\ \cancel{\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\supset \Rightarrow)\ \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} \qquad (\Rightarrow \supset)\ \frac{}{\Gamma, B \Rightarrow A \supset B, \Delta}$$

$$\frac{}{\Gamma \Rightarrow A \supset A, \Delta} \quad \frac{}{\Gamma \Rightarrow B \supset (A \supset B), \Delta} \quad \frac{}{\Gamma \Rightarrow (A \wedge B) \supset A, \Delta} \quad \frac{}{\Gamma \Rightarrow (A \wedge B) \supset B, \Delta}$$

$$\frac{}{\Gamma, A \vee A \Rightarrow A, \Delta} \quad \frac{}{\Gamma, A \vee (A \wedge B) \Rightarrow A, \Delta} \quad \frac{}{\Gamma, (A \wedge B) \vee A \Rightarrow A, \Delta}$$

$$\frac{}{\Gamma, \bot \Rightarrow \Delta} \quad \frac{}{\Gamma \Rightarrow \bot \supset A, \Delta} \quad \frac{}{\Gamma, \bot \vee A \Rightarrow A, \Delta} \quad \frac{}{\Gamma, A \vee \bot \Rightarrow A, \Delta}$$

Analytic No cut-elimination

Semantics for Pure Calculi

- Pure calculi correspond to *two-valued valuations* [Béziau '01].
- Each pure rule is read as a semantic condition.
- **G**-legal valuations: satisfy all semantic conditions.

Example

$$\frac{A \Rightarrow}{\Rightarrow \neg A} \qquad \frac{A \Rightarrow}{\neg\neg A \Rightarrow} \qquad \frac{\Rightarrow A \quad \Rightarrow \neg A}{\neg(A \wedge \neg A) \Rightarrow} \qquad \frac{\neg A \Rightarrow \quad \neg B \Rightarrow}{\neg(A \wedge B) \Rightarrow}$$

Corresponding semantic conditions:

1. If $v(A) = \mathrm{F}$ then $v(\neg A) = \mathrm{T}$
2. If $v(A) = \mathrm{F}$ then $v(\neg\neg A) = \mathrm{F}$
3. If $v(A) = \mathrm{T}$ and $v(\neg A) = \mathrm{T}$ then $v(\neg(A \wedge \neg A)) = \mathrm{F}$
4. If $v(\neg A) = \mathrm{F}$ and $v(\neg B) = \mathrm{F}$ then $v(\neg(A \wedge B)) = \mathrm{F}$

This semantics is non-deterministic.

Soundness and Completeness

Theorem

The sequent $\Gamma \Rightarrow \Delta$ is provable in **G** iff every **G**-legal valuation is a model of $\Gamma \Rightarrow \Delta$.

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The sequent $\Gamma \Rightarrow \Delta$ is provable in **G** using only formulas of $\mathcal{F}$ iff every **G**-legal valuation whose domain is $\mathcal{F}$ is a model of $\Gamma \Rightarrow \Delta$.

Soundness and Completeness

Theorem

The sequent $\Gamma \Rightarrow \Delta$ is provable in **G** using only formulas of $\mathcal{F}$ iff every **G**-legal valuation whose domain is $\mathcal{F}$ is a model of $\Gamma \Rightarrow \Delta$.

Definition

G is semantically analytic if every **G**-legal **partial** valuation whose domain is closed under subformulas can be extended to a **full G**-legal valuation.

Example

Consider the rules $\frac{\Rightarrow A}{\neg A \Rightarrow}$ and $\frac{\Rightarrow A}{\Rightarrow \neg A}$.

The partial valuation $\lambda x \in \{p\}.\mathrm{T}$ cannot be extended.

Theorem

A calculus is analytic iff it is semantically analytic.

Extending Partial Valuations

- Classical logic enjoys a simple extension method: enumeration + step-by-step extension
- Does this work for other logics?

