Convergence to Strong Equilibrium in Network Design Games

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Motivation

- Companies seeking to construct physical transmission channels (e.g. signal cables).
- The cost of each link is a function of the number of users.
- Load on a link has a positive effect.
- Agents are selfish (cost minimizers).
A network is a directed graph \((V, E)\), a source node \(s\), and a sink node \(t\), so that every node and every edge is on a path from \(s\) to \(t\).

Example:
Network design games

The model

- In a network $D = (V, E, s, t)$, all agents have the same origin $s$, and arbitrary destinations.
- Each agent $i$ selects a path from $s$ to its destination $d_i$.
- The cost-per-agent on edge $e$ is $c_e(\ell)$, where $\ell$ is the load.
- $c_e(\ell)$ is non-increasing: If $\ell < k$ then $c_e(\ell) \geq c_e(k)$.
- Cost of agent $i$ is the sum of the costs of edges in $i$’s path.

Example. Fair cost sharing: Each edge $e$ has a constant cost $x_e$, and $c_e(\ell) = \frac{x_e}{\ell}$.
Beneficial Coalitional Deviation (BCD)

A deviation in which each member of the coalition strictly decreases its cost.
Solution concept

Beneficial Coalitional Deviation (BCD)

A deviation in which each member of the coalition strictly decreases its cost.

Notation: \((c_e(1), c_e(2))\)

Cost = 4, Cost = 4
Beneficial Coalitional Deviation (BCD)

A deviation in which each member of the coalition strictly decreases its cost.

Notation: \((c_e(1), c_e(2))\)

Cost = 2, Cost = 2
Solution concept

Beneficial Coalitional Deviation (BCD)

A deviation in which each member of the coalition strictly decreases its cost.

**Notation:** \((c_e(1), c_e(2))\)

![Diagram](image)

Cost = 2, Cost = 2

**Strong equilibrium (SE)**

An outcome where no BCD exists.
Existence of strong equilibrium

[Holzman and Monderer, 2014] A strong equilibrium does not necessarily exist

Cost = 3
Cost = 1
Existence of strong equilibrium

[Holzman and Monderer, 2014] A strong equilibrium does not necessarily exist

\[ d^1 \]
\[ d^2 \]
\[ s \]

(2, 1)
(3, 0)

Cost = 2
Cost = 0
Existence of strong equilibrium

[Epstein et al., 2007] and [Holzman and Monderer, 2014] studied in which network topologies a strong equilibrium is guaranteed to exist.

Definition

A *series parallel* network is a network that can be produced by a sequence of the operations: create, parallel composition, and series composition.
Existence of strong equilibrium

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<tr>
<th>Network construction operations</th>
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Existence of strong equilibrium

Network construction operations

(1) Create:

\[
s \xrightarrow{} t
\]
Existence of strong equilibrium

Network construction operations

(2) Parallel composition: Consider two networks.

Collapse sources to a new source $s$, and sinks to a new sink $t$:
(3) Series composition: Consider two networks.

Collapse $t_1$ and $s_2$ to a single intermediate node $v$: 
Existence of strong equilibrium

**Definition**

A game is *good* if it is a network design game with non-increasing cost functions, played on a series parallel network.

**Theorem ([Epstein et al., 2007, Holzman and Monderer, 2014])**

A *good game always has a strong equilibrium*. 
Convergence to equilibrium

Problem: Beneficial coalitional deviations (BCDs) may cycle:
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- Cost = 15
- Cost = 14
**Problem:** Beneficial coalitional deviations (BCDs) may cycle: Consider fair cost sharing with two agents:

- Cost = 11
- Cost = 11
Convergence to equilibrium

**Problem:** Beneficial coalitional deviations (BCDs) may cycle: Consider fair cost sharing with two agents:

- Cost = 10
- Cost = 17
**Problem:** Beneficial coalitional deviations (BCDs) may cycle: Consider fair cost sharing with two agents:

- Cost = 15
- Cost = 14
Goal

Construct short BCD sequences that converge to a strong equilibrium in good games.
Related work

Convergence to strong equilibrium

- On Strong Equilibria in the Max Cut Game [Gourvès and Monnot, 2009]
- Strong Nash Equilibria in Games with the Lexicographical Improvement Property [Harks et al., 2009]
- Computing pure Nash and Strong Equilibria in Bottleneck Congestion Games [Harks et al., 2010]
- Strong Price of Anarchy, Utility Games and Coalitional Dynamics [Bachrach et al., 2014]
Definition (Agent domination)

Agent $i$ dominates agent $j$ if there is a path from $d^j$ to $d^i$.

