Computational Models

- Deterministic Time Classes
- NonDeterministic Time Classes
- Relationship between Deterministic and Nondeterministic Time
- The classes P and NP
- Examples of Problems in P and in NP
- Verifiability
- The class coNP

- Sipser, Chapter 7

Definition

Let M be a deterministic TM that halts on all inputs. The running time of M is a function

$$f : \mathcal{N} \rightarrow \mathcal{N}$$

where $f(n)$ is the maximum number of steps that M uses on any on input of length n.

If $f(n)$ is the running time of M, we say that M runs in time $f(n)$, and that M is an $f(n)$-TM.

Running Time

- The exact running time of most algorithms is quite complex, better to estimate it.
- Informally, we want to focus on “important" parts only.
- Example:In $6n^3 + 2n^2 + 20n + 45$, the term n^3 is the most important one.

Asymptotic Notation

Consider functions

$$f, g : \mathcal{N} \rightarrow \mathcal{R}^+$$

We say that

$$f(n) = O(g(n))$$

if there exist positive integers c and n_0, such that

$$f(n) \leq c \cdot g(n)$$

for all $n \geq n_0$.

Examples

- $5n^3 + 2n = O(n^3)$
- $617n^4 + 234n^3 + 20n^2 + 5n = O(n^4)$
- $sin(x) = O(1)$
- $2 = O(1)$.

Time Complexity

Consider

$$A = \{0^m 1^m \mid m \geq 0\}$$

Clearly this language is decidable.

Question: How much time does a single-tape TM need to decide it?

Time Complexity

M_1: On input string w,

1. Scan across tape and reject if 0 is found to the right of a 1.
2. While both 0s and 1s appear on tape, repeat the following:
   - scan across tape, crossing of a single 0 and a single 1 in each pass.
3. If no 0s and 1s remain, accept, otherwise reject.

Analysis (1)

We consider the three stages separately. Let n denote the input length.

1. Scan across tape and reject if 0 is found to the right of a 1. If not, return to starting point.

- Scanning requires n steps.
- Re-positioning head requires n steps.
- Total is $2n = O(n)$ steps.

Analysis (2)

2. While both 0s and 1s appear on tape, repeat the following

- scan across tape, crossing of a single 0 and a single 1 in each pass.

- Each scan requires $O(n)$ steps.
- Since each scan crosses off two symbols, the number of scans is at most $n/2$.
- Total number of steps is $(n/2) \cdot O(n) = O(n^2)$.

Analysis (3)

3. If 0s still remain after all 1s have been crossed out, or vice-versa, reject. Otherwise, if the tape is empty, accept.

- Single scan requires $O(n)$ steps.
- Total is $O(n)$ steps.

Final Analysis

Total cost for stages

1. $O(n)$
2. $O(n^2)$
3. $O(n)$

which is $O(n^2)$

Deterministic Time (Reminder)

Let M be a deterministic TM, and let

$$t : \mathcal{N} \longrightarrow \mathcal{N}$$

We say that M runs in time $t(n)$ if

- For every input x of length n,
- the number of steps that M uses,
- is at most $t(n)$.

Time Classes Definition

Let

$$t : \mathcal{N} \longrightarrow \mathcal{N}$$

be a function.

Definition:

$$\text{DTIME}(t(n)) \quad = \quad \{L \mid L \text{ is a language, decided by an } O(t(n))\text{-time DTM}\}$$

Note that $t(n)$ run time is also required for strings that are not in L.

Can we do it faster?

- Using a naive algorithm, we have seen that $A = \{0^m 1^m \mid m \geq 0\} \in$ DTIME(n^2).
- Can we do it faster?
- Using a more sophisticated, "binary representation algorithm", we can show: $A \in$ DTIME$(n \log n)$.
- In fact this is the best one can do on a single tape TM.

The Improvement

M_2 on input string w:

1. Scan across tape and reject if 0 is found to the right of 1.
2. Repeat the following while both 0 and 1 appear on tape:
   (a) scan across tape, checking whether total number of 0s plus 1s is even or add. If odd, reject.
   (b) scan across tape, crossing off every other 0 (starting with the first), and every other 1 (starting with the first) in each pass.
3. If no 0s or 1s remain, accept, otherwise reject.

Analysis

First, we verify that M_2 indeed halts.

