A Multiple-Conclusion Calculus for First-Order Gödel Logic

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Abstract. We present a multiple-conclusion hypersequent system for the standard first-order Gödel logic. We provide a constructive, direct, and simple proof of the completeness of the cut-free part of this system, thereby proving both completeness for its standard semantics, and the admissibility of the cut rule in the full system. The results also apply to derivations from assumptions (or "non-logical axioms"), showing that such derivations can be confined to those in which cuts are made only on formulas which occur in the assumptions. Finally, the results about the multiple-conclusion system are used to show that the usual single-conclusion system for the standard first-order Gödel logic also admits (strong) cut-admissibility.

1 Introduction

In [15] Gödel introduced a sequence $\{G_n\}$ ($n \geq 2$) of n-valued matrices in the language of propositional intuitionistic logic. He used these matrices to show some important properties of intuitionistic logic. An infinite-valued matrix G_ω in which all the $G_n s$ can be embedded was later introduced by Dummett in [13]. G_ω, in turn, can naturally be embedded in a matrix $G_{[0,1]}$, the truth-values of which are the real numbers between 0 and 1 (inclusive). It has not been difficult to show that the logics of G_ω and $G_{[0,1]}$ are identical, and both are known today as "Gödel logic".[1] Later it has been shown that this logic is also characterized as the logic of *linear* intuitionistic Kripke frames (see e.g. [14]). Gödel logic is probably the most important intermediate logic, i.e. a logic between intuitionistic logic and classical logic, which turns up in several places. Recently it has again attracted a lot of attention because of its recognition as one of the three most basic fuzzy logics [16].

Gödel logic can be naturally extended to the first-order framework. In particular, the standard first-order Gödel logic (the logic based on $[0, 1]$ as the set of truth-values) has been introduced and investigated in [21] (where it was called "intuitionistic fuzzy logic"). The Kripke-style semantics of this logic is provided by the class of all linearly ordered Kripke frames with *constant domains.*

[1] It is also called Gödel-Dummett logic, because it was first introduced and axiomatized in [13]. The name Dummett himself has used is *LC*.

A cut-free Gentzen-type formulation for Gödel logic was first given by Sonobe in [18]. Since then several other such calculi which employ ordinary sequents have been proposed (see [10, 1, 11, 5, 12]). All these calculi have the drawback of using some ad-hoc rules of a nonstandard form, in which several occurrences of connectives are involved. In contrast, in [2] a cut-free Gentzen-type proof system **HG** for propositional Gödel logic was introduced, which does not have this drawback. **HG** uses (single-conclusion) *hypersequents* (a natural generalization of Gentzen's original (single-conclusion) sequents), and it has exactly the same logical rules as the usual Gentzen-type system for propositional intuitionistic logic. **HG** was furthermore extended by Baaz, Ciabattoni, Fermüller, and Zach to provide appropriate proof systems for extensions of propositional Gödel logic with quantifiers of various types and modalities (see [6] for a survey). In particular, an extension of **HG** for the standard first-order Gödel logic (called **HIF**) was introduced in [8]. Following the work that started in [2], the framework of hypersequents was used by Metcalfe, Ciabattoni, and others for other fuzzy logics (like Łukasiewicz infinite-valued logic), and nowadays it is the major framework for the proof theory of fuzzy logics (see [17]).

Until recently, in all the works about **HG** and its extensions the proofs of completeness (either for the Gödel's many-valued semantics or for the Kripke semantics) and the proofs of cut-elimination have completely been separated. Completeness has been shown for the full calculus (including cut), while cut-elimination has been proved syntactically by some type of induction on complexity of proofs.[2] On the contrary, the recent [4] provided for the first time a constructive, direct, and simple proof of the completeness of the cut-free part of **HG** for its intended semantics (thereby proving both completeness of the calculus and the admissibility of the cut rule in it)[3]. However, [4] did not deal with the first-order extension of **HG**, and it was not clear how to adapt its completeness proof to the first-order case.

In this paper we present a hypersequent system for the standard first-order Gödel logic, for which it is possible to provide a purely semantic, simple (and easy to verify) proof of cut-admissibility. As usual, this proof is actually a completeness proof of the cut-free part of our system for its intended semantics. To overcome the difficulties encountered in adapting the proof of [4] to the first-order case, we move to the *multiple-conclusion* framework. The proposed system, which we call **MCG**, is a multiple-conclusion hypersequent system, which can be seen as a combination of **HIF** and the well-known multiple-conclusion sequent system for intuitionistic logic (called **LJ′** in [20]). Our results apply also to derivations from assumptions, as we actually prove *strong* cut-admissibility,

[2] The syntactic methods are notoriously prone to errors, especially (but certainly not only) in the case of hypersequent systems. Thus the first proof (in [8]) of cut-elimination for **HIF** was erroneous. There has also been a gap in the proof given in [2] in its handling of the case of disjunction. Many other examples, also for ordinary sequential calculi, can be given.

[3] A semantic proof of cut-admissibility for **HG** has been given in [9]. However, a complicated algebraic phase semantics was used there, and the proof is not constructive.

proving that derivations can be confined to those in which cuts are made only on formulas which occur in the assumptions. Finally, at the end of the paper we return to the original single-conclusion system **HIF** for Gödel logic, and use our results about **MCG** to provide a new, semantic proof that this system too admits (strong) cut-admissibility.

2 Preliminaries

Let $\mathcal{L}$ be a first-order language. We assume that the set of free variables and the set of bounded variables are disjoint. We use the metavariable a to range over the free variables, x to range over the bounded variables, p to range over the predicate symbols of $\mathcal{L}$, c to range over its constant symbols, and f to range over its function symbols. The sets of $\mathcal{L}$-terms and $\mathcal{L}$-formulas are defined as usual, and are denoted by $trm_{\mathcal{L}}$ and $frm_{\mathcal{L}}$, respectively. $trm^{cl}_{\mathcal{L}}$ and $frm^{cl}_{\mathcal{L}}$ respectively denote the sets of closed $\mathcal{L}$-terms and closed $\mathcal{L}$-formulas. Given an $\mathcal{L}$-formula ψ, a free-variable a, and an $\mathcal{L}$-term t, we denote by $\psi\{t/a\}$ the $\mathcal{L}$-formula obtained from ψ by replacing all occurrences of a by t.

