Computational Models - Lecture 2

- Non-Deterministic Finite Automata (NFA)
- Closure of Regular Languages Under $\bigcup, \circ, \ ^*$
- Regular expressions
- Equivalence with finite automata
- Sipser's book, 1.1-1.3

DFA Formal Definition (reminder)

A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma \to Q$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Languages and DFA (reminder)

The language $L(M)$ of a DFA M over Σ is the set of all strings over Σ that M accepts.

Note that

- M may accept many strings, but
- M accepts only one language.

A language is called regular if some deterministic finite automaton accepts it.

The Regular Operations (reminder)

Let A and B be languages.

The union operation:

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

The concatenation operation:

$$A \circ B = \{xy | x \in A \text{ and } y \in B\}$$

The star operation:

$$A^* = \{x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A\}$$

Claim: Closure Under Union (reminder)

If A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Approach to Proof:

- some M_1 accepts A_1
- some M_2 accepts A_2
- construct M that accepts $A_1 \cup A_2$.
- in our construction, states of M were Cartesian product of M_1 and M_2 states.

What About Concatenation?

Thm: If L_1, L_2 are regular languages, so is $L_1 \circ L_2$.

Example: $L_1 = \{\text{good}, \text{bad}\}$ and $L_2 = \{\text{boy}, \text{girl}\}$.

$$L_1 \circ L_2 = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$$

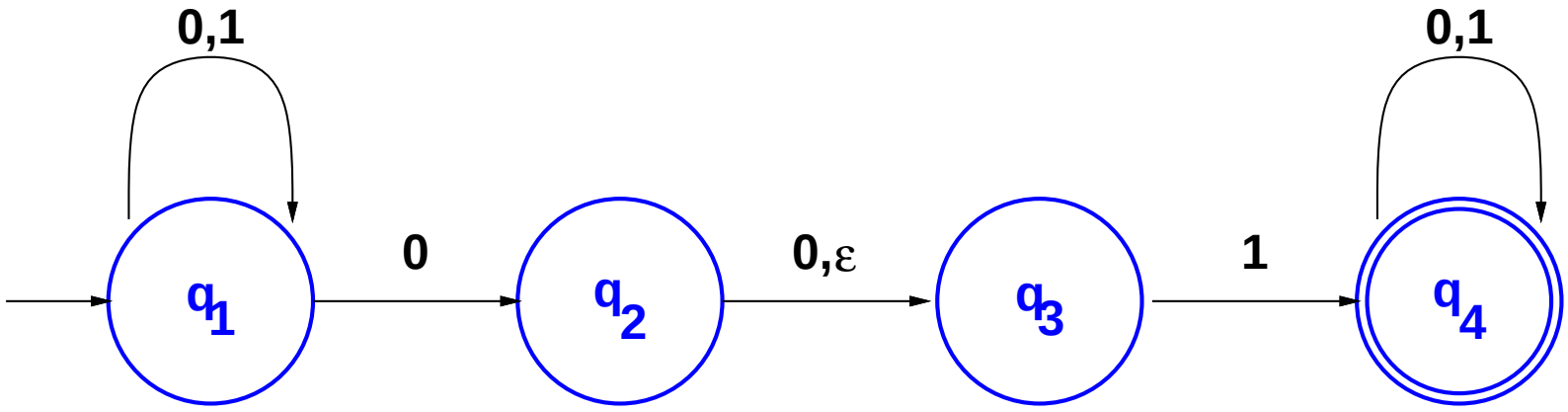
This is much harder to prove.

Idea: Simulate M_1 for a while, then switch to M_2.

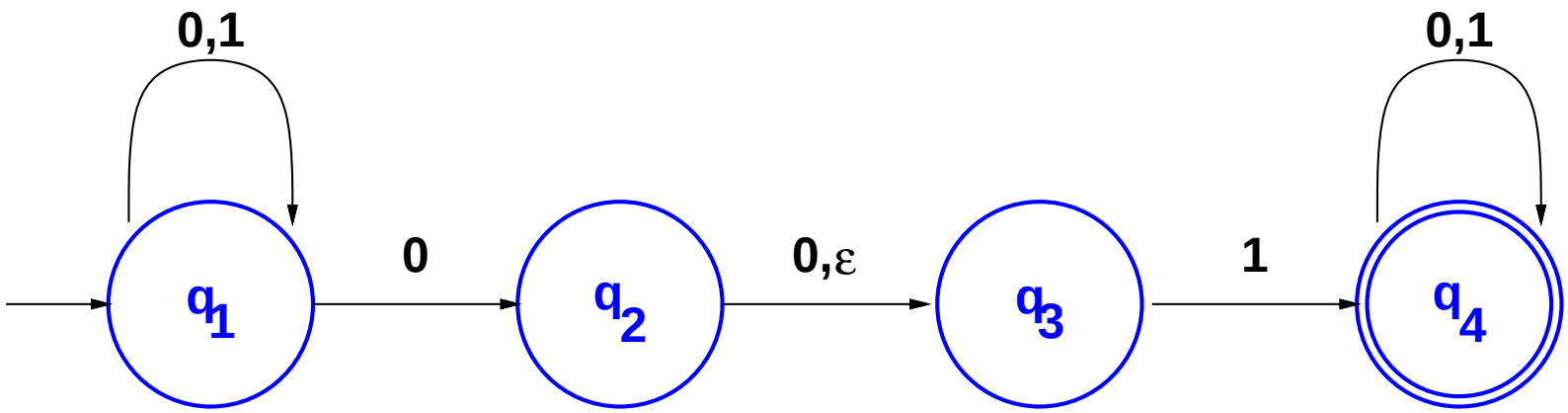
Problem: But when do you switch?

Seems hard to do with DFAs.
This leads us into non-determinism.

Non-Deterministic Finite Automata



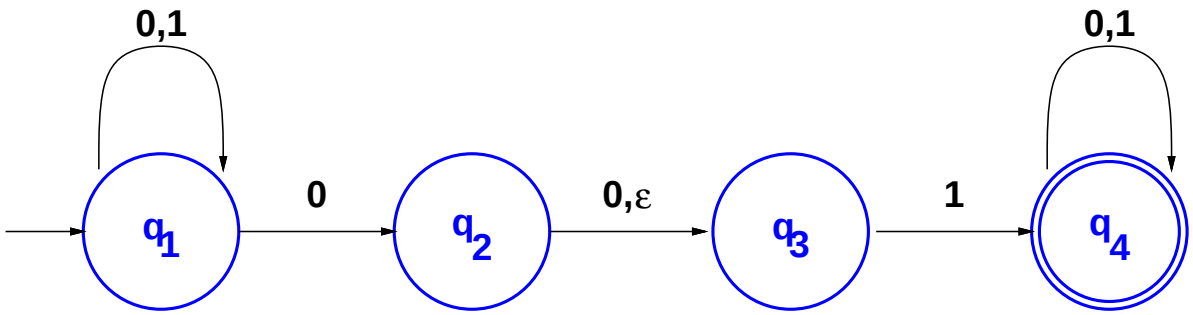
Non-Deterministic Finite Automata



- an NFA may have more than one transition labeled with the same symbol,
- an NFA may have no transitions labeled with a certain symbol, and
- an NFA may have transitions labeled with ε, the empty string.

Comment: Every DFA is also a non-deterministic finite automata (NFA).