- On each scan in step 2a:
  - The total number of 0s ic cut in half,
  - and if there was a remainder, it is discarded.
  - Same for 1s.
- Example: start with 13 0s and 13 1s,
  - first pass: 6 0s and 6 1s are left
  - second pass: 3 0s and 3 1s are left
  - third pass: one 0 and one 1 are left
  - then no 0s and 1s are left.

Further Improvements?

Question: Can the running time be made $o(n \log n)$?

Answer: Not on a single tape TM.

Question: But why do we have to stick with single tape TMs?

Answer: We don't!

A Two Tape TM

M_3: on input string w

1. Scan across tape and reject if 0 is found to the right of a 1.
2. Scan across 0s to first 1, copying 0s to tape 2.
3. Scan across 1s on tape 1 until the end. For each 1, cross off a 0. If no 0s left, $reject$.
4. If any 0s left, $reject$, otherwise $accept$.

Question: What is the running time?

Complexity

Deciding $\{0^n 1^n\}$:

- single-tape M_1: $O(n^2)$.
- single-tape M_2: $O(n \log n)$ (fastest possible!).
- two-tape M_3: $O(n)$.

Important difference between complexity and computability:

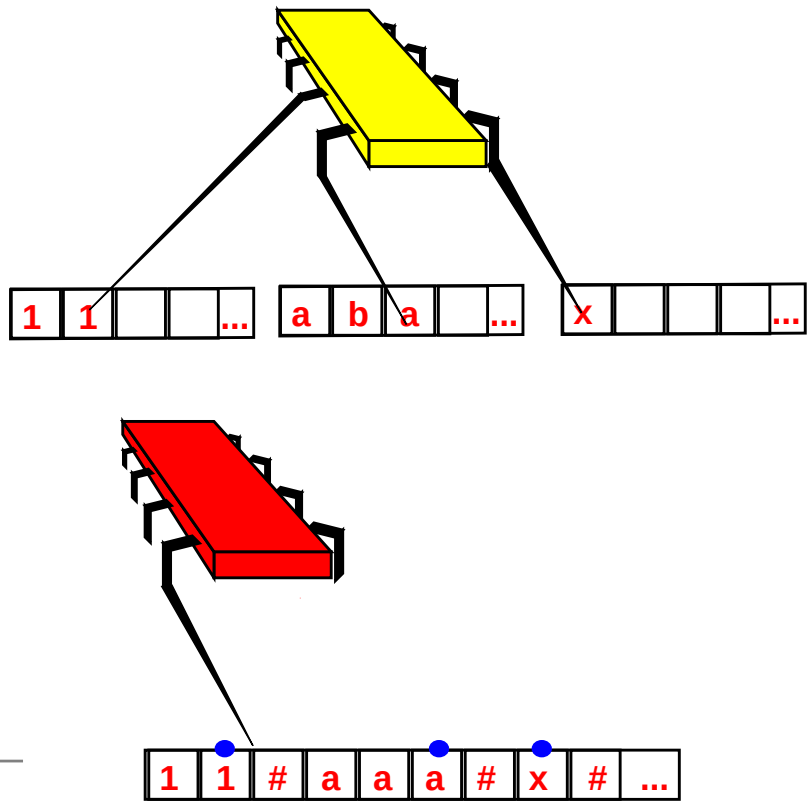
- Computability: all reasonable models equivalent (Church-Turing)
- Complexity: choice of model does affect run-time.

Q: By how much does model affect complexity?

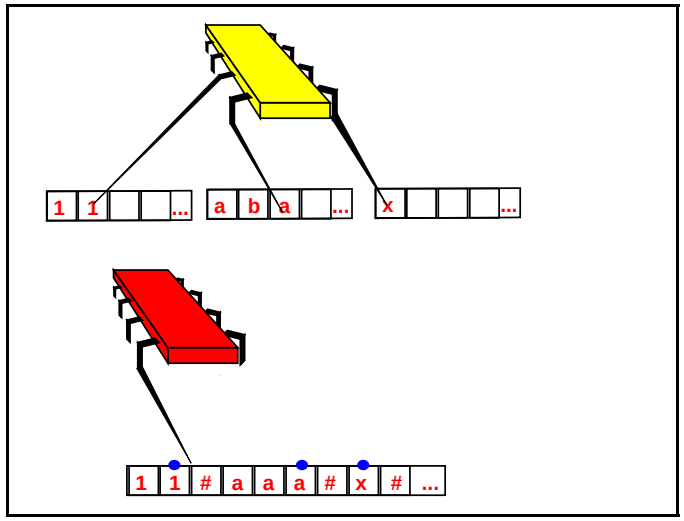
Models and Complexity

Let $t(n)$ be a function where $t(n) \geq n$, and let $L \subseteq \Sigma^*$ be a language.