2.1 Proof-Theoretical Preliminaries

Definition 1. A *sequent* is an ordered pair of finite sets of $\mathcal{L}$-formulas. A *hypersequent* is a finite set of sequents.

Given a set $\mathcal{H}$ of hypersequents, we denote by $frm[\mathcal{H}]$ the set of formulas that appear in $\mathcal{H}$. We shall use the usual sequent notation $\Gamma \Rightarrow \Delta$, and the usual hypersequent notation $s_1 \mid \ldots \mid s_n$. We also employ the standard abbreviations, e.g. $\Gamma, \psi \Rightarrow \Delta$ instead of $\Gamma \cup \{\psi\} \Rightarrow \Delta$, and $H \mid s$ instead of $H \cup \{s\}$.

Definition 2. *A sequent* $\Gamma \Rightarrow \Delta$ *is* single-conclusion *if* Δ *contains at most one formula. A* $\mathcal{L}$*-hypersequent* $s_1 \mid \ldots \mid s_n$ *is* single-conclusion *if* $s_1, \ldots, s_n$ *are all single-conclusion.*

Next we review the single-conclusion hypersequent system **HIF** for the standard first-order Gödel logic from [8].[4]

$$\varphi \Rightarrow \varphi \qquad \bot \Rightarrow$$

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow E}{H \mid \Gamma, \psi \Rightarrow E} \qquad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow}{H \mid \Gamma \Rightarrow \psi} \qquad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow E}$$

$$(com)\ \frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow E_1 \qquad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow E_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow E_1 \mid \Gamma_2, \Gamma_1' \Rightarrow E_2}$$

[4] What we present is actually an equivalent version of the system presented in [8]. Thus $\neg\varphi$ is defined here as $\varphi \supset \bot$, while the density rule is not present, since it can be eliminated. Other insignificant differences are due to the facts that we define hypersequents as *sets* of sequents rather than as *multisets*, and that we use multiplicative versions of the rules rather than additive ones.

$$(cut)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \varphi \quad H_2 \mid \Gamma_2, \varphi \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow E}$$

$$(\supset\Rightarrow)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \psi_1 \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow E}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \supset \psi_2 \Rightarrow E} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \psi_1 \Rightarrow \psi_2}{H \mid \Gamma \Rightarrow \psi_1 \supset \psi_2}$$

$$(\vee\Rightarrow)\ \frac{H_1 \mid \Gamma_1, \psi_1 \Rightarrow E_1 \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow E_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \vee \psi_2 \Rightarrow E_1, E_2}$$

$$(\Rightarrow\vee_1)\ \frac{H \mid \Gamma \Rightarrow \psi_1}{H \mid \Gamma \Rightarrow \psi_1 \vee \psi_2} \qquad (\Rightarrow\vee_2)\ \frac{H \mid \Gamma \Rightarrow \psi_2}{H \mid \Gamma \Rightarrow \psi_1 \vee \psi_2}$$

$$(\wedge\Rightarrow_1)\ \frac{H \mid \Gamma, \psi_1 \Rightarrow E}{H \mid \Gamma, \psi_1 \wedge \psi_2 \Rightarrow E} \qquad (\wedge\Rightarrow_2)\ \frac{H \mid \Gamma, \psi_2 \Rightarrow E}{H \mid \Gamma, \psi_1 \wedge \psi_2 \Rightarrow E}$$

$$(\Rightarrow\wedge)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \psi_1 \quad H_2 \mid \Gamma_2 \Rightarrow \psi_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \psi_1 \wedge \psi_2}$$

$$(\forall\Rightarrow)\ \frac{H \mid \Gamma, \varphi\{t/a\} \Rightarrow E}{H \mid \Gamma, \forall x(\varphi\{x/a\}) \Rightarrow E} \qquad (\Rightarrow\forall)\ \frac{H \mid \Gamma \Rightarrow \varphi}{H \mid \Gamma \Rightarrow \forall x(\varphi\{x/a\})}$$

$$(\exists\Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow E}{H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow E} \qquad (\Rightarrow\exists)\ \frac{H \mid \Gamma \Rightarrow \varphi\{t/a\}}{H \mid \Gamma \Rightarrow \exists x(\varphi\{x/a\})}$$

The rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ must obey the eigenvariable condition: a must not occur in the lower hypersequent. E, E_1 and E_2 denote here sets of formulas containing at most one formula. Note that the sets of formulas denoted by $\Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2'$ need not to be disjoint. Also note that in $(\vee\Rightarrow)$: either $E_1 = E_2$ or one of them is empty.

2.2 Semantic Preliminaries

In this paper we use the usual Kripke-style semantics for the standard first-order Gödel logic, rather than the many-valued one. There are two differences between this semantics and the Kripke-style semantics of first-order intuitionistic logic. First, for Gödel logic we use *linearly* ordered Kripke frames. Second, for first-order Gödel logic we need to use a constant domain, i.e. the same domain in each world, rather than the expanding domains used for intuitionistic logic.[5]

Definition 3. An $\mathcal{L}$*-structure* M is a pair $\langle D, I \rangle$ where D is a nonempty domain and I is an interpretation of constants and function symbols of $\mathcal{L}$, such that $I(c) \in D$ for every constant symbol c of $\mathcal{L}$, and $I(f) \in D^n \to D$ for every n-ary function symbol f of $\mathcal{L}$.

Definition 4. A $\langle \mathcal{L}, D \rangle$-predicate interpretation is a function assigning a subset of D^n to every n-ary predicate symbol of $\mathcal{L}$.

Definition 5. An $\mathcal{L}$*-frame* is a tuple $\mathcal{W} = \langle W, \leq, M, \{I_w\}_{w \in W} \rangle$ where:

1. W is a nonempty set linearly ordered by $\leq$.

[5] Currently no cut-free hypersequent calculus is known for the logic of linearly ordered Kripke frames with (non-constant) expanding domains.

2. $M = \langle D, I \rangle$ is an $\mathcal{L}$-structure.
3. For every $w \in W$, I_w is an $\langle \mathcal{L}, D \rangle$-predicate interpretation.
4. $I_u(p) \subseteq I_w(p)$ for every elements u, w of W such that $u \leq w$, and for every predicate symbol p.