Non-Deterministic Computation

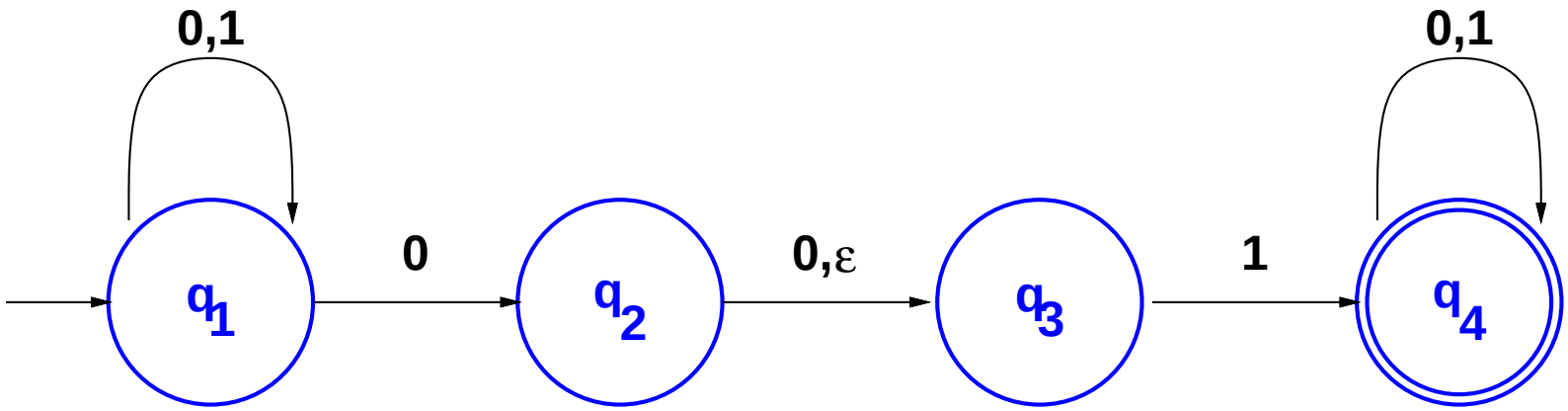


What happens when more than one transition is possible?

- the machine “splits” into multiple copies
- each branch follows one possibility
- together, branches follow all possibilities.
- If the input doesn’t appear, that branch “dies”.
- Automaton accepts if some branch accepts.

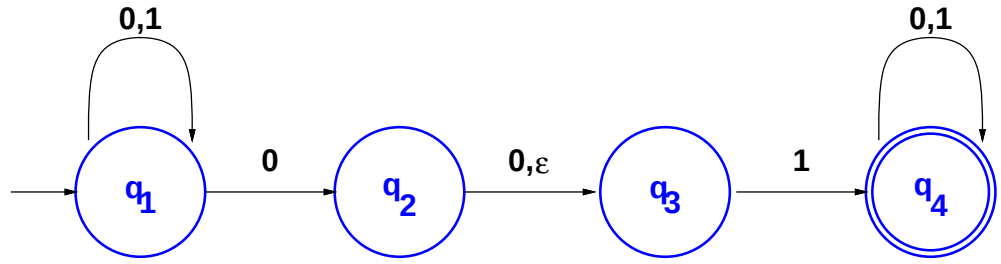
What does an ε transition do?

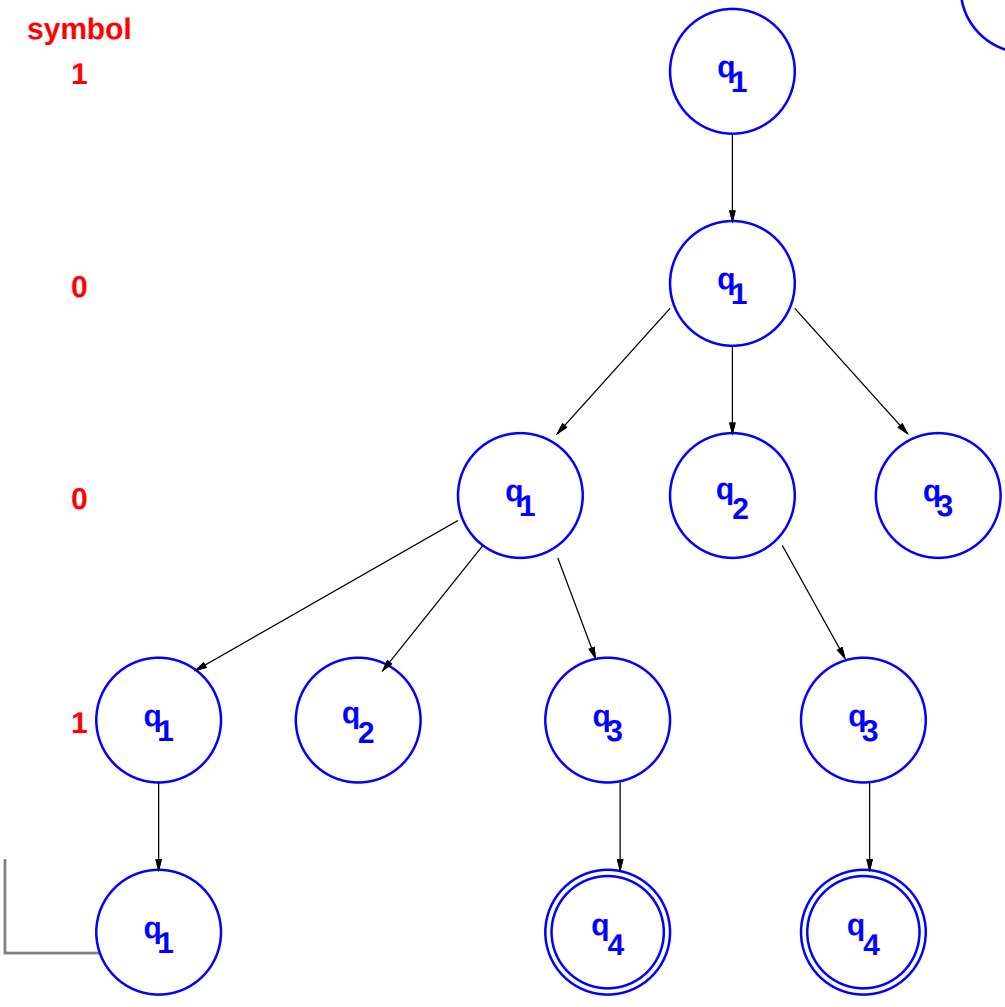
Non-Deterministic Computation



What happens on string 1001?

The String 1001





Why Non-Determinism?

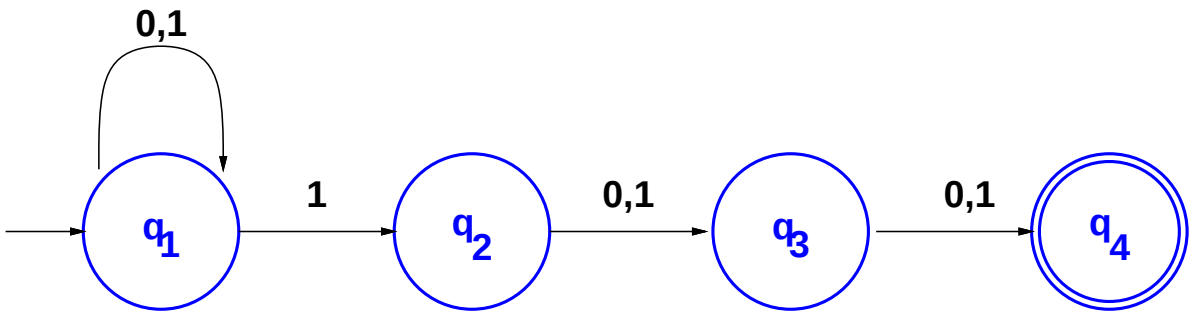
Theorem (to be proved soon): Deterministic and non-deterministic finite automata accept exactly the same set of languages.

Q.: So why do we need them?

A.: NFAs are often easier to design than equivalent DFAs.

