Convergence to Strong Equilibrium in Network Design Games

Michal Feldman and Ophir Friedler

Draft, June 8, 2015

Abstract. We study network design games in which each agent seeks to connect the source of the network to his destination, and the cost of each edge is non-increasing with the number of agents that use it. We study beneficial coalitional deviations (BCDs), a BCD is a joint deviation of a coalition in which each member decreases his cost. It was previously shown [4,5] that in this setting, if the network is a series-parallel network, then a strong equilibrium — an outcome of the game in which there is no BCD, is guaranteed to exist.

We start by showing that BCDs do not necessarily converge to a strong equilibrium. We then define a class of BCD sequences termed dominance based BCDs, and prove that dominance based BCDs converge to a strong equilibrium after at most $n$ iteration (where $n$ is the number of agents). Furthermore, we present an algorithm that efficiently computes dominance based BCDs.

1 Introduction

In a network design game [1] each agent seeks to connect two nodes in a directed network at a minimal cost. The strategies employed by each agent include all the paths that connect that agent’s two nodes (termed origin and destination). The paths may represent roads, internet cables, or even water pipelines. The cost of an edge is a function of the number of agents that use it.

An agent pays the total cost of the edges in its path, where an edge cost is a function of the number of agents using the edge. In this work we focus on non-increasing edge costs, where agents impose positive externalities on one another. Such settings emerge in cases where agents collectively construct a network and share the cost of the network links.

Each network design game possesses a pure Nash equilibrium (PNE): an outcome that is sustainable against unilateral deviations. However, a PNE is not necessarily stable against coalitional deviations: Therefore, this is an inadequate solution concept in settings where agents are capable of coordinating their actions. The most well studied solution concept that is stable against coalitional deviations is termed strong equilibrium (SE) [2]. An SE is an outcome where no beneficial coalitional deviation (BCD) exists (i.e., a deviation in which each member of the coalition strictly decreases its cost).

Epstein et al. [4] studied the existence and efficiency of SEs in non-increasing network design games. They showed that in a single-origin, any-destination
(SOAD) setting (i.e., where all agents have the same origin but may have arbitrary destinations) with a series-parallel (SP) network [4,5], an SE is guaranteed to exist. Holzman and Monderer [5] showed that this result is tight, i.e., for any network that is not SP, there exists a non-increasing SOAD network design game that does not admit an SE.

A natural question arises: Given an arbitrary outcome of an SOAD network design game with an SP network, can strategic agents converge to an SE via BCDs? and if yes, how fast?

Our contribution. We start by showing that there exist BCD sequences that do not converge to an SE. We then define a class of BCDs, termed dominance based BCDs. This class is based on the notion of domination between agents. In an SOAD setting, we say that agent $i$ is dominated by agent $j$ if there is a path from the destination of $i$ to the destination of $j$. Thus, domination is a partial order between the agents.

Dominance based BCDs proceed in the following manner: Take any (full) order of the agents consistent with the partial order. Every agent $i$, in its turn, computes the optimal profile for itself together with all the successive agents that can intersect its path (thus reducing its cost). We show that if such a coalitional deviation reduces $i$’s cost, then every agent in the coalition benefits from the deviation as well. Therefore, this is a BCD. We show that any sequence of dominance based BCDs converges to an SE within $n$ iterations at the most (where $n$ is the number of agents). Moreover, we present an algorithm that efficiently computes dominance based BCDs.

2 The Model and Preliminaries

A network is a tuple $(V, E, s, t)$ where $V$ is a set of nodes, $E$ is a set of directed edges between the nodes, $s$ is the source of the network and $t$ is the sink (i.e., $s$ and $t$ are two specific and different nodes). In a network, every vertex $v \in V$ and every edge $e \in E$ belongs to at least one path from $s$ to $t$. Given a path $R$, denote by $R_{u,v}$ the sub-path of $R$ from $u$ to $v$.

Given a network $D = (V, E, s, t)$, for each pair of vertices $u, v \in V$ The induced subnetwork $D_{u \rightarrow v}$ is the network with source $u$ and sink $v$ that consists of all the nodes and edges in $(V, E)$ that are on a path from $u$ to $v$.

A network design game is the tuple $G = (D = (V, E, s, t), N, \{(o^i, d^i)\}_{i \in N}, \{c_e(.)\}_{e \in E})$ where $D$ is a network, $N$ is the set of agents so that each agent $i$ possesses an origin $o^i \in V$ and a destination $d^i \in V$. The pair $(o^i, d^i)$ induces $i$’s strategy space $\sum_i$: All the paths from $o^i$ to $d^i$. An outcome of the game $P = (P^1, P^2, \ldots, P^n) \in \times_{i \in N} \sum_i$ consists of a strategy for each agent.