Definition 6. An $\langle \mathcal{L}, D \rangle$-evaluation is a function assigning an element in D to every free variable of $\mathcal{L}$. Given an $\langle \mathcal{L}, D \rangle$-evaluation e, a free variable a, and $d \in D$, we denote by $e_{[a:=d]}$ the $\langle \mathcal{L}, D \rangle$-evaluation which is identical to e except that $e_{[a:=d]}(a) = d$.

Given a structure $M = \langle D, I \rangle$, the M-extension of an $\langle \mathcal{L}, D \rangle$-evaluation e is a function $e' : trm_{\mathcal{L}} \to D$ defined as follows: $e'(c) = I(c)$ for every constant symbol c; $e'(a) = e(a)$ for every free variable a; and $e(f(t_1, \ldots, t_n) = I(f)(e'(t_1), \ldots, e'(t_n))$ for every function symbol f and $t_1, \ldots, t_n \in trm_{\mathcal{L}}$.

Definition 7. Let $\mathcal{W} = \langle W, \leq, M = \langle D, I \rangle, \{I_w\}_{w \in W} \rangle$ be an $\mathcal{L}$-frame, and e be an $\langle \mathcal{L}, D \rangle$-evaluation. The satisfaction relation $\vDash$ is recursively defined as follows:

1. $\mathcal{W}, w, e \vDash p(t_1, \ldots, t_n)$ iff $\langle e'(t_1), \ldots, e'(t_n) \rangle \in I_w(p)$, where e' is the M-extension of e.
2. $\mathcal{W}, w, e \nvDash \bot$.
3. $\mathcal{W}, w, e \vDash \psi_1 \supset \psi_2$ iff $\mathcal{W}, u, e \nvDash \psi_1$ or $\mathcal{W}, u, e \vDash \psi_2$ for every element $u \geq w$.
4. $\mathcal{W}, w, e \vDash \psi_1 \vee \psi_2$ iff $\mathcal{W}, w, e \vDash \psi_1$ or $\mathcal{W}, w, e \vDash \psi_2$.
5. $\mathcal{W}, w, e \vDash \psi_1 \wedge \psi_2$ iff $\mathcal{W}, w, e \vDash \psi_1$ and $\mathcal{W}, w, e \vDash \psi_2$.
6. $\mathcal{W}, w, e \vDash \forall x(\psi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \vDash \psi$ for every $d \in D$.
7. $\mathcal{W}, w, e \vDash \exists x(\psi\{x/a\})$ iff $\mathcal{W}, w, e_{[a:=d]} \vDash \psi$ for some $d \in D$.

$\vDash$ is extended to sequents as follows: $\mathcal{W}, w, e \vDash \Gamma \Rightarrow \Delta$ iff either $\mathcal{W}, w, e \nvDash \varphi$ for some $\varphi \in \Gamma$, or $\mathcal{W}, w, e \vDash \varphi$ for some $\varphi \in \Delta$.

It is a routine matter to prove the following proposition:

Proposition 1. *Let $\mathcal{W} = \langle W, \leq, M = \langle D, I \rangle, \{I_w\}_{w \in W} \rangle$ be an $\mathcal{L}$-frame, and e be an $\langle \mathcal{L}, D \rangle$-evaluation. Let ψ be an $\mathcal{L}$-formula, and u be an element of W such that $\mathcal{W}, u, e \vDash \psi$. Then, $\mathcal{W}, w, e \vDash \psi$ for every element w of W such that $u \leq w$.*

Definition 8. Let $\mathcal{W} = \langle W, \leq, M = \langle D, I \rangle, \{I_w\}_{w \in W} \rangle$ be an $\mathcal{L}$-frame.

1. $\mathcal{W}$ is a *model* of a hypersequent H iff for every $\langle \mathcal{L}, D \rangle$-evaluation e, there exists a component $s \in H$ such that $\mathcal{W}, w, e \vDash s$ for every $w \in W$.
2. $\mathcal{W}$ is a model of a set of hypersequents $\mathcal{H}$ iff it is a model of every $H \in \mathcal{H}$.

We define the semantic consequence relation between hypersequents:

Definition 9. Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. $\mathcal{H} \vdash^{Kr} H$ iff every $\mathcal{L}$-frame which is a model of $\mathcal{H}$ is also a model of H.

3 The Multiple-Conclusion System

The system **MCG** is the following (multiple-conclusion) hypersequent system:

Axioms:

$$\varphi \Rightarrow \varphi \qquad \bot \Rightarrow$$

Structural Rules:

$$(IW\Rightarrow)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma, \psi \Rightarrow \Delta} \quad (\Rightarrow IW)\ \frac{H \mid \Gamma \Rightarrow \Delta}{H \mid \Gamma \Rightarrow \Delta, \psi} \quad (EW)\ \frac{H}{H \mid \Gamma \Rightarrow \Delta}$$

$$(com)\ \frac{H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow \Delta_1 \quad H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow \Delta_1 \mid \Gamma_2, \Gamma_1' \Rightarrow \Delta_2} \qquad (split)\ \frac{H \mid \Gamma \Rightarrow \Delta_1, \Delta_2}{H \mid \Gamma \Rightarrow \Delta_1 \mid \Gamma \Rightarrow \Delta_2}$$

$$(cut)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \varphi \quad H_2 \mid \Gamma_2, \varphi \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$(\supset\Rightarrow)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \psi_1 \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \supset \psi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\supset)\ \frac{H \mid \Gamma, \psi_1 \Rightarrow \psi_2}{H \mid \Gamma \Rightarrow \psi_1 \supset \psi_2}$$

$$(\vee\Rightarrow)\ \frac{H_1 \mid \Gamma_1, \psi_1 \Rightarrow \Delta_1 \quad H_2 \mid \Gamma_2, \psi_2 \Rightarrow \Delta_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \psi_1 \vee \psi_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\vee)\ \frac{H \mid \Gamma \Rightarrow \Delta, \psi_1, \psi_2}{H \mid \Gamma \Rightarrow \Delta, \psi_1 \vee \psi_2}$$

$$(\wedge\Rightarrow)\ \frac{H \mid \Gamma, \psi_1, \psi_2 \Rightarrow \Delta}{H \mid \Gamma, \psi_1 \wedge \psi_2 \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \psi_1 \quad H_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi_2}{H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \psi_1 \wedge \psi_2}$$

$$(\forall\Rightarrow)\ \frac{H \mid \Gamma, \varphi\{t/a\} \Rightarrow \Delta}{H \mid \Gamma, \forall x(\varphi\{x/a\}) \Rightarrow \Delta} \qquad (\Rightarrow\forall)\ \frac{H \mid \Gamma \Rightarrow \Delta, \varphi}{H \mid \Gamma \Rightarrow \Delta, \forall x(\varphi\{x/a\})}$$

$$(\exists\Rightarrow)\ \frac{H \mid \Gamma, \varphi \Rightarrow \Delta}{H \mid \Gamma, \exists x(\varphi\{x/a\}) \Rightarrow \Delta} \qquad (\Rightarrow\exists)\ \frac{H \mid \Gamma \Rightarrow \Delta, \varphi\{t/a\}}{H \mid \Gamma \Rightarrow \Delta, \exists x(\varphi\{x/a\})}$$

The rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ must obey the eigenvariable condition: a must not occur in the lower hypersequent.