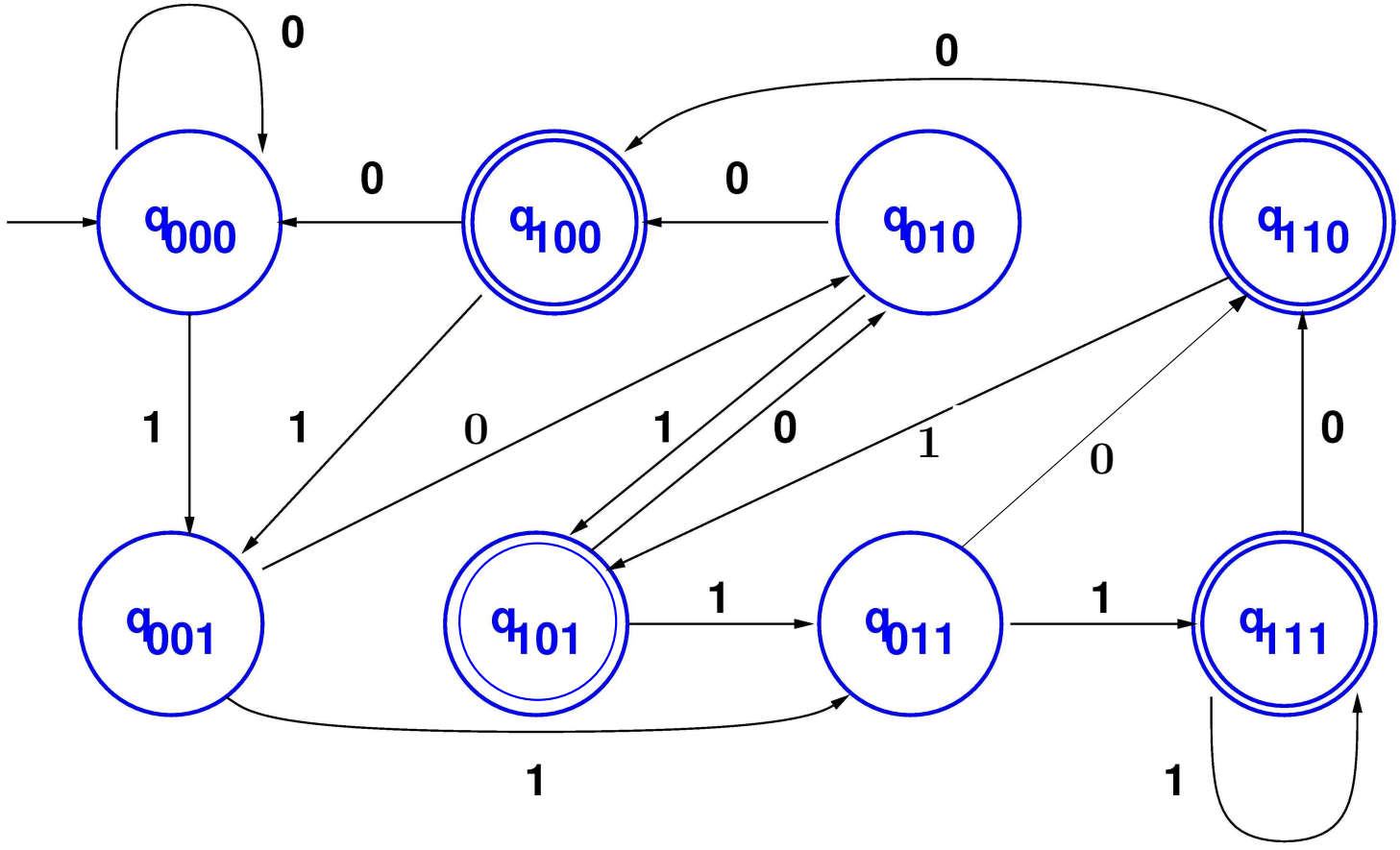
Example: Design a finite automaton that accepts all strings with a 1 in their third-to-the-last position?

Solving with NFA



- "Guesses" which symbol is third from the last, and
- checks that indeed it is a 1.
- If guess is premature, that branch "dies", and no harm occurs.

Solving with DFA



NFA – Formal Definition

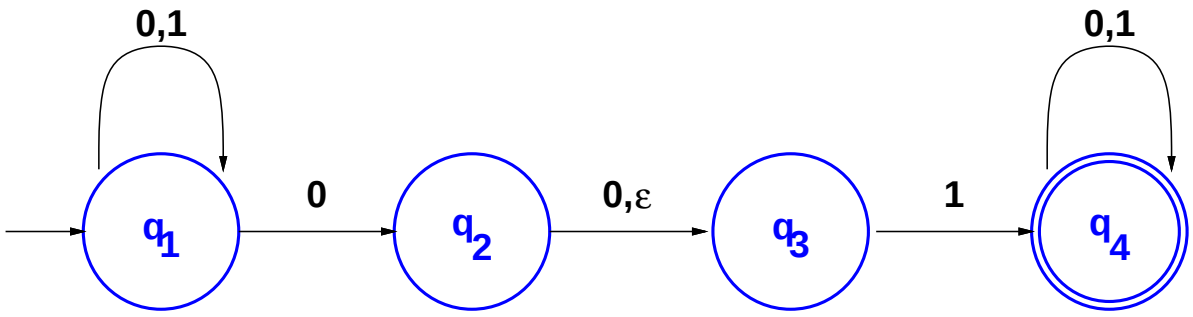
Transition function δ is going to be different.

- Let $\mathcal{P}(Q)$ denote the powerset of Q.
- Let Σ_ε denote $\Sigma \cup \{\varepsilon\}$.

A non-deterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- Q is a finite set called the states,
- Σ is a finite set called the alphabet,
- $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
- $q_0 \in Q$ is the start state, and
- $F \subseteq Q$ is the set of accept states.

Example



$N_1 = (Q, \Sigma, \delta, q_1, F)$

where

- $Q = \{q_1, q_2, q_3, q_4\}$, $\Sigma = \{0, 1\}$,
- δ is

	0	1	ε
q_1	$\{q_1, q_2\}$	$\{q_1\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

- q_1 is the start state, and $F = \{q_4\}$.

Formal Model of Computation

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA, and
- w be a string over Σ_ε that has the form $y_1 y_2 \cdots y_m$ where $y_i \in \Sigma_\varepsilon$.
- u be the string over Σ obtained from w by omitting all occurrences of ε.

Suppose there is a sequence of *states* (in Q), $r_0, \ldots, r_n$, such that

- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, y_{i+1})$, $0 \leq i < n$
- $r_n \in F$

Then we say that M accepts u.

Equivalence of NFAs and DFAs

- Given an NFA, N, we construct a DFA, M, that accepts the same language.
- Let us first assume that there are no ε transitions (we will deal with them later).
- Make DFA simulate all possible NFA states.
- As consequence of the construction, if the NFA has k states, the DFA has 2^k states (an exponential blow up).

Equivalence of NFAs and DFAs

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA accepting A.

Construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$.

- $Q' = \mathcal{P}(Q)$.
- For $R \in Q'$ and $a \in \Sigma$, let

$$\delta'(R, a) = \{q \in Q | q \in \delta(r, a) \text{ for some } r \in R\}$$

- $q_0' = \{q_0\}$
- $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$

♣

Notice: F' is a set whose elements are subsets of Q, so (as expected) F' is a subset of $\mathcal{P}(Q)$.

Dealing with ε-Transitions

For any state R of M, define $E(R)$ to be the collection of states reachable from R by ε transitions only.