Claim: If a $t(n)$-time multitape TM decides L, then there is an $O(t^2(n))$-time single tape TM that decides L.



Reminder: Simulating MultiTape TMs



On input $w = w_1 \cdots w_n$, single tape S:

- puts on its tape $\# \; \dot{w}_1 \; w_2 \cdots w_n \# \; \dot{\sqcup} \; \# \; \dot{\sqcup} \; \# \cdots \#$
- scans its tape from first $\#$ to $k+1$-st $\#$ to read symbols under "virtual" heads.
- rescans to write new symbols and move heads
- if S tries to move virtual head onto $\#$, then M takes "tape fault" and re-arranges tape.

Complexity of Simulation

For each step of M, S performs

- two scans
- up to k rightward shifts

On input of length n, M makes $O(t(n))$ many steps, so active portion of each tape is $O(t(n))$ long.

Total number of steps S makes:

- $O(t(n))$ steps to simulate one step of M.
- Total simulation $O(t(n)) \times O(t(n)) = O(t^2(n))$.
- Initial tape arrangement $O(n)$.
- Grand total: $O(n) + O(t^2(n)) = O(t^2(n))$ steps,
- under the reasonable assumption (why?) that $t(n) > n$.

Non-Deterministic Time

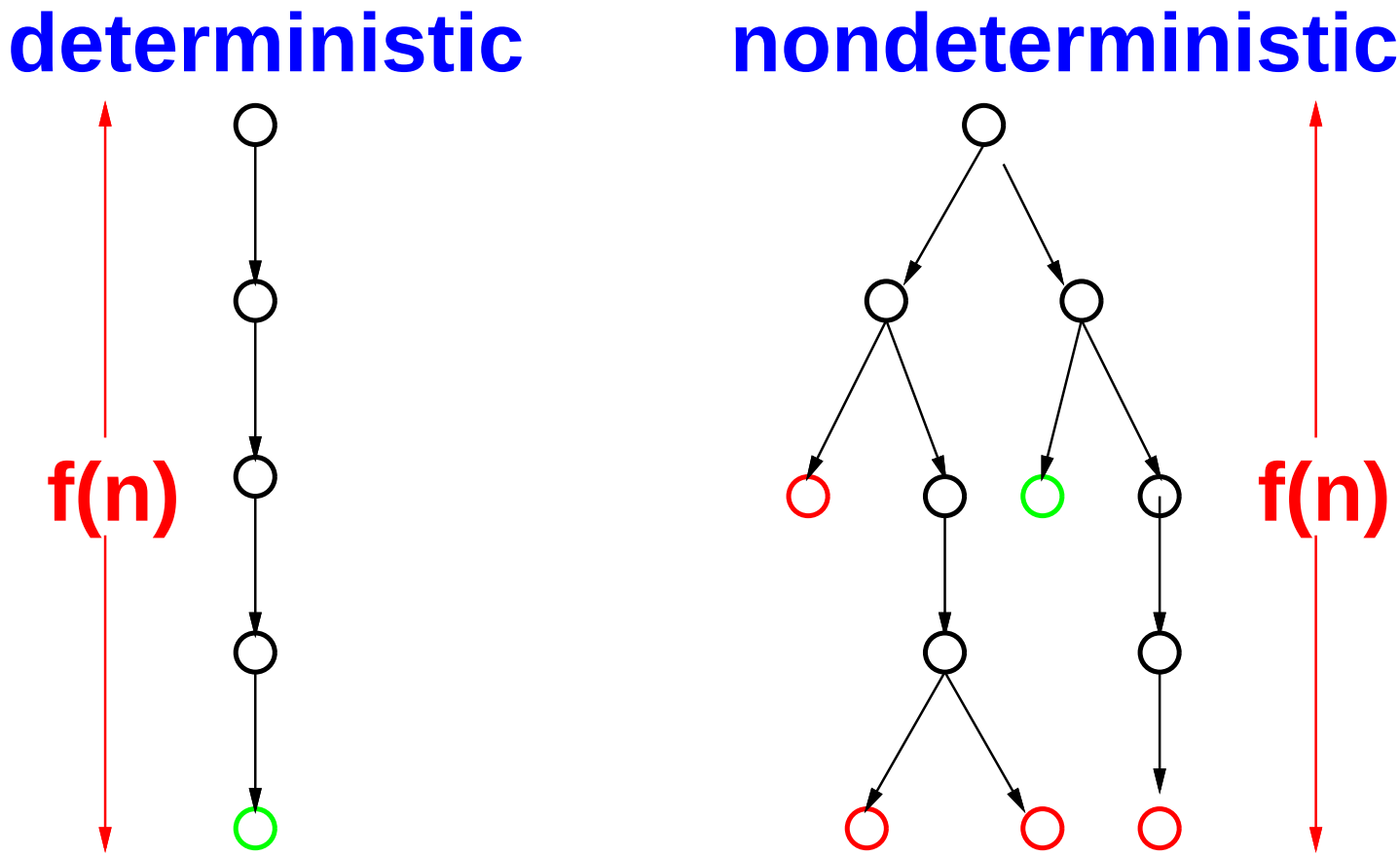
Let N be a non-deterministic TM, and let