Remark 1. The main difference between **MCG** and the system **HIF** of [8] is the fact that **MCG** employs *multiple-conclusion* hypersequents. Among other things, this involves having full internal weakening $(\Rightarrow IW)$ on the right, and allowing also right context formulas in all the rules, except for $(\Rightarrow\supset)$. Note that such formulas are *not* allowed in $(\Rightarrow\supset)$, and so this rule looks exactly like its single-conclusion counterpart. The communication rule is also strengthened, allowing arbitrary finite sets of formula in the right-hand side of its premises. In addition, an extra (right) split rule is added, allowing to split the formulas in the right side of one component into two different components.

Definition 10. Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents. $\mathcal{H} \vdash H$ if there exists a derivation of H from $\mathcal{H}$ in **MCG**. Given a set $\mathcal{E}$ of $\mathcal{L}$-formulas, we write $\mathcal{H} \vdash^{\mathcal{E}} H$ if there exists a derivation of H from $\mathcal{H}$ in **MCG** in which the cut-formula of every application of the rule is in $\mathcal{E}$.

Remark 2. In this notation, strong cut-admissibility means that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$ whenever $\mathcal{H} \vdash H$. Usual cut-admissibility is obtained as a special case when $\mathcal{H} = \emptyset$.

The following usual lemma will be used in the sequel.

Lemma 1. *Let $\mathcal{H} \cup \{H\}$ be a set of hypersequents, c be a constant symbol not occurring in $\mathcal{H} \cup \{H\}$, and a be a free variable. Then, $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$ iff $\mathcal{H}\{c/a\} \vdash^{frm[\mathcal{H}]} H\{c/a\}$.*

4 Soundness, Completeness and Cut-Admissibility

In this section we prove the soundness and completeness theorem for **MCG** with respect to the Kripke semantics of first-order Gödel logic (presented in Section 2). Completeness is proved for **MCG** without the cut-rule, and so it also proves cut-admissibility. We begin with the soundness theorem.

Theorem 1. *Let $\mathcal{H} \cup \{H\}$ be a set of closed hypersequent. If $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$ then $\mathcal{H} \vdash^{Kr} H$.*

Proof (Outline). Let $\mathcal{W} = \langle W, \leq, M = \langle D, I \rangle, \{I_w\}_{w \in W} \rangle$ be an $\mathcal{L}$-frame which is a model of $\mathcal{H}$. We show that for every $\langle \mathcal{L}, D \rangle$-evaluation e, there exists a component $s \in H$ such that $\mathcal{W}, w, e \vDash s$ for every $w \in W$. Since the axioms of **MCG** and the assumptions of $\mathcal{H}$ trivially have this property, it suffices to show that this property is preserved also by applications of the rules of **MCG**. This is a routine matter. We do here only the case of (com).

Suppose that $H = H_1 \mid H_2 \mid \Gamma_1, \Gamma_2' \Rightarrow \Delta_1 \mid \Gamma_2, \Gamma_1' \Rightarrow \Delta_2$ is derived from the hypersequents $H_1 \mid \Gamma_1, \Gamma_1' \Rightarrow \Delta_1$ and $H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \Delta_2$ using (com). Assume for contradiction that $\mathcal{W}$ is not a model of H. Thus there exists an $\langle \mathcal{L}, D \rangle$-evaluation e, such that for every $s \in H$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In particular, for every $s \in H_1 \cup H_2$, there exists $w \in W$ such that $\mathcal{W}, w, e \nvDash s$. In addition, there exist $w_1 \in W$ such that $\mathcal{W}, w_1, e \nvDash \Gamma_1, \Gamma_2' \Rightarrow \Delta_1$, and $w_2 \in W$ such that $\mathcal{W}, w_2, e \nvDash \Gamma_2, \Gamma_1' \Rightarrow \Delta_2$. By definition, $\mathcal{W}, w_1, e \vDash \psi$ for every $\psi \in \Gamma_1 \cup \Gamma_2'$, $\mathcal{W}, w_1, e \nvDash \psi$ for every $\psi \in \Delta_1$, $\mathcal{W}, w_2, e \vDash \psi$ for every $\psi \in \Gamma_2 \cup \Gamma_1'$, and $\mathcal{W}, w_2, e \nvDash \psi$ for every $\psi \in \Delta_2$. Since $\leq$ is linear, either $w_1 \leq w_2$ or $w_2 \leq w_1$. Assume w.l.o.g that $w_1 \leq w_2$. Then by Proposition 1, $\mathcal{W}, w_2, e \vDash \psi$ for every $\psi \in \Gamma_2'$. It follows that $\mathcal{W}, w_2, e \nvDash \Gamma_2, \Gamma_2' \Rightarrow \Delta_2$. But this implies that $\mathcal{W}$ is not a model of $H_2 \mid \Gamma_2, \Gamma_2' \Rightarrow \Delta_2$. □

To prove completeness, we use *extended sequents* and *extended hypersequents*, defined as follows:

Definition 11. An *extended sequent* is an ordered pair of (possibly infinite) sets of $\mathcal{L}$-formulas, written: $\mathcal{T} \Rightarrow \mathcal{U}$. Given two extended sequents $\mu_1 = \mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $\mu_2 = \mathcal{T}_2 \Rightarrow \mathcal{U}_2$, we write $\mu_1 \sqsubseteq \mu_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$.