$E(R) = \{q \in Q | q$ can be reached from some $r \in R$ by 0 or more ε transitions$\}$

Define **transition function**:

$\delta'(R, a) = \{q \in Q |\ q \in E(\delta(r, a))$ for some $r \in R\}$

Change **start state** to

$$q_0' = E(\{q_0\})$$

Equivalence of NFAs and DFAs

Formally, use induction on m to show that if $y_1 y_2 \cdots y_m$ is a string over Σ^*, and the set of all possible states that $N = (Q, \Sigma, \delta, q_0, F)$ could reach on it is $R \subseteq Q$, then the deterministic DFA, M, reaches state R on $y_1 y_2 \cdots y_m$. Then, use the definition of acceptance by N, and of accept states for M.

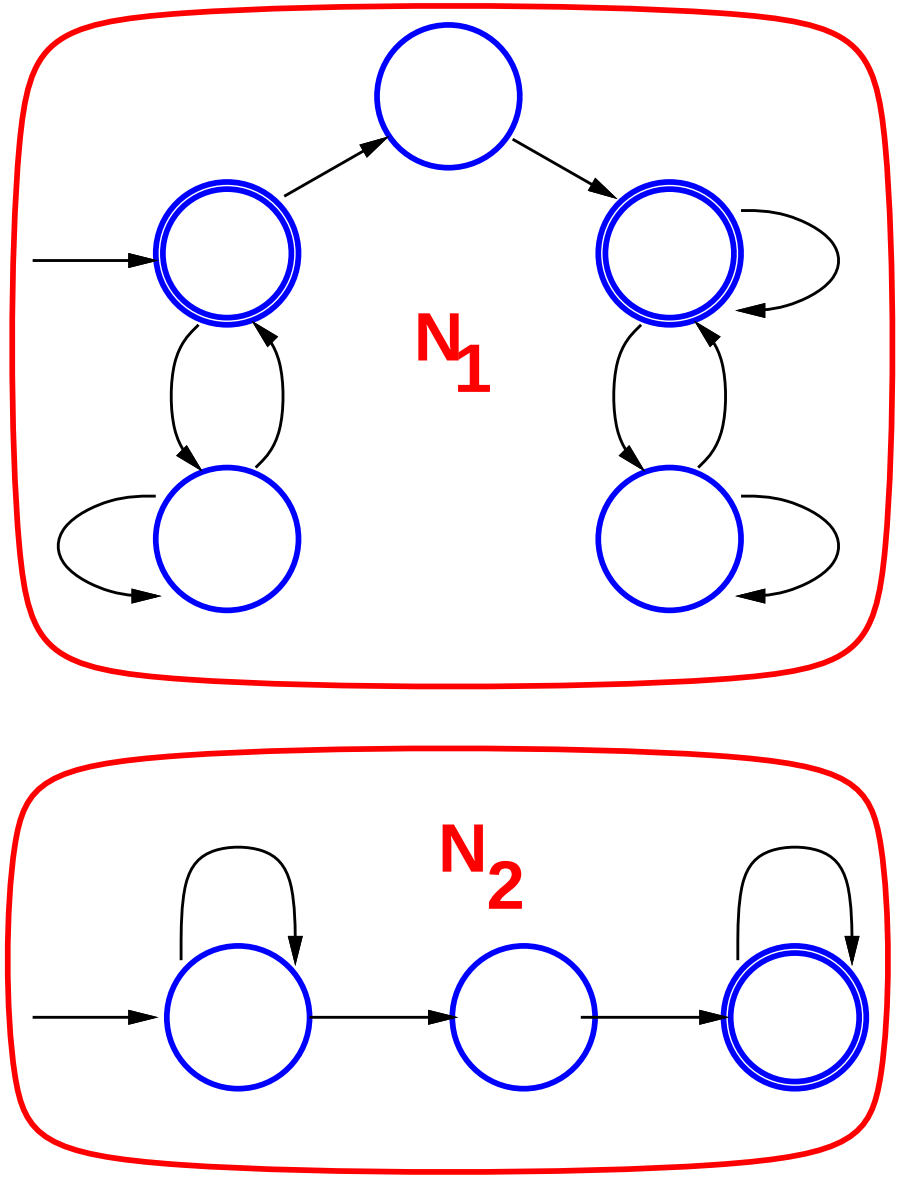
Regular Languages, Revisited

By definition, a language is regular if it is accepted by some DFA.

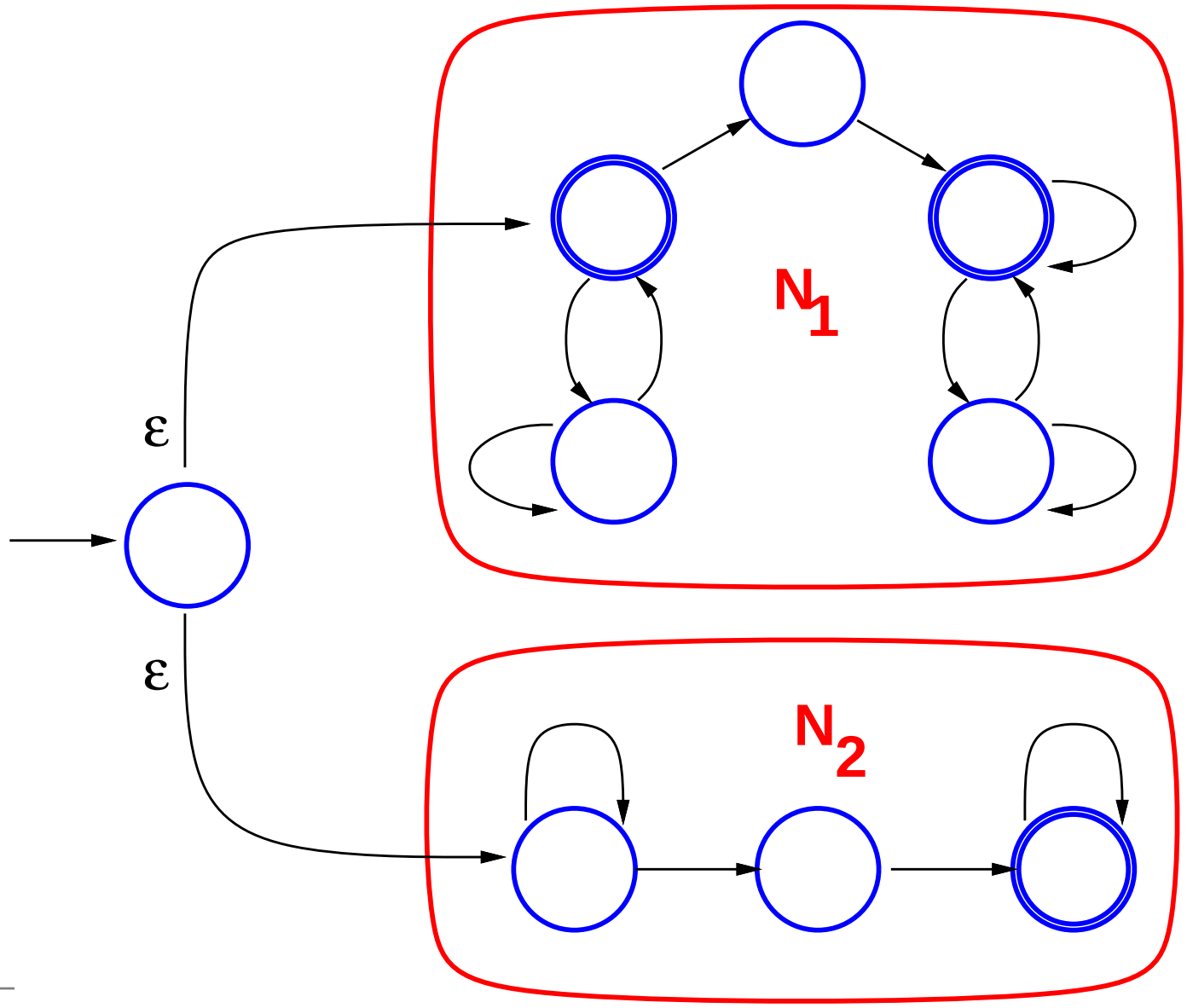
Corollary: A language is regular if and only if it is accepted by some NFA.

