Automated Support for the Investigation of Paraconsistent and Other Logics

Agata Ciabattoni

Vienna University of Technology

joint work with O. Lahav, L. Spendier and A. Zamansky

Non-classical logics

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Their usefulness depend on the existence of:

(i) an **analytic calculus**, i.e. where proof search proceeds by step-wise decomposition of the formulas to be proved

(ii) an **intuitive semantics** that provides insight into the logic

State of the art

Finding an analytic calculus and useful semantics for a logic

- Ad hoc procedures

  $\Rightarrow$ (Too?) Many papers are written on similar results

  $\Rightarrow$ Error-prone task (e.g. cut-elimination)

  $\Rightarrow$ Many open problems

State of the art

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General and automated procedures are desirable!

Paraconsistent logics

"Inconsistency-tolerant" non-classical logics.

- Within classical logic, contradictions entail everything

$$A, \neg A \vdash B$$

- *Paraconsistent logics* allow "contradictory" but non-trivial theories.
- Many applications in Computer Science: integration of information from multiple sources, negotiations among agents with conflicting goals, ...

Examples: C-systems

Internalize the concepts of consistency inside the object language via the operator $\circ$ with intuitive meaning $\circ A$: "A is consistent".

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Positive fragment of classical propositional logic CL^+ extended with suitable combinations of

$(\mathbf{n_1})$	$\psi \vee \neg\psi$	$(\mathbf{n_2})$	$\psi \supset (\neg\psi \supset \varphi)$
$(\mathbf{c})$	$\neg\neg\psi \supset \psi$	$(\mathbf{e})$	$\psi \supset \neg\neg\psi$
$(\mathbf{n}^{\mathbf{l}}_{\wedge})$	$\neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)$	$(\mathbf{n}^{\mathbf{r}}_{\wedge})$	$(\neg\psi \vee \neg\varphi) \supset \neg(\psi \wedge \varphi)$
$(\mathbf{n}^{\mathbf{l}}_{\vee})$	$\neg(\psi \vee \varphi) \supset (\neg\psi \wedge \neg\varphi)$	$(\mathbf{n}^{\mathbf{r}}_{\vee})$	$(\neg\psi \wedge \neg\varphi) \supset \neg(\psi \vee \varphi)$
$(\mathbf{n}^{\mathbf{l}}_{\supset})$	$\neg(\psi \supset \varphi) \supset (\psi \wedge \neg\varphi)$	$(\mathbf{n}^{\mathbf{r}}_{\supset})$	$(\psi \wedge \neg\varphi) \supset \neg(\psi \supset \varphi)$
$(\mathbf{b})$	$\psi \supset (\neg\psi \supset (\circ\psi \supset \varphi))$	$(\mathbf{r}_{\diamond})$	$\circ(\psi \diamond \varphi) \supset (\circ\psi \vee \circ\varphi)$
$(\mathbf{k})$	$\circ\psi \vee (\psi \wedge \neg\psi)$	$(\mathbf{i})$	$\neg\circ\psi \supset (\psi \wedge \neg\psi)$
$(\mathbf{o}^{\mathbf{1}}_{\diamond})$	$\circ\psi \supset \circ(\psi \diamond \varphi)$	$(\mathbf{o}^{\mathbf{2}}_{\diamond})$	$\circ\varphi \supset \circ(\psi \diamond \varphi)$
$(\mathbf{a}_{\diamond})$	$(\circ\psi \wedge \circ\varphi) \supset \circ(\psi \diamond \varphi)$	$(\mathbf{a}_{\neg})$	$\circ\psi \supset \circ\neg\psi$

with $\diamond = \wedge, \vee, \supset$

Motivating work

First modular approach for C-systems (A. Avron, B. Konikowska, and A. Zamansky, LICS 2012)

Starting from a Hilbert calculus for a C-system:

(Step 1) Define a *suitable semantics*
$\Rightarrow$ needs ingenuity!

(Step 2) Use the obtained semantics to define a *sequent calculus*

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Can we do more and can we do it in an automated way?