Definition 12. An *extended hypersequent* is a (possibly infinite) set of extended sequents. Given two extended hypersequents Ω_1, Ω_2, we write $\Omega_1 \sqsubseteq \Omega_2$ (and say that Ω_2 *extends* Ω_1) if for every extended sequent $\mu_1 \in \Omega_1$, there exists $\mu_2 \in \Omega_2$ such that $\mu_1 \sqsubseteq \mu_2$.

We shall use the same notations for extended sequents and extended hypersequents. For example, we write $\Omega \mid s$ instead of $\Omega \cup \{s\}$.

Definition 13. An extended sequent $\mathcal{T} \Rightarrow \mathcal{U}$ admits *the witness property* if the following hold:

1. If $\forall x(\psi\{x/a\}) \in \mathcal{U}$ then there exists a constant c such that $\psi\{c/a\} \in \mathcal{U}$.
2. If $\exists x(\psi\{x/a\}) \in \mathcal{T}$ then there exists a constant c such that $\psi\{c/a\} \in \mathcal{T}$.

Definition 14. Let Ω be an extended hypersequent, and $\mathcal{H}$ be a set of hypersequents.

1. Ω is called *closed* if it consists of extended sequents consisting only of closed $\mathcal{L}$-formulas.
2. Ω is called *$\mathcal{H}$-consistent* if $\mathcal{H} \nvdash^{frm[\mathcal{H}]} H$ for every hypersequent $H \sqsubseteq \Omega$.
3. Let ψ be an $\mathcal{L}$-formula. Ω is called *internally $\mathcal{H}$-maximal with respect to ψ* if for every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:
   (a) If $\psi \notin \mathcal{T}$ then $\Omega \mid \mathcal{T}, \psi \Rightarrow \mathcal{U}$ is not $\mathcal{H}$-consistent.
   (b) If $\psi \notin \mathcal{U}$ then $\Omega \mid \mathcal{T} \Rightarrow \mathcal{U}, \psi$ is not $\mathcal{H}$-consistent.
4. Ω is called *internally $\mathcal{H}$-maximal* if it is internally $\mathcal{H}$-maximal with respect to any closed $\mathcal{L}$-formula.
5. Let s be a sequent of the form $\psi_1 \Rightarrow \psi_2$. Ω is called *externally $\mathcal{H}$-maximal with respect to s* if either $\{s\} \sqsubseteq \Omega$, or $\Omega \mid s$ is not $\mathcal{H}$-consistent.
6. Ω is called *externally $\mathcal{H}$-maximal* if it is externally $\mathcal{H}$-maximal with respect to any closed sequent of the form $\psi_1 \Rightarrow \psi_2$.
7. Ω admits *the witness property* if every $\mu \in \Omega$ admits the witness property.
8. Ω is called *$\mathcal{H}$-maximal* if it is closed, $\mathcal{H}$-consistent, internally $\mathcal{H}$-maximal, externally $\mathcal{H}$-maximal, and it admits the witness property.

Obviously, every hypersequent is an extended hypersequent, and so all of these properties apply to (usual) hypersequents as well.

The following three propositions are easily proved in the presence of the internal and external weakening rules:

Proposition 2. *A usual hypersequent H is $\mathcal{H}$-consistent iff $\mathcal{H} \nvdash^{frm[\mathcal{H}]} H$.*

Proposition 3. *Let Ω be an extended hypersequent, which is internally $\mathcal{H}$-maximal with respect to a formula ψ. For every $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$:*

1. *If $\psi \notin \mathcal{T}$, then $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma, \psi \Rightarrow \Delta$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.*
2. *If $\psi \notin \mathcal{U}$, then $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid \Gamma \Rightarrow \Delta, \psi$ for some hypersequent $H \sqsubseteq \Omega$ and sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$.*

Proposition 4. *Let Ω be an extended hypersequent, which is externally $\mathcal{H}$-maximal with respect to a sequent s. Then, either $s \sqsubseteq \Omega$, or there exists a hypersequent $H \sqsubseteq \Omega$ such that $\mathcal{H} \vdash^{frm[\mathcal{H}]} H \mid s$.*

A certain $\mathcal{H}$-maximal extended hypersequent serves as the set of worlds in the refuting frame built in the completeness proof. Lemma 4 below ensures the existence of that extended hypersequent. In turn, for the proof of Lemma 4 we need Lemmas 2 and 3 below.

Lemma 2. *Assume $\mathcal{L}$ has an infinite number of constant symbols. Let $\mathcal{H}$ be a set of hypersequents, and $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent closed hypersequent. Then there exists a $\mathcal{H}$-consistent closed hypersequent H' of the form $\Gamma'_1 \Rightarrow \Delta'_1 \mid \ldots \mid \Gamma'_n \Rightarrow \Delta'_n$, such that $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$, and H' admits the witness property.*

Lemma 3. *Assume $\mathcal{L}$ has an infinite number of constant symbols. Let $\mathcal{H}$ be a set of hypersequents, and $H = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a $\mathcal{H}$-consistent closed hypersequent. Let ψ be a closed $\mathcal{L}$-formula, and s be a closed sequent of the form $\psi_1 \Rightarrow \psi_2$. Then there exists a $\mathcal{H}$-consistent closed hypersequent H', such that:*

- *$H' = \Gamma'_1 \Rightarrow \Delta'_1 \mid \ldots \mid \Gamma'_{n'} \Rightarrow \Delta'_{n'}$, where $n' \in \{n, n+1\}$, $\Gamma_i \subseteq \Gamma'_i$ and $\Delta_i \subseteq \Delta'_i$ for every $1 \leq i \leq n$.*
- *H' is internally $\mathcal{H}$-maximal with respect to ψ.*
- *H' is externally $\mathcal{H}$-maximal with respect to s.*
- *H' admits the witness property.*

Lemma 4. *Assume $\mathcal{L}$ has an infinite number of constant symbols. Let $\mathcal{H}$ be a set of hypersequents. Every $\mathcal{H}$-consistent closed hypersequent H can be extended to a $\mathcal{H}$-maximal extended hypersequent Ω.*

Using Lemma 4, we turn to the completeness of the cut-free fragment of **MCG**.