This is an alternative way of characterizing regular languages.
We will now use the equivalence to show that regular languages are closed under the regular operations (union, concatenation, star).

Closure Under Union (alternative proof)



Regular Languages Closed Under Union



Regular Languages Closed Under Union

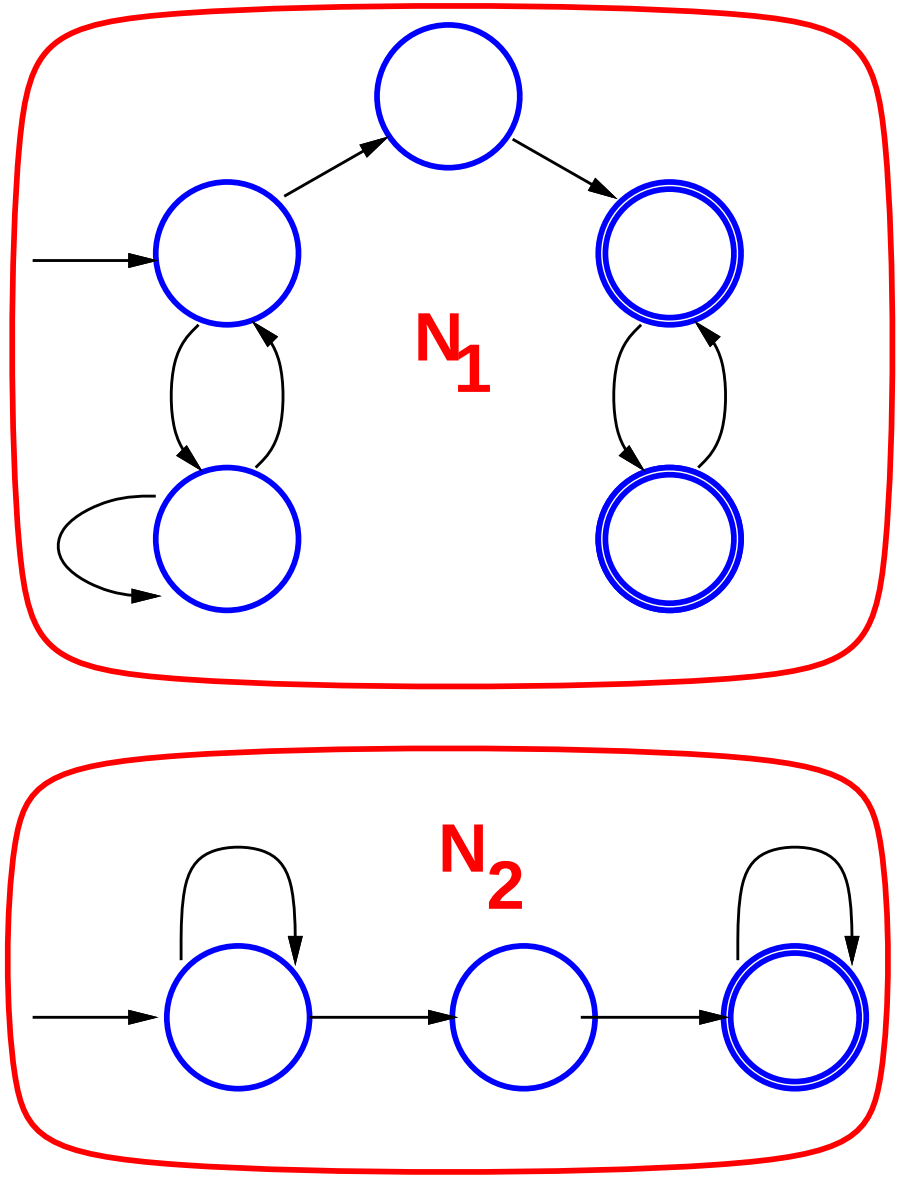
Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept L_1, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept L_2.

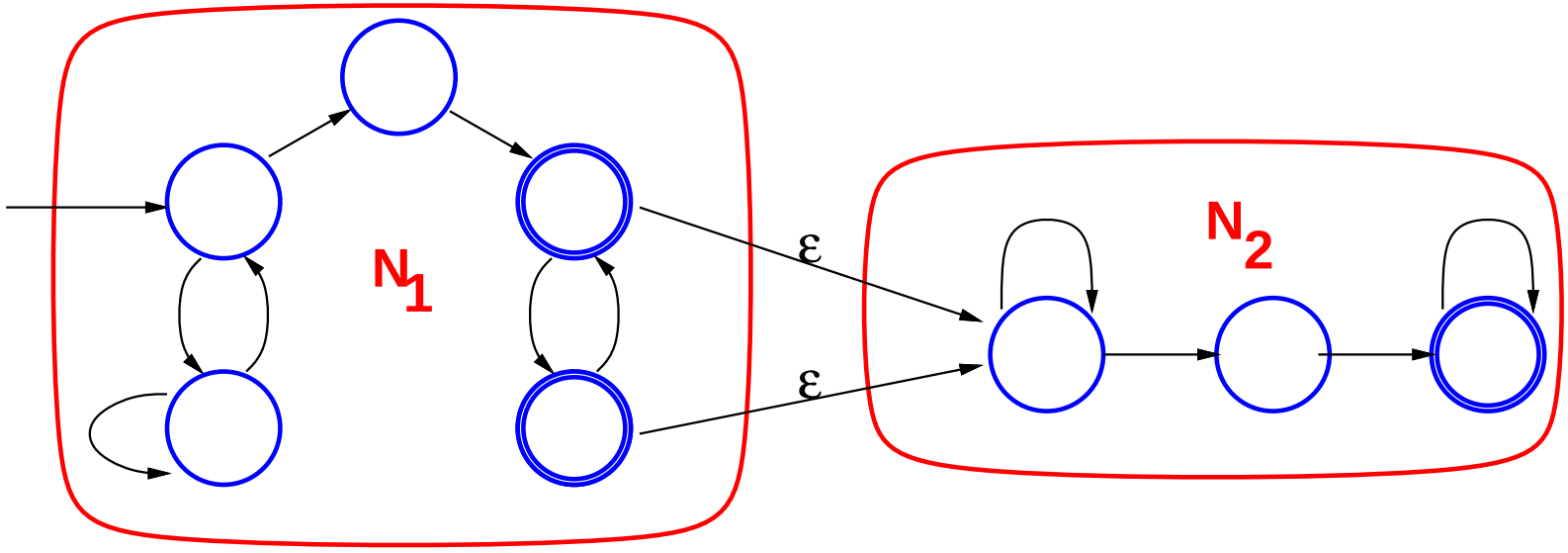
Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0\} \cup Q_1 \cup Q_2$
- Σ is the same, q_0 is the start state
- $F = F_1 \cup F_2$
- $$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Regular Languages Closed UnderConcatenation



Regular LanguagesClosed Under Concatenation



Remark: Final states are exactly those of N_2.

Regular LanguagesClosed Under Concatenation

Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept L_1, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept L_2.