Each edge $e$ is associated with a cost function $c_e : N \rightarrow \mathbb{R}$ that measures the cost with respect to the load $l_e(P) = |\{i : e \in P^i\}|$ on the edge in an outcome $P$ of the game, i.e., the cost of an edge $e$ is $c_e(l_e(P))$. The cost of a path $u$ in outcome $P$ is $C_u(P)$ the total cost of $u$’s edges, i.e., $C_u(P) = \sum_{e \in u} c_e(l_e(P))$. An agent $i$’s cost in outcome $P$ is the cost of $i$’s path, i.e., $c_i(P) = C_{P^i}(P)$. The optimal outcome of a game $G$ from agent $i$’s point of view is denoted by
opt_i(G) = \min_{P \in \Sigma_i} c_i(P). In this paper we focus on the class non-increasing cost functions (denoted by DEC), i.e., for every edge e, if \( k \geq l \) then \( c_e(k) \leq c_e(l) \). A well studied special case is fair cost sharing, where \( c_e \) is a constant, and given a load of \( l \) on edge \( e \), the cost of \( e \) is \( c_e \).

Given two outcomes \( P \) and \( R \), the outcome \((R^i, P^{-i})\) denotes the outcome where \( P^i \) is replaced by \( R^i \) (e.g., \( P = (P^i, P^{-i}) \)). Given a set of agents \( A \subseteq V \), the outcome \((R^A, P^{-A})\) denotes the outcome where all agents in \( A \) play by \( R \), and all agents in \( A^c \) play by \( P \), where \( A^c = V \setminus A \) (e.g., \( P = (P^A, P^{-A}) \)).

The following origin-destination classes of agents are considered in the literature: In SOSD (same-origin same-destination) all origins coincide and all destinations coincide. In SOAD (same-origin any-destinations) all origins coincide and destinations may be distinct. In AOAD (any-origins any-destinations) origins as well as destinations may be distinct. Clearly \( \text{SOSD} \subseteq \text{SOAD} \subseteq \text{AOAD} \).

**Beneficial coalitional deviation.** Given an outcome \( P \), and a coalition \( J \), a beneficial coalitional deviation (BCD) is a deviation (i.e., a strategy profile) \( R^J \) of the agents in \( J \), such that in outcome \((R^J, P^{-J})\) the cost of each agent \( j \in J \) is strictly less than in \( P \), i.e., \( c_j(R^J, P^{-J}) < c_j(P) \).

**Graph theoretic preliminaries.** The following operations are used for graph construction:

- Identification: The identification operation allows to collapse two nodes to one. Given a graph \( G = (V, E) \), the identification of two nodes \( v_1, v_2 \in V \) results in the new graph with nodes \( V' = V \setminus \{v_1, v_2\} \cup \{w\} \) and edges \( E' \) consist of edges from \( E \) where \( v_1 \) and \( v_2 \) are replaced by \( w \).
- CREATE: Create a network with a single edge \( s \to t \).
- PARALLEL: Given two series parallel networks \( A = (V_A, E_A, s_a, t_a) \) and \( B = (V_B, E_B, s_b, t_b) \), form a new network \( C \) by identifying \( s_a \) with \( s_b \) and \( t_a \) with \( t_b \). This is also called the parallel composition of \( A \) and \( B \).
- SERIES: Given two series parallel networks \( A = (V_A, E_A, s_a, t_a) \) and \( B = (V_B, E_B, s_b, t_b) \), form a new network \( C \) by identifying \( t_a \) with \( s_b \). This is also called the series composition of \( A \) and \( B \).

A multiple-edge network is a network that can be produced by a sequence of the operations CREATE and PARALLEL. A series parallel network is a network that can be produced by a sequence of the operations CREATE, PARALLEL, SERIES. A multi-extension parallel network is a network that can be produced by a sequence of the operations CREATE, PARALLEL, SERIES, such that in each SERIES operation, one of the networks is a multiple-edge network.

**Subdivision.** A subdivision of a network \( D \) is a network obtained from \( D \) by replacing every edge by a path of one or more edges. It was shown in \cite{5} and restated in Proposition \cite{1} that a network is series parallel if and only if there is no subnetwork which is isomorphic to a subdivision of the Braess graph (Fig \cite{1}).
Proposition 1. (§) Let $D$ be a network. Then $D$ is series-parallel if and only if $D$ has no subnetwork which is isomorphic to a subdivision of the network $W$ (Fig 1).

A direct corollary of Proposition 1 is that given a series-parallel network, any subnetwork is also a series-parallel network.