Theorem 2. *Let $\mathcal{H}_0 \cup \{H_0\}$ be a set of closed hypersequent. If $\mathcal{H}_0 \vdash^{Kr} H_0$ then $\mathcal{H}_0 \vdash^{frm[\mathcal{H}]} H_0$.*

Proof. Assume $\mathcal{H}_0 \not\vdash^{\mathcal{E}} H_0$, where $\mathcal{E} = frm[\mathcal{H}_0]$. We construct an $\mathcal{L}$-frame $\mathcal{W}$ which is a model of $\mathcal{H}_0$ but not of H_0. First, assume (w.l.o.g) that $\mathcal{L}$ has an infinite number of constant symbols (if not, then we add infinitely many constant symbols, and obviously $\mathcal{H}_0 \not\vdash^{\mathcal{E}} H_0$ still holds). By Lemma 4, there exists a $\mathcal{H}_0$-maximal extended hypersequent Ω such that $H_0 \sqsubseteq \Omega$.

Define $\mathcal{W} = \langle W, \leq, M, \{I_w\}_{w \in W}\rangle$, as follows:

- $W = \Omega$ (obviously, W is not empty).
- For every $\mathcal{T}_1 \Rightarrow \mathcal{U}_1, \mathcal{T}_2 \Rightarrow \mathcal{U}_2 \in W$, $\mathcal{T}_1 \Rightarrow \mathcal{U}_1 \leq \mathcal{T}_2 \Rightarrow \mathcal{U}_2$ iff $\mathcal{T}_1 \subseteq \mathcal{T}_2$.
- $M = \langle D, I\rangle$ where D is the set of all closed $\mathcal{L}$-terms, $I(c) = c$ for every constant c, and $I(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for every n-ary function symbol f and $t_1, \ldots, t_n \in D$.
- $\langle t_1, \ldots, t_n\rangle \in I_{\mathcal{T} \Rightarrow \mathcal{U}}(p)$ iff $p(t_1, \ldots, t_n) \in \mathcal{T}$ for every n-ary predicate symbol p and $t_1, \ldots, t_n \in D$.

We first prove that $\langle W, \leq\rangle$ is linearly ordered:

Partial Order Obviously $\leq$ is reflexive and transitive. To see that it is also anti-symmetric, let $w_1, w_2 \in W$ such that $w_1 \leq w_2$ and $w_2 \leq w_1$. Assume $w_1 = \mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $w_2 = \mathcal{T}_2 \Rightarrow \mathcal{U}_2$. By definition, $\mathcal{T}_1 = \mathcal{T}_2$ in this case. Assume for contradiction that $\mathcal{U}_1 \neq \mathcal{U}_2$, and let $\psi \in \mathcal{U}_1 \setminus \mathcal{U}_2$ (w.l.o.g.). Since Ω is internally $\mathcal{H}$-maximal, there exist a hypersequent $H \sqsubseteq \Omega$ and a sequent $\Gamma \Rightarrow \Delta \sqsubseteq w_2$, such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \Delta, \psi$. Using the split rule, we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \psi \mid \Gamma \Rightarrow \Delta$. But, $\Gamma \Rightarrow \psi \sqsubseteq w_1$, and this contradicts Ω's consistency. Hence $\mathcal{U}_1 = \mathcal{U}_2$, and so $w_1 = w_2$.

Linearity Let $\mathcal{T}_1 \Rightarrow \mathcal{U}_1, \mathcal{T}_2 \Rightarrow \mathcal{U}_2 \in W$. Assume for contradiction that $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$. Let $\psi_1 \in \mathcal{T}_1 \setminus \mathcal{T}_2$ and $\psi_2 \in \mathcal{T}_2 \setminus \mathcal{T}_1$. By Ω's internal maximality, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$ and sequents $\Gamma_1 \Rightarrow \Delta_1 \sqsubseteq \mathcal{T}_1 \Rightarrow \mathcal{U}_1$ and $\Gamma_2 \Rightarrow \Delta_2 \sqsubseteq \mathcal{T}_2 \Rightarrow \mathcal{U}_2$ such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid \Gamma_1, \psi_2 \Rightarrow \Delta_1$ and such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H_2 \mid \Gamma_2, \psi_1 \Rightarrow \Delta_2$. By applying (com) to these two hypersequents we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid H_2 \mid \Gamma_1, \psi_1 \Rightarrow \Delta_1 \mid \Gamma_2, \psi_2 \Rightarrow \Delta_2$. But this contradicts Ω's consistency.

The following claims are proved by a standard structural induction:

- For every $\langle \mathcal{L}, D\rangle$-evaluation e and $t \in D$, $e'(t) = t$.
- For every $\langle \mathcal{L}, D\rangle$-evaluation e, $t \in D$, an $\mathcal{L}$-formula ψ, a free variable a, and $w \in W$: $\mathcal{W}, w, e_{[a:=t]} \vDash \psi$ iff $\mathcal{W}, w, e \vDash \psi\{t/a\}$.

Next we prove that the following hold for every $w = \mathcal{T} \Rightarrow \mathcal{U} \in W$, and $\langle \mathcal{L}, D\rangle$-evaluation e:

(a) If $\theta \in \mathcal{T}$ then $\mathcal{W}, w, e \vDash \theta$.
(b) If $\theta \in \mathcal{U}$ then $\mathcal{W}, w, e \not\vDash \theta$.

(a) and **(b)** are proved together using a simultaneous induction on the complexity of θ. Here we do three crucial cases.

Let $w = \mathcal{T} \Rightarrow \mathcal{U} \in W$ and let e be an $\langle \mathcal{L}, D\rangle$-evaluation.

- Suppose θ is a closed atomic formula $p(t_1, \ldots, t_n)$. By definition, $\mathcal{W}, w, e \vDash \theta$ iff $\langle e'(t_1), \ldots, e'(t_n)\rangle \in I_w(p)$, where e' is the M-extension of e (see Definition 6). By a previous claim, $e'(t_i) = t_i$ for every $1 \leq i \leq n$. And so, our construction ensures that $\mathcal{W}, w, e \vDash \theta$ iff $\theta \in \mathcal{T}$. This proves (**a**). For (**b**), note that $\psi \Rightarrow \psi$ is an axiom (for every $\mathcal{L}$-formula ψ), and since Ω is $\mathcal{H}$-consistent, $\theta \in \mathcal{U}$ implies $\theta \notin \mathcal{T}$.
- Suppose $\theta = \psi_1 \supset \psi_2$.
  1. Assume that $\theta \in \mathcal{T}$. We show that for every element $w' \in W$ such that $w \leq w'$ either $\mathcal{W}, w', e \not\vDash \psi_1$ or $\mathcal{W}, w', e \vDash \psi_2$.
     Let $w' = T' \Rightarrow \mathcal{U}' \in W$ such that $w \leq w'$ (and so, $\mathcal{T} \subseteq \mathcal{T}'$). By the induction hypothesis, it suffices to show that either $\psi_1 \in \mathcal{U}'$ or $\psi_2 \in \mathcal{T}'$. Assume otherwise. Then by Ω's internal maximality, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, and sequents $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2 \sqsubseteq \mathcal{T}' \Rightarrow \mathcal{U}'$ such