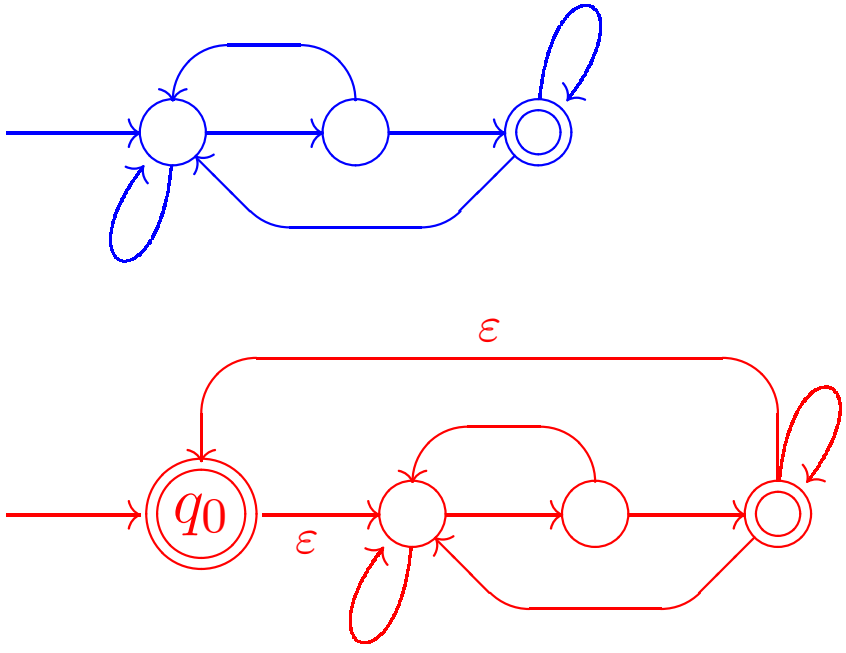
Define $N = (Q, \Sigma, \delta, q_1, F_2)$:

- $Q = Q_1 \cup Q_2$
- q_1 is the start state of N
- F_2 is the set of accept states of N

$$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\ \delta_2(q,a) & q \in Q_2 \end{cases}$$

Regular Languages Closed Under Star

N_1 accepts R_1. Wanna build NFA for $R = (R_1)^*$.



Regular Languages Closed Under Star

Suppose $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts L_1.
Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = \{q_0\} \cup Q_1$
- q_0 is the new start state.
- $F = \{q_0\} \cup F_1$

$$\delta'(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,\varepsilon) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$

Summary

- Regular languages are closed under
  - union
  - concatenation
  - star
- Non-deterministic finite automata
  - are equivalent to deterministic finite automata
  - but much easier to use in some proofs and constructions.

Regular Expressions

- A regular expression describes a language by means of single symbols from Σ, ϵ, and $\emptyset$, combined with $\cup, *$, and possibly with parentheses.
- Examples:
  - $\mathbf{0}$ and $\mathbf{1}$ are shorthand for $\{0\}$ and $\{1\}$
  - $(\mathbf{0} \cup \mathbf{1})$ is shorthand for $\{0, 1\}$.
  - $\mathbf{0}^*$ is shorthand for $\{0\}^*$.
  - $\mathbf{0}^*\mathbf{1}\mathbf{0}^*$ is shorthand for the set of all strings over $\Sigma = \{0, 1\}$ having exactly a single $\mathbf{1}$.
- Regular expressions are often used in text editors or shell scripts.

Regular Expressions – Formal Definition

The set of regular expressions over Σ is defined as follows:

- For every $a \in \Sigma$, $\mathbf{a}$ is a regular expression.
- $\boldsymbol{\epsilon}$ is a regular expression.
- $\boldsymbol{\emptyset}$ is a regular expression.
- For regular expressions $\mathbf{R_1}$ and $\mathbf{R_2}$, $(\mathbf{R_1} \cup \mathbf{R_2})$, $(\mathbf{R_1} \circ \mathbf{R_2})$ and $(\mathbf{R_1^*})$ are regular expressions.

Regular Expressions – Formal Definition

The relation between regular expressions and the languages they represent is established by a function $\mathcal{L}$, such that if $\mathbf{R}$ is a regular expression, then $\mathcal{L}(\mathbf{R})$ is the language represented by $\mathbf{R}$:

$\mathbf{R}$	$\mathcal{L}(\mathbf{R})$
$\mathbf{a}$	$\{a\}$
$\boldsymbol{\epsilon}$	$\{\varepsilon\}$
$\boldsymbol{\emptyset}$	$\emptyset$
$(\boldsymbol{R_1} \cup \boldsymbol{R_2})$	$\mathcal{L}(\boldsymbol{R_1}) \cup \mathcal{L}(\boldsymbol{R_2})$
$(\boldsymbol{R_1} \circ \boldsymbol{R_2})$	$\mathcal{L}(\boldsymbol{R_1}) \circ \mathcal{L}(\boldsymbol{R_2})$
$(\boldsymbol{R_1}^*)$	$\mathcal{L}(\boldsymbol{R_1})^*$

Some Shorter Notations

- Every language which can be represented by a regular expression, can be represented by infinitely many of them. For example:
  - $\mathcal{L}((\boldsymbol{R} \cup \emptyset)) = \mathcal{L}(\boldsymbol{R})$
  - $\mathcal{L}(((\boldsymbol{R_1} \cup \boldsymbol{R_2}) \cup \boldsymbol{R_3})) = \mathcal{L}((\boldsymbol{R_1} \cup (\boldsymbol{R_2} \cup \boldsymbol{R_3})))$
- Since union and concatenation are associative operations, we omit the extra parentheses in regular expressions.
- E.g., we treat $\boldsymbol{R_1} \cup \boldsymbol{R_2} \cup \boldsymbol{R_3}$ as a regular expression, although “officially" it is not.

Examples

Let $\Sigma = \{a_1, \ldots, a_n\}$ be an alphabet.

Notation: $\boldsymbol{\Sigma = (a_1 \cup a_2 \cup \ldots \cup a_n)}$

- $\mathcal{L}(\boldsymbol{\Sigma^*})$ - all strings.
- $\mathcal{L}(\boldsymbol{\Sigma^* 1})$ - all strings ending in 1.
- $\mathcal{L}(\boldsymbol{0\Sigma^* \cup \Sigma^* 1})$ - strings starting with 0 or ending in 1.
- $\mathcal{L}(\boldsymbol{\Sigma \cup \emptyset})$ - Σ
- $\mathcal{L}(\boldsymbol{\Sigma \cup \epsilon})$ - $\{a_1, a_2, \ldots, a_n, \varepsilon\}$
- $\mathcal{L}(\boldsymbol{\Sigma \circ \emptyset})$ - $\emptyset$
- $\mathcal{L}(\boldsymbol{\Sigma \circ \epsilon})$ - Σ

And More Examples

Let $\Sigma = \{0, 1\}$. Recall the notation $\mathbf{\Sigma} = (0 \cup 1)$.

- $\mathcal{L}(\mathbf{0^*10^*}) = \{\text{w}|\ \text{w includes a single 1}\}$
- $\mathcal{L}(\mathbf{\Sigma^*1\Sigma^*}) = \{\text{w}|\ \text{w includes at least one 1}\}$
- $\mathcal{L}((\mathbf{\Sigma\Sigma})^*) = \{\text{w}|\ \text{w has even length}\}$
- $\mathcal{L}(\mathbf{0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1}) =$
  $\{\text{w}|\ \text{w starts and ends with the same symbol}\}$
- $\mathcal{L}((\mathbf{0} \cup \varepsilon)\mathbf{1^*}) = \mathcal{L}(\mathbf{01^* \cup 1^*})$
- $\mathcal{L}(\mathbf{1^*\emptyset}) = \emptyset$

Remarkable Fact

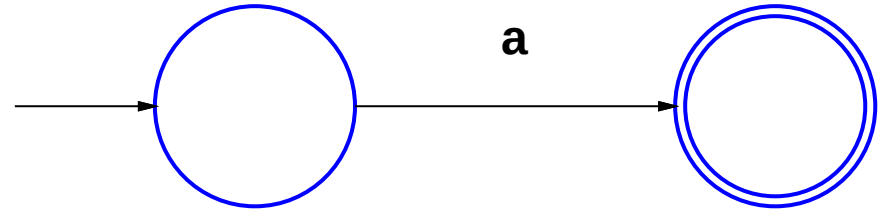
Thm.: A language, L, is described by a regular expression, $\mathbf{R}$, if and only if L is regular.