Our results

We identified a formal grammar generating (infinitely many) axioms in the language of CL^+ with *new unary connectives*.
Example

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$(\mathbf{a}_{\diamond})$	$(\circ\psi \wedge \circ\varphi) \supset \circ(\psi \diamond \varphi)$	$(\mathbf{a}_{\neg})$	$\circ\psi \supset \circ\neg\psi$

. . . and (infinitely) many more axioms . . .

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(Step 1) to extract a corresponding **sequent calculus**

(Step 2) use the calculus to define a **suitable semantics** for the logic, the semantics is used:
- ⇒ to show the decidability of the logic
- ⇒ to check the analyticity of the calculus

Step 1: From axioms to logical rules

Example

Axiom $(n^l_\wedge)$	$\neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)$
Invertibility	$\dfrac{}{\Rightarrow \neg(\psi \wedge \varphi) \supset (\neg\psi \vee \neg\varphi)}$
Invertibility	$\dfrac{}{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi \vee \neg\varphi}$
Ackermann Lemma	$\dfrac{}{\neg(\psi \wedge \varphi) \Rightarrow \neg\psi, \neg\varphi}$
Equivalent logical rule	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \qquad \Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \varphi) \Rightarrow \Delta}$

The generated rules

Type 1: $\dfrac{\mathcal{Q}}{\Gamma, \star(\psi \diamond \varphi) \Rightarrow \Delta}$ $\dfrac{\mathcal{Q}}{\Gamma \Rightarrow \star(\psi \diamond \varphi), \Delta}$

Type 2: $\dfrac{\mathcal{P}}{\Gamma, \star\star\psi \Rightarrow \Delta}$ $\dfrac{\mathcal{P}}{\Gamma \Rightarrow \star\star\psi, \Delta}$

Type 3: $\dfrac{\mathcal{P}}{\Gamma, \star\psi \Rightarrow \Delta}$ $\dfrac{\mathcal{P}}{\Gamma \Rightarrow \star\psi, \Delta}$

- Premises $\mathcal{Q}$ contain (a subset of) $\{\psi, \star\psi, \varphi, \star\varphi\}$
- Premises $\mathcal{P}$ contain (a subset of) $\{\psi, \star\psi\}$

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Adding *logical* rules to LK^+: cut-free calculus?

Useful semantics

Definition

A partial non-deterministic matrix (PNmatrix) $\mathcal{M}$ consists of:

(i) a set $\mathcal{V}_{\mathcal{M}}$ of truth values,

(ii) a subset of $\mathcal{V}_{\mathcal{M}}$ of designated truth values, and

(iii) a truth-table $\diamond_{\mathcal{M}} : \mathcal{V}_{\mathcal{M}}^n \to P(\mathcal{V}_{\mathcal{M}})$ for every n-ary connective $\diamond$.

PNmatrices generalise the notion of non-deterministic matrices (A. Avron, 2001) by allowing *empty sets* in the truth tables.

Why non-determinism?

Standard rules for classical negation and conjunction:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$$

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

	$\neg$
1	**0**
0	**1**

		$\wedge$
1	**1**	**1**
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

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	$\neg$
1	**0**
0	**???**

		$\wedge$
1	**1**	???
1	**0**	**0**
0	**1**	**0**
0	**0**	**0**

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$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta} \qquad \frac{\qquad}{\qquad}$$

$$\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad \frac{\qquad}{\qquad}$$

	$\neg$
1	$\{\mathbf{0}\}$
0	$\{\mathbf{1},\mathbf{0}\}$

		$\wedge$
1	**1**	$\{\mathbf{1},\mathbf{0}\}$
1	**0**	$\{\mathbf{0}\}$
0	**1**	$\{\mathbf{0}\}$
0	**0**	$\{\mathbf{0}\}$

Step 2: Extracting PNmatrices

- Truth values $\mathcal{V}_{\mathcal{M}}$: tuples of size $=$ $\#$ of unary connectives $+1$

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- Truth values $\mathcal{V}_{\mathcal{M}}$: tuples of size $=$ $\#$ of unary connectives $+1$
- New rules reduce the level of non-determinism