that $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid \Gamma_1 \Rightarrow \Delta_1, \psi_1$, and $\mathcal{H}_0 \vdash^{\mathcal{E}} H_2 \mid \Gamma_2, \psi_2 \Rightarrow \Delta_2$. By applying ($\supset\Rightarrow$) we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \theta \Rightarrow \Delta_1, \Delta_2$. But since $\theta \in \mathcal{T}$, $\theta \in \mathcal{T}'$ and so $H_1 \mid H_2 \mid \Gamma_1, \Gamma_2, \theta \Rightarrow \Delta_1, \Delta_2 \sqsubseteq \Omega$. This contradicts Ω's consistency.

2. Assume that $\theta \in \mathcal{U}$.
First we claim that $\mathcal{H}_0 \not\vdash^{\mathcal{E}} H \mid \psi_1 \Rightarrow \psi_2$ for every hypersequent $H \sqsubseteq \Omega$. To see this assume for contradiction that there exists a hypersequent $H \sqsubseteq \Omega$, such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \psi_1 \Rightarrow \psi_2$. By applying ($\Rightarrow\supset$) to this hypersequent we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Rightarrow \theta$. But this contradicts Ω's consistency. Therefore, by Ω's external maximality, $\psi_1 \Rightarrow \psi_2 \sqsubseteq \Omega$. Thus there exists an extended sequent $\mathcal{T}' \Rightarrow \mathcal{U}' \in \Omega$, such that $\psi_1 \in \mathcal{T}'$ and $\psi_2 \in \mathcal{U}'$. By the induction hypothesis, $\mathcal{W}, \mathcal{T}' \Rightarrow \mathcal{U}', e \models \psi_1$ and $\mathcal{W}, \mathcal{T}' \Rightarrow \mathcal{U}', e \not\models \psi_2$. It follows that if $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{W}, w, e \not\models \theta$ and we are done.
Assume now that $\mathcal{T} \not\subseteq \mathcal{T}'$. By linearity, $\mathcal{T}' \subseteq \mathcal{T}$, and so $\psi_1 \in \mathcal{T}$. By the induction hypothesis, $\mathcal{W}, w, e \models \psi_1$. Now notice that $\psi_2 \in \mathcal{U}$. To see this assume for contradiction that $\psi_2 \notin \mathcal{U}$. Then by Ω's internal maximality, there exist a hypersequent $H \sqsubseteq \Omega$, and a sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \Delta, \psi_2$. By applying the split rule we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \psi_2$. By applying internal weakening we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \Delta \mid \Gamma, \psi_1 \Rightarrow \psi_2$. Finally, by ($\Rightarrow\supset$) we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \theta$. But this contradicts Ω's consistency. By the induction hypothesis, $\mathcal{W}, w, e \not\models \psi_2$. This again implies that $\mathcal{W}, w, e \not\models \theta$.

– Suppose $\theta = \forall x(\psi\{x/a\})$.
  1. Assume that $\mathcal{W}, w, e \not\models \theta$. We show that $\theta \notin \mathcal{T}$. By definition, there exists some $t \in D$ such that $\mathcal{W}, w, e_{[a:=t]} \not\models \psi$. By a previous claim, it follows that $\mathcal{W}, w, e \not\models \psi\{t/a\}$. By the induction hypothesis, $\psi\{t/a\} \notin \mathcal{T}$. Now, using Ω's internal maximality, there exist a hypersequent $H \sqsubseteq \Omega$ and a sequent $\Gamma \Rightarrow \Delta \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma, \psi\{t/a\} \Rightarrow \Delta$. By applying ($\forall\Rightarrow$), we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H \mid \Gamma, \theta \Rightarrow \Delta$. Since Ω is $\mathcal{H}$-consistent, $\theta \notin \mathcal{T}$.
  2. Assume that $\theta \in \mathcal{U}$. By Ω's witness property, there exists a constant symbol c such that $\psi\{c/a\} \in \mathcal{U}$. From the induction hypothesis it follows that $\mathcal{W}, w, e \not\models \psi\{c/a\}$. By a previous claim, it follows that $\mathcal{W}, w, e_{[a:=c]} \not\models \psi$. Since $c \in D$, by definition, $\mathcal{W}, w, e \not\models \theta$.

It remains to show that $\mathcal{W}$ is a model of $\mathcal{H}_0$ but not of H_0. First, notice that for every $\psi \in \mathit{frm}[\mathcal{H}]$ and $\mathcal{T} \Rightarrow \mathcal{U} \in \Omega$, either $\psi \in \mathcal{T}$ or $\psi \in \mathcal{U}$. To see this, note that otherwise, by Ω's internal maximality, there exist hypersequents $H_1, H_2 \sqsubseteq \Omega$, and sequents $\Gamma_1 \Rightarrow \Delta_1, \Gamma_2 \Rightarrow \Delta_2 \sqsubseteq \mathcal{T} \Rightarrow \mathcal{U}$, such that $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid \Gamma_1, \psi \Rightarrow \Delta_1$ and $\mathcal{H}_0 \vdash^{\mathcal{E}} H_2 \mid \Gamma_2 \Rightarrow \Delta_2, \psi$. Now using a (legal) application of the cut rule, we obtain $\mathcal{H}_0 \vdash^{\mathcal{E}} H_1 \mid H_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, but this contradicts Ω's consistency.