$\Longrightarrow$ construct an NFA accepting $\mathbf{R}$.

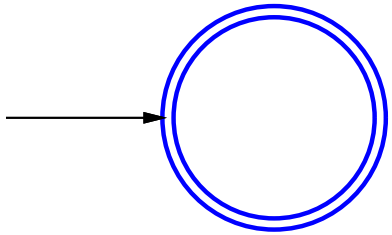
$\Longleftarrow$ Given a regular language, L, construct an equivalent regular expression.

Given R, Build NFA Accepting It ($\Longrightarrow$)

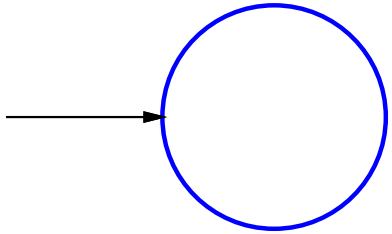
1. $\boldsymbol{R = \mathbf{a}}$, for some $a \in \Sigma$



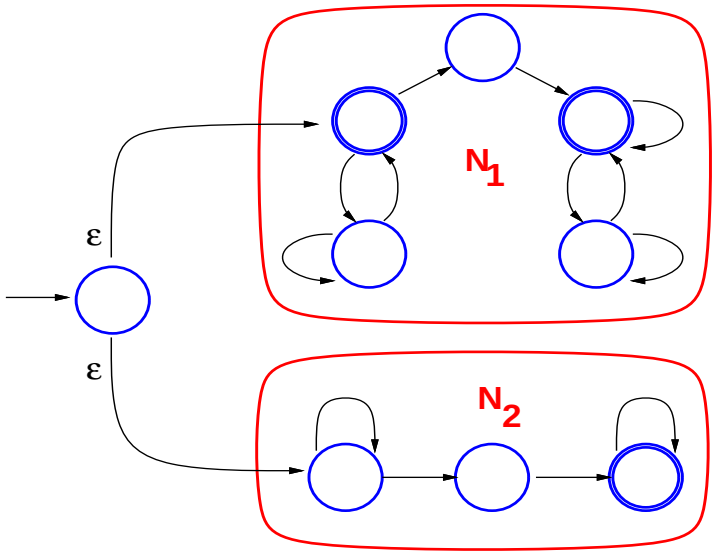
2. $\boldsymbol{R = \epsilon}$



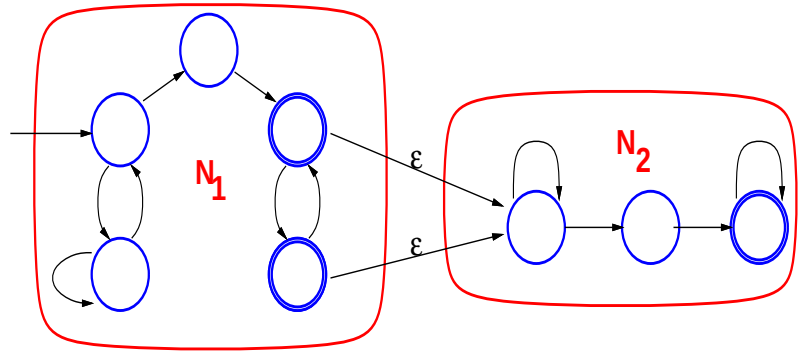
3. $\boldsymbol{R = \emptyset}$



Given R, Build NFA Accepting It ($\Longrightarrow$)

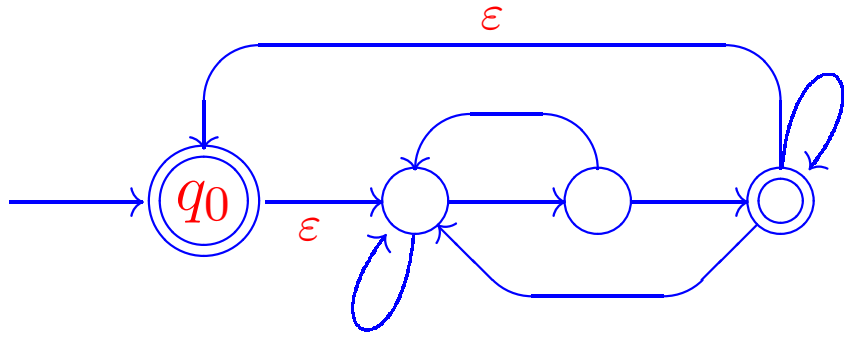


$\boldsymbol{R = (R_1 \cup R_2)}$



$\boldsymbol{R = (R_1 \circ R_2)}$

$\boldsymbol{R = (R_1)^*}$



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Regular Expression from an NFA ($\Longleftarrow$)

We now define generalized non-deterministic finite automata (GNFA).
An NFA:

- Each transition labeled with a symbol or ε,
- reads zero or one symbols,
- takes matching transition, if any.

A GNFA:

- Each transition labeled with a regular expression,
- reads zero or more symbols,
- takes transition whose regular expression matches string, if any.

GNFAs are natural generalization of NFAs.

A Special Form of GNFA

- Start state has outgoing arrows to every other state, but no incoming arrows.
- Unique accept state has incoming arrows from every other state, but no outgoing arrows.
- Except for start and accept states, an arrow goes from every state to every other state, including itself.

Easy to transform any GNFA into special form.

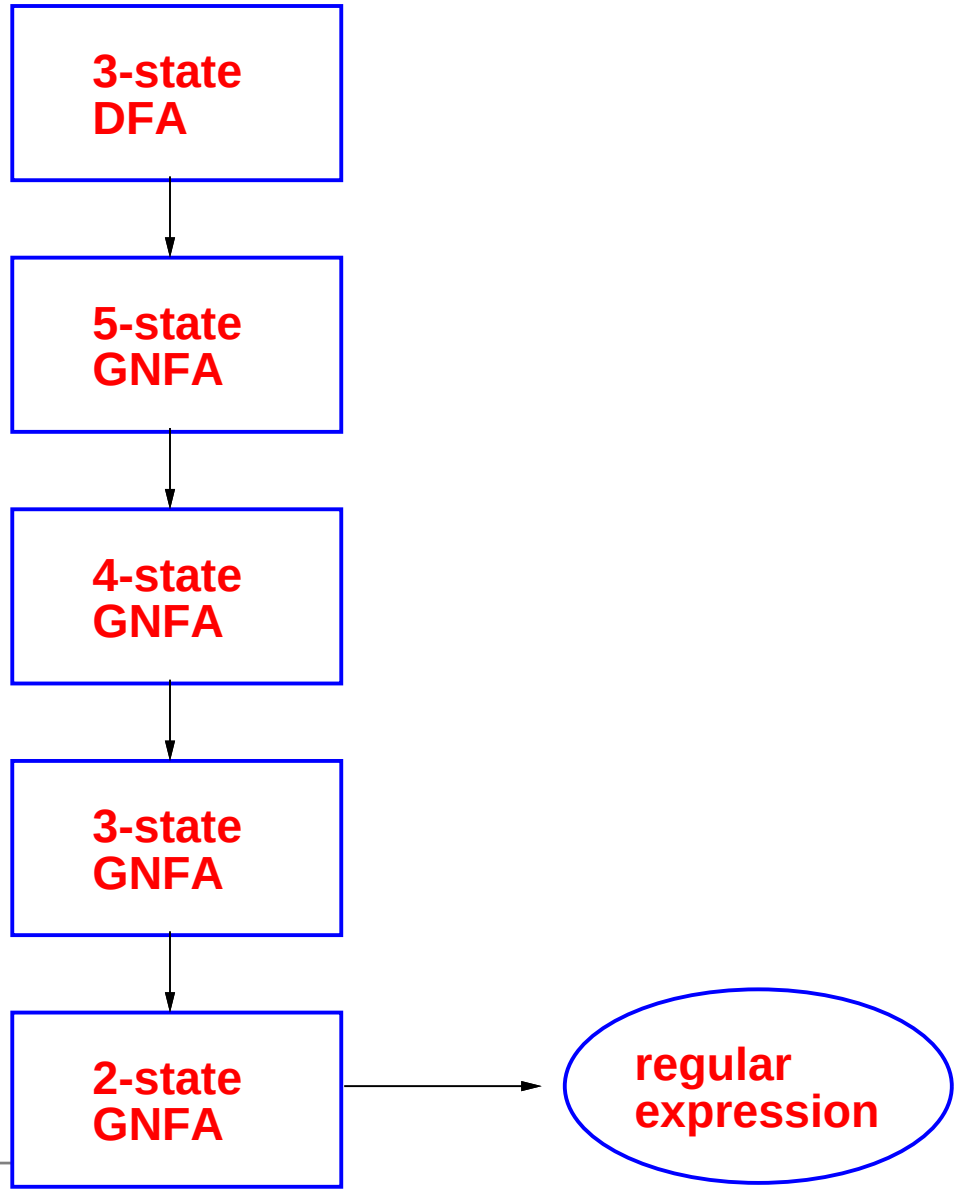
Really? How? . . .