If the path $\sim$ is non-empty, then $i$ strictly dominates $j$. 
Lemma (Intersecting vertex [Epstein et al., 2007])

*For a series-parallel network:*
Lemma (Intersecting vertex [Epstein et al., 2007])

For a series-parallel network:

\[ s \rightarrow p \rightarrow d^1 \rightarrow t \]

\[ d^2 \]
Lemma (Intersecting vertex [Epstein et al., 2007])

For a series-parallel network:

Either: (a) there is a path from $d^1$ to $d^2$. 
Lemma (Intersecting vertex [Epstein et al., 2007])

For a series-parallel network:

Or: (b) there exists a vertex $y \in p$:

- $y$ is on every path from $s$ to $d^2$.
- Every $y \rightarrow d^2$ and $p$ are edge disjoint.
Agent $i$ forms a BCD by selecting a path $p$: 

$$s \xrightarrow{p} d^1 \xrightarrow{} d^2 \xrightarrow{} d^3$$
Agent *i* forms a BCD by selecting a path *p*:

Every agent that does not strictly dominate *i* joins, from *s* to its intersecting vertex **if it benefits**.
Agent $i$ forms a BCD by selecting a path $p$:

Every agent that does not strictly dominate $i$ joins, from $s$ to its intersecting vertex if it benefits.
Agent \( i \) *forms* a BCD by selecting a path \( p \):

Every agent that does not strictly dominate \( i \) joins, from \( s \) to its intersecting vertex *if it benefits*. In a *best response with respect to* \( i \), the path \( p \) is chosen so that \( i \)'s cost is minimal.
Lemma

*In a good game, after a best response with respect to an undominated agent* \( i \), \( i \) incurs an optimal cost.*
Computing a best response for an undominated agent

- Current outcome is $P$, undominated agent $i$.
- $N[e]$: The agents that have a path through edge $e$.

Compute a minimal cost path $O$ from $s$ to $d^i$ when the edge weights are $c_e(|N[e]|)$.

![Diagram](https://via.placeholder.com/150)

Rename agents so that if $j < k$ then the intersecting vertex $y_j$ is before $y_k$ (w.r.t $O$). At first, the deviating coalition is everyone.
Computing a best response for an undominated agent

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- \( N[e] \): The agents that have a path through edge \( e \).

Compute a minimal cost path \( O \) from \( s \) to \( d^i \) when the edge weights are \( c_e(|N[e]|) \).

Rename agents so that if \( j < k \) then the intersecting vertex \( y_j \) is before \( y_k \) (w.r.t \( O \)). At first, the deviating coalition is everyone.
For each \( j \): If \( j \) doesn’t benefit by deviating to \( s \overset{O}{\sim} y_j \), replace \( s \overset{O}{\sim} y_j \) with \( s \overset{P^j}{\sim} y_j \), and remove \( \leq j \) from the deviating coalition.
Lemma

In a good game, after a best response with respect to an undominated agent $i$, $i$ incurs an optimal cost.
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Corollary

In a strong equilibrium, every undominated agent incurs an optimal cost.

Therefore, after a best response with respect to an undominated agent, the agent will not be part of any BCD.
Lemma

In a good game, after a best response with respect to an undominated agent \( i \), \( i \) incurs an optimal cost.

Corollary

In a strong equilibrium, every undominated agent incurs an optimal cost.

Therefore, after a best response with respect to an undominated agent, the agent will not be part of any BCD.

But, it will not necessarily stay this way.
Bad best responses

Red agent forms a best response:

- $s \rightarrow u$ with $(1, 1)$
- $s \rightarrow u$ with $(2, 2)$
- $s \rightarrow u$ with $(3, 1)$
- $u \rightarrow d^j$ with $(2)$
- $u \rightarrow d^j$ with $(3)$
- $u \rightarrow d^i$ with $(2)$
- $u \rightarrow d^i$ with $(3)$

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Red agent forms a best response:
Bad best responses

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\[ \begin{align*}
\text{(1, 1)} & \quad \text{(2, 2)} & \quad \text{(3, 1)} \\
\end{align*} \]

\[ \begin{align*}
\text{(2)} & \quad \text{(3)} & \quad \text{(3)} \\
\end{align*} \]
And then the blue agent forms a best response:
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And then the blue agent forms a best response:

Now the red agent is no longer incurs an optimal cost.
Definition (Minimal effort)

Say agent $i$ forms the BCD between outcomes $P$ and $Q$:

Outcome $P$:

- $s$
- $u$
- $v$
- $d^i$
Definition (Minimal effort)

Say agent $i$ forms the BCD between outcomes $P$ and $Q$:

**Outcome $Q$:**

\[ s \rightarrow u \rightarrow v \rightarrow d^i \]
Definition (Minimal effort)

Say agent $i$ forms the BCD between outcomes $P$ and $Q$:

**Outcome $Q$:**

The cost of $u \sim v$ in $P$ is **higher** than the cost of $u \sim v$ in $Q$.
Definition (Minimal effort)

Say agent $i$ forms the BCD between outcomes $P$ and $Q$:

**Outcome $Q$:**

The cost of $u \sim v$ in $P$ is **higher** than the cost of $u \sim v$ in $Q$.

Lemma

*In every good game, there exists a best response with respect to $i$, in which $i$ performs minimal effort.*
Adjusting a best response to perform minimal effort

- $P$ = outcome before deviation
- $Q$ = outcome after deviation
- $i$ = agent that formed the best response.
- $J$ = deviating coalition.

If $s; v$ disobeys minimal effort, replace with $s; v^+$, and remove from $J$ all agents with an intersecting vertex in $s; v$. 

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Adjusting a best response to perform minimal effort

- $P = \text{outcome before deviation}$
- $Q = \text{outcome after deviation}$
- $i = \text{agent that formed the best response.}$
- $J = \text{deviating coalition.}$

If $s \overset{Q^i}{\sim} v$ disobey minimal effort, replace with $s \overset{P^i}{\sim} v$, and remove from $J$ all agents with an intersecting vertex in $s \overset{Q^i}{\sim} v$. 
**Main theorem**

**Lemma**

*In a sequence of best responses with minimal effort, after an undominated agent* $i$ *forms a best response, no agent deviates away from an edge in* $i$*'s path throughout the entire sequence.*

Therefore $i$ will *never* be part of a BCD.
Main theorem

Lemma

In a sequence of best responses with minimal effort, after an undominated agent $i$ forms a best response, no agent deviates away from an edge in $i$’s path throughout the entire sequence.

Therefore $i$ will never be part of a BCD.

Definition

A dominance based BCD sequence, is a sequence of best responses of minimal effort, when the agents forming the best responses are ordered from the most dominant to the least.

Theorem

Every dominance based BCD sequence converges to an SE in every good game.
Let an undominated agent $i$ perform a best response with minimal effort.
Proof of the main theorem

- Let an undominated agent \( i \) perform a best response with minimal effort.
- Let \( \mathcal{G}^1 \) be the game with the agents \( N \setminus \{i\} \), and the cost function of every edge \( e \) in \( i \)'s becomes \( \tilde{c}_e(x) = c_e(x + 1) \).
Proof of the main theorem

- Let an undominated agent $i$ perform a best response with minimal effort.
- Let $G^1$ be the game with the agents $N \setminus \{i\}$, and the cost function of every edge $e$ in $i$’s becomes $\tilde{c}_e(x) = c_e(x + 1)$.
- By previous lemma, the game $G^1$ has the same BCDs as the original game (as long as only best responses are performed).
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By previous lemma, the game $\mathcal{G}^1$ has the same BCDs as the original game (as long as only best responses are performed).

Repeat the same argument for an undominated agent in $\mathcal{G}^2, \mathcal{G}^3, \ldots$. 

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- By previous lemma, the game $\mathcal{G}^1$ has the same BCDs as the original game (as long as only best responses are performed).
- Repeat the same argument for an undominated agent in $\mathcal{G}^2, \mathcal{G}^3, \ldots$.
- In the game $\mathcal{G}^{n-1}$, there is one agent. After its best response, there are no BCDs. Therefore, this is a SE.
Conclusions about good games

Not all BCD sequences converge to a strong equilibrium in good games.
Conclusions about good games

1. Not all BCD sequences converge to a strong equilibrium in good games.
2. We can find BCD sequences that converge after at most $n$ iterations.
Conclusions about good games

1. Not all BCD sequences converge to a strong equilibrium in good games.

2. We can find BCD sequences that converge after at most $n$ iterations.

3. All undominated agents incur an optimal cost in every strong equilibrium.
Thank you