Now let $H \in \mathcal{H}_0$, and let e be an $\langle \mathcal{L}, D\rangle$-evaluation. Since obviously $\mathcal{H}_0 \vdash^{\mathcal{E}} H$, Lemma 2 implies that $H \not\sqsubseteq \Omega$. Thus there exists a sequent $s \in H$, such that $s \not\sqsubseteq \mu$ for every $\mu \in \Omega$. We prove that $\mathcal{W}, w, e \models s$ for every $w \in W$. Let $w \in W$. Assume $w = \mathcal{T} \Rightarrow \mathcal{U}$ and $s = \Gamma \Rightarrow \Delta$. Since $s \not\sqsubseteq w$, there either exists $\psi \in \Gamma$ such that $\psi \notin \mathcal{T}$, or $\psi \in \Delta$ such that $\psi \notin \mathcal{U}$. This implies that there either exists $\psi \in \Gamma$ such that $\psi \in \mathcal{U}$, or $\psi \in \Delta$ such that $\psi \in \mathcal{T}$. By **(a)** and **(b)**, either there exists $\psi \in \Gamma$ such that $\mathcal{W}, w, e \not\models \psi$, or there exists $\psi \in \Delta$ such that $\mathcal{W}, w, e \models \psi$. Therefore, $\mathcal{W}, w, e \models s$.

We end the proof by showing that $\mathcal{W}$ is not a model of H_0. Let e be an arbitrary $\langle \mathcal{L}, D\rangle$-evaluation, and let $\Gamma \Rightarrow \Delta \in H_0$. Since $H_0 \sqsubseteq \Omega$ there exists an extended sequent $w = \mathcal{T} \Rightarrow \mathcal{U} \in \Omega$ such that $\Gamma \Rightarrow \Delta \sqsubseteq w$. By **(a)**, for every $\psi \in \Gamma$, $\mathcal{W}, w, e \models \psi$. By **(b)**, for every $\psi \in \Delta$, $\mathcal{W}, w, e \not\models \psi$. Thus, $\mathcal{W}, w, e \not\models \Gamma \Rightarrow \Delta$. □

Finally, we state the two main corollaries, easily obtained from the two previous theorems.

Corollary 1 (Strong Soundness and Completeness).
MCG *is strongly sound and complete with respect to the Kripke semantics of the standard first-order Gödel logic, i.e.* $\mathcal{H} \vdash H$ *iff* $\mathcal{H} \vdash^{Kr} H$ *for every set* $\mathcal{H} \cup \{H\}$ *of closed hypersequent.*

Corollary 2 (Strong Cut-Admissibility).
MCG *admits strong cut-admissibility, i.e.* $\mathcal{H} \vdash H$ *iff* $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$*, for every set* $\mathcal{H} \cup \{H\}$ *of closed hypersequent.*

Remark 3. In [21] the following density rule was introduced and used to axiomatize standard first-order Gödel logic:

$$\frac{\Gamma \Rightarrow \varphi \vee (\psi \supset p) \vee (p \supset \theta)}{\Gamma \Rightarrow \varphi \vee (\psi \supset \theta)}$$

where p (a metavariable for an atomic formula) does not occur in the conclusion. In [19] this rule was proved to be admissible (using a semantic proof). The (single-conclusion) hypersquential version of this rule has the form (see [6]):

$$\frac{H \mid \Gamma \Rightarrow p \mid \Delta, p \Rightarrow \psi}{H \mid \Gamma, \Delta \Rightarrow \psi}$$

By Corollary 1, this rule is admissible in **MCG**.

5 Cut-Admissibility for HIF

In this section we study the relation between **MCG** and the single-conclusion system **HIF**, and derive a semantic cut-admissibility proof for the system **HIF** itself. Denote by $\vdash_{\leq 1}$ the provability relation (between sets of single-conclusion hypersequents, and single-conclusion hypersequents) induced by **HIF** (see Section 2).

Definition 15. Given a hypersequent H, $H^{\leq 1}$ is the single-conclusion hypersequent $\bigcup_{\Gamma \Rightarrow \Delta \in H} \{\Gamma \Rightarrow E \mid E \subseteq \Delta\}$, where E denotes sets of $\mathcal{L}$-formulas which are either singletons or empty. Let $\mathcal{H}^{\leq 1} = \{H^{\leq 1} \mid H \in \mathcal{H}\}$.

The following theorem provides the relation between **MCG** and **HIF**.

Theorem 3. *For every set of hypersequents* $\mathcal{H} \cup \{H\}$*, and set* $\mathcal{E}$ *of* $\mathcal{L}$*-formulas,* $\mathcal{H} \vdash^{\mathcal{E}} H$ *iff* $\mathcal{H}^{\leq 1} \vdash^{\mathcal{E}}_{\leq 1} H^{\leq 1}$.

The proof of this theorem is done as usual by induction on the length of the proof in **MCG**. The most problematic case (dealing with the rule $(\exists\Rightarrow)$) follows from Lemma 30 in [6].

Corollary 3 (Strong Cut-Admissibility for HIF).
HIF *admits strong cut-admissibility, i.e.* $\mathcal{H} \vdash_{\leq 1} H$ *iff* $\mathcal{H} \vdash^{frm[\mathcal{H}]}_{\leq 1} H$*, for every set* $\mathcal{H} \cup \{H\}$ *of closed single-conclusion hypersequents.*

Proof. One direction is trivial. For the converse, assume $\mathcal{H} \vdash_{\leq 1} H$. In this case, obviously, $\mathcal{H} \vdash H$. By Corollary 2, $\mathcal{H} \vdash^{frm[\mathcal{H}]} H$. Theorem 3 implies that $\mathcal{H}^{\leq 1} \vdash^{frm[\mathcal{H}]}_{\leq 1} H^{\leq 1}$. Now, notice that for a single-conclusion hypersequent H, $H^{\leq 1} = H \cup \{\Gamma \Rightarrow \emptyset \mid \Gamma \Rightarrow \varphi \in H\}$, and obviously $H^{\leq 1} \vdash^{\emptyset}_{\leq 1} H$ and $H \vdash^{\emptyset}_{\leq 1} H^{\leq 1}$. It now follows that $\mathcal{H} \vdash^{frm[\mathcal{H}]}_{\leq 1} H$. □

6 Further Research

We believe that a (multiple-conclusion) hypersequent system can also be used to provide a similar semantic proof of strong cut-admissibility in Gentzen's LJ. Many other (multiple or single-conclusion) hypersequent systems for various propositional and first-order fuzzy logics and intermediate logics have only syntactic proofs of (usual) cut-elimination theorem (see e.g. [17]). It should be interesting to find for them too more simple semantic proofs and derive corresponding strong cut-admissibility theorems. For other fuzzy logics, Kripke-style semantics might not suffice.

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