Converting DFA to Regular Expression ($\Longleftarrow$)

Strategy – sequence of equivalent transformations

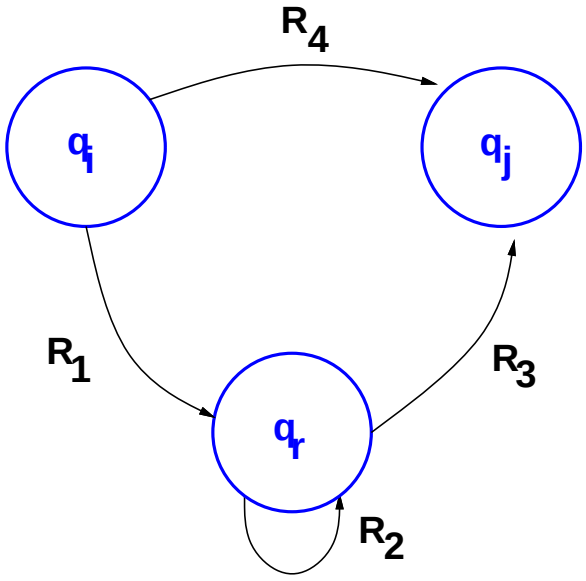
- given a k-state DFA
- transform into $(k+2)$-state GNFA
  - while GNFA has more than 2 states, transform it into equivalent GNFA with one fewer state
- eventually reach 2-state GNFA (states are just start and accept).
- label on single transition is the desired regular expression.

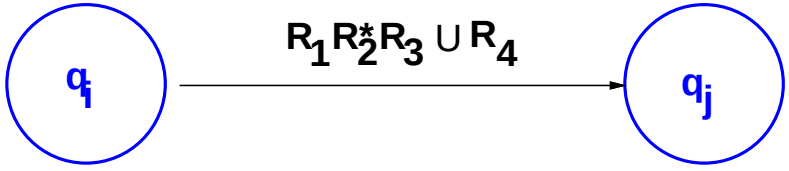
Converting Strategy ($\Longleftarrow$)



Removing One State

We remove one state q_r, and then repair the machine by altering regular expression of other transitions.





Formal Treatment – GNFA Definition

- q_s is start state.
- q_a is accept state.
- $\mathcal{R}$ is collection of regular expressions over Σ.

The transition function is

$$\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \rightarrow \mathcal{R}$$

If $\delta(q_i, q_j) = R$, then arrow from q_i to q_j has label R.

Arrows connect every state to every other state except:

- no arrow from q_a
- no arrow to q_s

Formal Definition

A generalized deterministic finite automaton (GNFA) is $(Q, \Sigma, \delta, q_s, q_a)$, where

- Q is a finite set of states,
- Σ is the alphabet,
- $\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \rightarrow \mathcal{R}$ is the transition function.
- $q_s \in Q$ is the start state, and
- $q_a \in Q$ is the unique accept state.

A Formal Model of GNFA Computation

A GNFA accepts a string $w \in \Sigma^*$ if there exists a *parsing* of w, $w = w_1 w_2 \cdots w_k$, where each $w_i \in \Sigma^*$, and there exists a sequence of states $q_0, \ldots, q_k$ such that

- $q_0 = q_s$, the start state,
- $q_k = q_a$, the accept state, and
- for each i, $w_i \in \mathcal{L}(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.
- (namely w_i is an element of the language described by the regular expression R_i.)

The CONVERT Algorithm

Given GNFA G, convert it to equivalent GNFA G'.

- let k be the number of states of G.
- If $k = 2$, return the regular expression labeling the only arrow.
- If $k > 2$, select any q_r distinct from q_s and q_a.
- Let $Q' = Q - \{q_r\}$.
- For any $q_i \in Q' - \{q_a\}$ and $q_j \in Q' - \{q_s\}$, let
  - $R_1 = \delta(q_i, q_r)$, $R_2 = \delta(q_r, q_r)$,
  - $R_3 = \delta(q_r, q_j)$, and $R_4 = \delta(q_i, q_j)$.
- Define $\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$.
- Denote the resulting $k - 1$ states GNFA by G'.

The CONVERT Procedure

We define the recursive procedure CONVERT$(\cdot)$:

Given GNFA G.

- Let k be the number of states of G.
- If $k = 2$, return the regular expression labeling the only arrow of G.
- If $k > 2$, let G' be the $k - 1$ states GNFA produced by the algorithm.

- Return CONVERT(G').

Correctness Proof of Construction

Theorem: G and CONVERT(G) accept the same language.

Proof: By induction on number of states of G

Basis: When there are only 2 states, there is a single label, which characterizes the strings accepted by G.

Induction Step: Assume claim for $k-1$ states, prove for k.

Let G' be the $k-1$ states GNFA produced from G by the algorithm.

G and G' accept the same language

By the induction hypothesis, G' and CONVERT(G') accept the same language.

On input G, the procedure returns CONVERT(G').

So to complete the proof, it suffices to show that G and G' accept the same language.

Three steps:

1. If G accepts the string w, then so does G'.
2. If G' accepts the string w, then so does G.
3. Therefore G and G' are equivalent.

Step One

Claim: If G accepts w, then so does G':

- If G accepts w, then there exists a "path of states" $q_s, q_1, q_2, \ldots, q_a$ traversed by G on w, leading to the accept state q_a.
- If q_r does not appear on path, then G' accepts w because the the new regular expression on each edge of G' contains the old regular expression in the "union part".
- If q_r does appear, consider the regular expression corresponding to $\ldots q_i, q_r, \ldots, q_r, q_j \ldots$.
  The new regular expression $(R_{i,r})(R_{r,r})^*(R_{r,j})$ linking q_i and q_j encompasses any such string.
- In both cases, the claim holds.

Steps Two and Three

Claim: If G' accepts w, then so does G.

Proof: Each transition from q_i to q_j in G' corresponds to a transition in G, either directly or through q_r. Thus if G' accepts w, then so does G.

- This completes the proof of the claim that $L(G) = L(G')$.
- Combined with the induction hypothesis, this shows that G and the regular expression CONVERT(G) accept the same language.
- This, in turn, proves our *remarkable claim*:
  A language, L, is described by a regular expression, R, if and only if L is regular. ♣