$$f : \mathcal{N} \longrightarrow \mathcal{N}$$

We say that N runs in time $f(n)$ if

- For every input x of length n,
- the maximum number of steps that N uses,
- on any branch of its computation tree on x,
- is at most $f(n)$.

Deterministic vs. Non-Deterministic



Notice that non-accepting branches must reject within $f(n)$ many steps.

Models and Complexity

Claim: Suppose N is a nondeterministic TM that runs in time $t(n)$ and decides the language L.

Then there is an $2^{O(t(n))}$-time deterministic TM, D, that decided L.

Note contrast with multi-tape result.

Simulation

Let N be a non-deterministic TM running in $t(n)$ time. Want to describe the deterministic TM, D, simulating N.

Basic idea of simulation:

- D tries all possible branches.
- If D finds any accepting state, it accepts.
- If all branches reject, D rejects.
- Notice N has no looping branches, so exactly one of two possibilities must occur.

Simulation Details

N's computation is a tree:

- root is starting configuration,
- each node has bounded fanout $\leq b$ (why?),
- each branch has length $\leq t(n)$,
- total number of leaves at most $b^{t(n)}$,
- total number of nodes in tree $O\left(b^{t(n)}\right)$,
- time to arrive from root to any node is $O(t(n))$.
- $\Longrightarrow$ Time to visit all nodes is

$$O\left(t(n) \times b^{t(n)}\right) = O\left(2^{O(t(n))}\right) .$$

Remark

Breadth-first search used in simulation

- Inefficiently traverses from root to visit each node.
- Can be improved upon by using depth-first search (why is it OK now?) or other tree search strategies.
- Still, doing this may save constants, but nothing substantial (why?)

Important Distinction

- At most polynomial gap in time to perform tasks between different deterministic models (single- vs. multi-tape TMs, TM vs. RAM, etc.)
- compared to
- Apparently exponential gap in time to perform tasks between deterministic and non-deterministic models.

The Good, the Bad, and the Ugly

Complexity differences:
Polynomial is small; Exponential is large

	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.00001 second	.00004 second	.00009 second	.00016 second	.00025 second	.00036 second
n^3	.00001 second	.00008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minute	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	$2 \cdot 10^8$ centuries	$1.3 \cdot 10^{13}$ centuries

Polynomial is Good, Exponential is Bad

Claim: All "reasonable" models of computation are polynomially equivalent.

Any one can simulate another with only polynomial increase in running time.

Question: Is a given problem solvable in

- Linear time? model-specific.
- Polynomial time? model-independent.
- We are interested in computation, not in models *per se*!

The Class P

Definition: P is the set of languages decidable in polynomial time on deterministic TMs.

$$P = \bigcup_{c \geq 0} \text{DTIME}(n^c)$$

- The class P is important because:
- Invariant for all TMs with any number of tapes.
- Invariant for all models of computation polynomially equivalent to TMs.
- Roughly corresponds to realistically solvable (tractable) problems.

The Class P

- Invariant for all models of computation polynomially equivalent to deterministic TMs
  - not affected by particulars of model . . .
  - go ahead, have another tape, they're pretty small and inexpensive . . .