Type 3: $\dfrac{\mathcal{P}}{\Gamma, \star\psi \Rightarrow \Delta}$ reduce the set of truth values $\mathcal{V}_{\mathcal{M}}$

Type 2: $\dfrac{\mathcal{P}}{\Gamma, \star_i \star_j \psi \Rightarrow \Delta}$ determine truth tables for $\star_j$

Type 1: $\dfrac{\mathcal{Q}}{\Gamma, \star(\psi \diamond \varphi) \Rightarrow \Delta}$ determine truth tables for $\diamond$

Example: Extracting a PNmatrix I

CL^+ with one new unary connective $\neg$:

$$\mathcal{V}_\mathcal{M} := \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

- Rules introducing unary connectives reduce $\mathcal{V}_\mathcal{M}$, e.g.:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\psi, \Delta}$$

$$\mathcal{V}_\mathcal{M} := \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

Example: Extracting a PNmatrix II

CL^+ with one new unary connective $\neg$:

$$\mathcal{V}_{\mathcal{M}} := \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

- Rules introducing unary connectives reduce $\mathcal{V}_{\mathcal{M}}$, e.g.:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\psi, \Delta}$$

$$\mathcal{V}_{\mathcal{M}} := \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

Example: Extracting a PNmatrix III

$$\mathcal{V}_{\mathcal{M}} := \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

- Rules introducing formulas of the form $\neg\neg\psi$ determine the truth table for $\neg$, e.g.:

$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta}$$

$\neg$	
$\langle 0,1\rangle$	$\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$
$\langle 1,0\rangle$	$\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$
$\langle 1,1\rangle$	$\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$

Example: Extracting a PNmatrix IV

$$\mathcal{V}_{\mathcal{M}} := \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$$

- Rules introducing formulas of the form $\neg\neg\psi$ determine the truth table for $\neg$, e.g.:

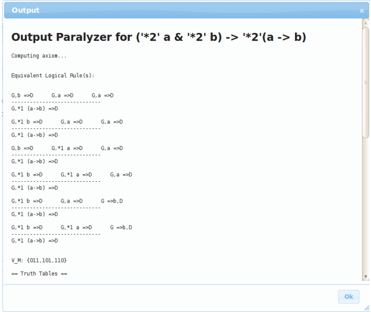
$$\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta}$$

$\neg$	
$\langle 0,1\rangle$	$\{\langle 1,0\rangle\}$
$\langle 1,0\rangle$	$\{\langle 0,1\rangle\}$
$\langle 1,1\rangle$	$\{\langle 1,0\rangle, \langle 1,1\rangle\}$

Paralyzer (PARAconsistent logic anaLYZER)

Input: Axioms $\mathcal{A}$ according to our grammar. Output:

- Proof Theory: sequent calculus for CL^{+} with $\mathcal{A}$
- Semantics: truth tables (using PNmatrices)



http://www.logic.at/staff/lara/tinc/webparalyzer/paralyzer.html

Semantics at work

Theorem

All our logics are decidable.

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A generated calculus is "analytic" iff the corresponding PNmatrix does not have empty sets in the truth tables of the connectives.

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What to do if there is an empty set?

- Transform the PNmatrix into a *finite family* of Nmatrices
- Apply the method in (A. Avron et al., 2006) and produce a *family* of cut-free sequent calculi.

Open problems and work in progress

- Extend the grammar
  E.g. (Kamide 2009, 2012, Kamide and Wansing 2012)
  $\neg \sim (\alpha \wedge \beta) \supset \neg \sim \alpha \vee \neg \sim \beta$,
  $\sim\sim\sim (\alpha \rightarrow \beta) \supset \sim\sim \alpha \wedge \sim\sim\sim \beta$,
  $\sim^j (\alpha \wedge \beta) \supset \sim^j \alpha \vee \sim^j \beta$ (with j odd), ...
- Consider "intuitionistic"-based paraconsistent logics
- First-order logics

The big picture

Theory and tools for the investigation of non-classical logics

- Systematic introduction of analytic calculi
- Their exploitation
  - new semantic foundations (e.g. this work)
  - decidability proofs (e.g. this work)
  - standard completeness
  - properties of algebraic structures
  - ...

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