Fig. 1. Network $W$ (Braess graph)

3 Converging Dynamics

In this section we study the convergence of BCDs to an SE. For coherency, most of the proofs are deferred to the appendix. In § the notion of strong networks was defined. A network $D$ is $\Gamma$-strong with respect to a subclass $\Gamma$ of games on $D$, if every game in $\Gamma$ has a SE. Recall that DEC is the class of non-increasing cost functions.

Theorem 1. A network $D$ is SOAD-DEC-strong if and only if $D$ is series parallel.

Good games. Throughout this paper, we call a non-increasing network design game a good game if the agents are in an SOAD setting and the network is series-parallel.

In this section we study the convergence of coalitional dynamics to a SE for the class of good games. Since we consider a restricted class of networks, it is of interest to check whether Nash equilibria and strong equilibria coincide. The following proposition answers this negatively:

Proposition 2. There exists a non-increasing network design game on a parallel-edges network with a Nash equilibrium that is not a strong equilibrium. Even for two agents sharing their origin and destination.

The separation between NEs and SEs exists even for very simple good games that are also fair cost sharing.

Proposition 3. There exists a fair cost sharing game on an extension parallel network with a Nash equilibrium that is not a strong equilibrium, even for three agents in an SOAD setting.
Therefore, coordination is necessary for agents to converge to a SE. Furthermore, coordination alone is also insufficient in order to converge to a SE. Proposition 4 shows that BCDs do not necessarily converge, even in a very simple setting.

**Proposition 4.** There exists a fair cost sharing game on an extension parallel network with two agents that have the same origins and destinations, with an infinite sequence of BCDs.

Then it is of interest to study cases in which BCDs do converge, and what is their rate of convergence.

Lemma 1 (followed directly from the intersecting vertex lemma from [4]) provides an important property of series-parallel networks that will be used extensively.

**Lemma 1.** (Intersecting vertex) Let $D = (V, E, s, t)$ be a series-parallel network. For two destination nodes $d^1, d^2$, and a path $Q$ from $s$ to $d^1$, one of the following holds:

1. There is a path from $d^1$ to $d^2$.
2. There exists a vertex $y \in Q$ so that for any path $Q' \text{ from } s \text{ to } d^2$, it holds that $y \in Q'$ and $Q'_{y,d^2}$ and $Q$ are edge disjoint (We call $y$ the intersecting vertex of $Q$ and $d^2$).

**Agent dominance.** Given a good game $G$, we say that an agent $i$ dominates agent $j$ (i.e. $i \prec j$) if there is a path from $d^j$ to $d^i$, and if the path is non-empty (i.e. $d^i \neq d^j$), we say that agent $i$ strictly dominates agent $j$. Since the network in $G$ is acyclic, agent domination $\prec$ is a partial order. An agent $i$ is undominated if no agent strictly dominates it. A full order $\prec_\pi$ is called a dominance based order if it is consistent with the partial order $\prec$.

**BCD formation.** An agent $i$ forms a BCD as follows: Consider a path $Q^i$ from $s$ to $d^i$. By Lemma 1, each agent $j$ that does not strictly dominate $i$, has an intersecting vertex $y_j$ of $Q^i$ and $d^i$. Therefore, agent $j$ may deviate to the path $Q^i_{s,y_j}$ (and from $y_j$ continues to $d^j$ as before). An agent $j$ will deviate only if it strictly benefits, when the rest of such agents deviate as well. Therefore, the coalition consists of agents that (1) do not strictly dominate $i$ and (2) strictly decrease their costs when jointly deviating (each agent $j$ to $Q^i_{s,y_j}$). Agent $i$ selects a path $Q^i$ so that the deviation decreases its cost.

**Minimal effort.** Let $P$ be the current outcome, and let $Q$ be the outcome after a BCD formed by $i$. Then agent $i$ performs minimal effort if for every pair of nodes $u, v \in P \cap Q^i$, if $P_{u,v} \neq Q^i_{u,v}$ then the cost of the path $P_{u,v}$ in outcome $P$ is strictly higher than the cost of the path $Q^i_{u,v}$ in outcome $Q$. 
Coalitional Dynamics. A best response with respect to agent $i$, is a BCD formed by $i$ that minimizes $i$’s cost. A dominance based BCD sequence, is a sequence of best responses with respect to agents according to some dominance based order $\prec_\pi$, from the most dominant to the least. Furthermore, we assume that the agents forming the BCDs are performing minimal effort. Lemma 2 shows that assuming this does not affect the quality of BCDs.

Lemma 2. In a good game $G$, for every best response with respect to $i$, there exists a best response with respect to $i$, in which $i$ performs minimal effort.

When we do not explicitly mention which agent formed a best response, we will just call the deviation a coalitional best response.

In Theorem 2, which is the main theorem of this section, we prove the convergence of dominance based BCDs to an SE in good games.