The Class P

- Roughly corresponds to realistically solvable (tractable) problems.
  - actually depends on context
  - going from exponential to polynomial algorithm usually requires major insight,
  - if you find an inefficient polynomial algorithm, you can often find a more efficient one.

Examples: Problems in P

- Integer arithmetic: Addition, subtraction, multiplication, division with remainder.
- Modular arithmetic: Exponentiation (RSA), inverse.
- Integer Algorithms: Greatest common divisor (gcd).
- Operations research: Maximum network flow, linear programming,
- Algebra: Matrix multiplication, computing determinants, matrix inversion, solving systems of linear equations, factoring polynomials.
- Graph algorithms: DFS and DFS in graphs, minimum spanning trees, finding Eulerian path.

Typical analysis

- break algorithm into stages
- check that each stage is polynomial
- check that number of stages is polynomial
- Ergo, the algorithm is polynomial.

Encoding

For numbers

- binary is good
- unary is not realistic (exponentially longer)

For graphs

- list of nodes and edges (good)
- adjacency matrix (good)

Path

Given

- directed graph G
- nodes s and t
- is there a path from s to t?

$$\text{PATH} = \{\langle G, s, t\rangle | G \text{ has directed path from } s \text{ to } t\}$$

Complexity of PATH

Theorem:

$$\text{PATH} \in P$$

When in doubt, try brute force :-)

- let m be the number of nodes in G
- any path from s to t need not repeat nodes
- examine each path in G of length $\leq m$,
- check if it goes from s to t.

Question: What is the complexity of this algorithm?

Complexity of PATH

Theorem:

$$\text{PATH} \in P$$

1. Place mark on s
2. Repeat until no additional nodes marked:
   - scan edges of G.
   - If edge (a, b) found from marked node a to unmarked node b,
   - then mark node b.
3. If t marked, accept, otherwise reject.

Question: What is the complexity of this algorithm?

Complexity of PATH

1. Place mark on s
2. Repeat until no additional nodes marked:
   - scan edges of G.
   - If edge (a, b) found from marked a to unmarked b,
   - then mark b.
3. If t marked, accept, otherwise reject.

How many stages?

- Stages 1 and 3 run once.
- Stage 2 runs at most m times, because each time (except last) it marks at least one new node.

Total number of stages is polynomial.

Path

1. Place mark on s
2. Repeat until no additional nodes marked:
   - scan edges of G.
   - If edge (a, b) found from marked a to unmarked b,
   - then mark b.
3. If t marked, accept, otherwise reject.

How much is each stage?

- Stages 1 and 3 polynomial.
- Stage 2 scans and marks nodes in graph, also polynomial.

Total time complexity is polynomial. ♣

Relative Primality

Two numbers are relatively prime if 1 is the largest integer that evenly divides them both.

- 10 and 21 are relatively prime
- 10 and 22 are not.

Definition:

$$\text{RELPRIME} = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$$

Relative Primality

RELPRIME $= \{\langle x, y\rangle |\ x$ and y are relatively prime$\}$

First Idea: Search through all possible divisors of x, y and test divisibility.

If x, y in unary:

- size of $\langle x\rangle$ is x
- testing all potential divisors of x, y is polynomial

If x, y in binary:

- size of $\langle x\rangle$ is $\log x$
- testing all potential divisors of x, y is exponential

Important Notation: Such algorithm is called pseudo-polynomial.

GCD

RELPRIME $= \{\langle x, y\rangle |\ x$ and y are relatively prime$\}$

Euclid's greatest common divisor algorithm, E:

On input $\langle x, y\rangle$

1. Repeat until $y = 0$
   - $x \leftarrow x \bmod y$
   - exchange x and y
2. output x

R for RELPRIME : on input $\langle x, y\rangle$

- Run E on $\langle x, y\rangle$
- if the result is 1, accept, otherwise reject.

Euclid's Algorithm

Enough to check that Euclid's Algorithm is polynomial.

- each execution of Stage 1 cuts x by at least half (check details)
- after each loop $x < y$
- values swapped
- number of stages is $\min(\log_2 x, \log_2 y)$

total running time is polynomial.

Correctness holds (but should be verified).