Example

☺ $$\frac{A \Rightarrow A}{\Rightarrow A \supset A}$$ ☺ $$\frac{A, A \wedge B \Rightarrow A, B}{A \Rightarrow (A \wedge B) \supset A, B}$$

☹ $$\frac{\Rightarrow A}{\Rightarrow \neg A}$$ ☹ $$\frac{B \vee C, A \Rightarrow B}{B \vee C \Rightarrow A \supset B}$$

The classical extension method works for calculi that consist of safe applications of rules of **LK**.

Liberal Analyticity

Definition (k-subformulas)

- A is a k-subformula of $\neg A$.
- $\neg^k A_i$ is a k-subformula of $A_1 \diamond A_2$.

Example

$\neg\neg A$ is a 2-subformula of $A \wedge B$.

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Example

$\neg\neg A$ is a 2-subformula of $A \wedge B$.

Definition (k-analyticity)

A calculus is *k-analytic* if $\vdash \Gamma \Rightarrow \Delta$ implies that there is a derivation of $\Gamma \Rightarrow \Delta$ using only k-subformulas of $\Gamma \cup \Delta$.

k-safe applications

$$\frac{A, A \wedge B \Rightarrow A, B}{A \Rightarrow (A \wedge B) \supset A, B} \qquad \frac{\neg\neg A, A \wedge B \Rightarrow A, \neg B}{\neg\neg A \Rightarrow (A \wedge B) \supset A, \neg B}$$

Theorem

A calculus whose rules are k-safe applications of **LK**-rules is k-analytic.

Example: A 1-analytic Pure Calculus for da Costa's Paraconsistent Logic $\mathbf{C}_1$ [Avron, Konikowska, Zamansky '12]

$$\cancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow \neg A, \Delta}{\Gamma, \neg(A \wedge \neg A) \Rightarrow \Delta} \qquad \color{red}{\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta}}$$

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$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta} \qquad \frac{\Gamma, A, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \supset B) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow A \wedge B, \color{blue}{\neg A, \neg B}, \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \color{blue}{\neg A, \neg B}, \Delta}$$

Why **LK**?

What basic properties of the rules of **LK** were used?

- The conclusion has the form $\Gamma \Rightarrow A, \Delta$ or $\Gamma, A \Rightarrow \Delta$
- The rest of the formulas in the rule are *k*-subformulas of A
- Right and left rules "play well" together:

  For any two contextless applications of the form

  $$\frac{s_1 \quad \ldots \quad s_n}{\Rightarrow A} \qquad \frac{s'_1 \quad \ldots \quad s'_m}{A \Rightarrow}$$

  we have $\quad s_1, \ldots, s_n, s'_1, \ldots, s'_m \vdash^{(cut)} \Rightarrow$

  Generalizes *coherence* (Avron, Lev '01,'05).

- Every such calculus has a valuation extension method.

Corollary

Every calculus that admits the basic properties is $k-$analytic.

A Sequent Calculus for First-Degree Entailment [Anderson,Belnap 75']

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Every calculus that admits the basic properties is $k-$analytic.

$$\cancel{\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}} \quad \cancel{\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \neg\neg A \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \neg\neg A, \Delta}$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta \quad \Gamma, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \wedge B) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \neg A, \neg B, \Delta}{\Gamma \Rightarrow \neg(A \wedge B), \Delta}$$

$$\frac{\Gamma, \neg A, \neg B \Rightarrow \Delta}{\Gamma, \neg(A \vee B) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \neg A, \Delta \quad \Gamma \Rightarrow \neg B, \Delta}{\Gamma \Rightarrow \neg(A \vee B), \Delta}$$

- Each conclusion has the form $\Rightarrow A$ or $A \Rightarrow$.
- All other formulas are 1-subformulas of A.
- The rules "play well" together.

Therefore, this calculus is 1-analytic.

Conclusions and Further Work

- We provided a general sufficient condition for analyticity in pure calculi.
- Useful for:
  - Verifying analyticity
  - Introducing new analytic calculi
  - Augmenting analytic calculi with more useful rules
- Further work:
  - Cut-elimination
  - Non-pure calculi (context restrictions)
  - First order logics

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Thank you!