Consequently, RELPRIME $\in P$. ♣

Decidability of Context-Free Languages

- Theorem: Every context-free language is a member of P.
- We already know that there is an algorithm for each CFL that decides it. Is there one that runs in polynomial time?
- First attempt: Let L be a CFL generated by a CFG in Chomsky Normal Form. Then any $w \in L$ of length has a derivation of $2n - 1$ steps. Try all possible derivations of this length, and if there is one of w - accept. Else - reject.
- But the number of derivations of k steps may be exponential in k.

Dynamic Programming Approach

- Technique of dynamic programming: accumulation of information about smaller subproblems to solve larger problems.
- Record the solution to any subproblem so that we don't need to solve it later in a table.
- The subproblem: determine whether each variable of G generates each substring of w.
- The (i, j)-th entry of the table contains all variables that generate the substring $w_i \ldots w_j$.
- For $j < i$, the entries are not used.

Dynamic Programming Approach - cont.

- The algorithm fills the entries of the table for each substring of w: first of length 1, then of length 2,...
- Suppose that the algorithm has determined which variables generate of substrings up to length k. To determine whether variable A generates a substring of length $k+1$, we split it to two non-empty parts in k possible ways.
- For each split, we examine each rule $A \rightarrow BC$ to see whether B generates the first part, and C generates the second part.

The Algorithm

Let G be a CFG in Chomsky Normal Form. On input $w = w_1 \ldots w_n$:

1. If $w = \epsilon$ and $S \rightarrow \epsilon$, accept.
2. For i = 1 to n and each variable A, test whether $A \rightarrow b$ is a rule, where $b = w_i$. If so, place A in table(i,i).
3. For $l = 2$ to n:
4. For $i = 1$ to $n - l + 1$, let $j = i + l - 1$ (j *is the end position of the substring*).
5. For $k = i$ to $j - 1$:
6. for each rule $A \rightarrow BC$: if table(i,k) contains B and table(k+1,j) contains C, put A in table(i,j).
7. If S is in table(1,n), accept. Else reject.

Analysis

- Stage 2 runs at most Vn times, where V is the number of variables of G and is a constant (independent of $|w| = n$) $\Rightarrow O(n)$
- The inner loop runs in $O(n^3)$ (the number of variables is a constant independent of $|w|$).
- Overall: $O(n^3)$

NTime Classes Definition

Let

$$f : \mathcal{N} \longrightarrow \mathcal{N}$$

be a function.

Definition:

$$\text{NTIME}(f(n)) = \{L \mid L \text{ is a language, decided by an } O(f(n))\text{-time NTM}\}$$

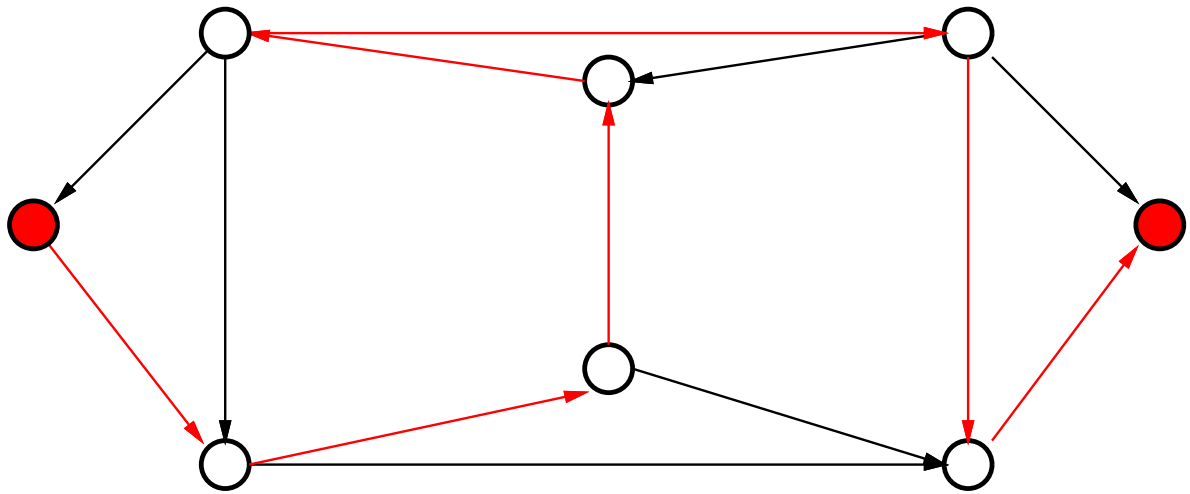
The Class NP

Definition: NP is the set of languages decidable in polynomial time on non-deterministic TMs.

$$NP = \bigcup_{c \geq 0} \text{NTIME}(n^c)$$

- The class NP is important because:
- Invariant for all TMs with any number of tapes.
- NP is insensitive to choice of reasonable non-deterministic computational model.
- Rougly corresponds to problems whose positive solutions cannot be efficiently generated ($\Rightarrow$ intractable), but can be efficiently verified.

Hamiltonian Path



A Hamiltonian path in a directed $\mathcal{G}$ visits each node exactly once.

Hamiltonian Path

$$\text{HAMPATH} = \{\langle G, s, t\rangle | G \text{ has Hamiltonian path from } s \text{ to } t\}$$

Question: How hard is it to decide this language?

Hamiltonian Path

$$\text{HAMPATH} = \{\langle G, s, t\rangle | G \text{ has Hamiltonian path from } s \text{ to } t\}$$

Easy to obtain exponential time algorithm:

- generate each potential path
- check whether it is Hamiltonian

The Class $\mathcal{NP}$

Here is an NTM that decides HAMPATH in poly time.

On input $\langle G, s, t \rangle$,

1. Guess and write down a list of numbers $p_1, \ldots, p_m$, where m is number of nodes in G, and $1 \leq p_i \leq m$.
2. Check for repetitions in list. If any found, *reject*.
3. Check whether $p_1 = s$ and $p_m = t$. If either does not hold, *reject*.
4. For $i, 1 \leq i \leq m - 1$, check whether (p_i, p_{i+1}) is an edge in G. If any is not, *reject*. Otherwise accept.

NP

On input $\langle G, s, t \rangle$,

1. Non-deterministically guess and write down a list of numbers $p_1, \ldots, p_m$.
2. Check that there are no repetitions.
3. Check whether $p_1 = s$ and $p_m = t$.
4. Check whether (p_i, p_{i+1}) is an edge in G.
5. If 1 through 4 satisfied, accept, else reject.

- Stage 1 polynomial time
- Stages 2 and 3 simple checks.
- Stage 4 simple poly-time too. $\surd$

Hamiltonian Path

This problem has one very interesting feature: polynomial verifiability.

- we don't know a fast way to find a Hamiltonian path
- but we can check whether a given path is Hamiltonian in polynomial time.

In other words,

- verifying correctness of a path is much easier
- than determining whether one exists

Composite Numbers

A natural number is composite if it is the product of two integers greater than one.

$$\text{COMPOSITES} = \{x \mid x = pq \text{ for integers } p, q > 1\}$$

- we don't know a polynomial-time algorithm for deciding this problem*
- But we can easily verify that a nubmer is composite.

Verifiability

Not all problems are polynomially verifiable.

There is no known way to verify $\overline{\text{HAMPATH}}$ in polynomial time.

In fact, we will see many examples where L is polynomially verifiable, but its complement, $\overline{L}$, is not known to be polynomially verifiable.

Verifiability

A verifier for a language $\mathcal{A}$ is an algorithm $\mathcal{V}$ where

$$\mathcal{A} = \{w \mid \mathcal{V} \text{ accepts } \langle w, c\rangle \text{ for some string } c\}$$

- The verifier uses the additional information c to verify $w \in \mathcal{A}$.
- We measure verifier run time by length of w.
- The string c is called a certificate (or proof) for w if $\mathcal{V}$ accepts $\langle w, c\rangle$.
- A polynomial verifier runs in polynomial time in $|w|$ (so $|c| \leq |w|^{O(1)}$).
- A language $\mathcal{A}$ is polynomially verifiable if it has a polynomial verifier.

Examples

For HAMPATH, a certificate for

$$\langle G, s, t \rangle \in \text{HAMPATH}$$

is simply the Hamiltonian path from s to t.

Can verify in time polynomial in $|\langle G \rangle|$ whether given path is Hamiltonian.

Examples

For COMPOSITES, a certificate for

$$x \in \text{COMPOSITES}$$

is simply one of its divisors.

Can verify in time polynomial in $|x|$ if given divisor indeed divides x.

NP and Verifiability

Theorem: A language is in NP if and only if it has a polynomial time verifier.

Proof – Intuition:

- NTM simulates verifier by guessing the certificate.
- Verifier simulates NTM by using accepting branch as certificate.