Theorem 2. Every dominance based BCD sequence converges to an SE in every good game $G$.

Proof. Consider a good game $G$. Let $\prec_\pi$ be a dominance based order. Rename the agents so that $i_1 \prec_\pi i_2 \ldots \prec_\pi i_n$. Assume the current outcome is $P$. Denote by $P(1)$ the outcome after the best response with respect to agent $i_1$. Lemma 3 states that in outcome $P(1)$ agent $i_1$ is at an optimal cost.

Lemma 3. In a good game $G$, let $\tilde{P}$ be the outcome after a best response with respect to an undominated agent $i$. Then $i$ is at an optimal cost in $\tilde{P}$, i.e., $c_i(\tilde{P}) = \text{opt}_i(G)$.

By optimality in Lemma 3 the agent $i_1$ will never be part of a BCD so long that the cost of his path does not change. Lemma 4 shows that if the rest of the BCDs are also coalitional best responses, then indeed the cost of $i_1$’s path does not change.

Lemma 4. In a sequence of coalitional best responses, after a best response with respect to an undominated agent $i$, no agent deviates away from an edge in $i$’s path throughout the entire sequence.

Consider the game $G^1$ with the agents $N^1 = \{i_2, \ldots i_n\}$, where the cost function of each edge is updated as follows:

$$
\begin{align*}
e \notin P^{i_1} & \Rightarrow \tilde{c}_e(x) = c_e(x) \\
e \in P^{i_1} & \Rightarrow \tilde{c}_e(x) = c_e(x + 1)
\end{align*}
$$

(I)

I.e., the edges on $i_1$’s path have $i_1$’s congestion “hardwired” to the cost functions. It is easy to see that if $G$ is a good game then $G^1$ is a good game. By Lemma 4 the agent $i_1$ will not be a part of another BCD (as long as only coalitional best responses are performed), because after the first BCD $i_1$ is already at an optimal cost $\text{opt}_{i_1}(G)$. As a result, (as long as only coalitional best responses are performed) the game $G^1$ will have the same set of BCDs as the game $G$. 

Denote by $P_{(2)}$ the outcome after the best response with respect to the agent $i_2$. Applying Lemma 3 to the game $G^1$, we get that in the outcome $P_{(2)}^{-i_1}$, agent $i_2$ is at an optimal cost in the game $G^1$. By Lemma 4 the agent $i_2$ will not be a part of another BCD (as long as only coalitional best responses are performed). As a result, the game $G^2$ with the agents $N^2 = \{i_3, \ldots, i_n\}$, where the cost function of each edge is updated in the same manner as (1) will have the same set of BCDs as $G^1$ (which in turn has the same set of BCDs as $G$).

Denote by $P_{(j)}$ the outcome after the best response with respect to the agent $i_j$. Applying Lemma 3 to the game $G^{j-1}$, we get that in outcome $P_{(j)}^{-\{i_1, \ldots, i_{j-1}\}}$, agent $i_j$ is at an optimal cost in the game $G^{j-1}$. By Lemma 4 the agent $i_j$ will not be a part of another BCD (as long as only coalitional best responses are performed). As a result, the game $G^j$ with the agents $N^j = \{i_{j+1}, \ldots, i_n\}$, where the cost function of each edge is updated in the same manner as (1) will have the same set of BCDs as $G^{j-1}$ (which in turn has the same set of BCDs as $G, G^1, \ldots, G^{j-2}$).

Denote by $P_{(n)}$ the outcome after the best response with respect to the agent $i_n$. According to the above, $i_n$ will not be part of another BCD. As a result, the game $G^{n-1}$ (that only has one agent — $i_n$) does not have any BCDs from $P_{(n)}$, which in turn implies that the games $G, G^1, \ldots, G^{n-2}$ do not have any BCDs, i.e., $P_{(n)}$ is a SE of the game $G$.

**Corollary 1.** Let $S$ be an SE of a good game $G$. For every agent $i$, let $N(i)$ be the agents that do not strictly dominate $i$, and $G^i = G[S \setminus N(i)]$. Then $S[N(i)]$ is an outcome of $G^i$ and $c_i(S[N(i)]) = \text{opt}_i(G^i)$.

Corollary 1 means that in an SE each agent $i$ incurs an optimal cost given the strategies of agents that strictly dominate $i$. Specifically, every undominated agent incurs an optimal cost in every SE. In [4,5] it was shown that there exists an SE that optimizes the cost of agents whose origins and destinations coincide with the source and the sink of the network respectively (termed complete agents). Corollary 1 can be viewed as an extension of this result. Furthermore, optimality is guaranteed for every undominated agent, and not only for complete agents.

**Computing dynamics in good games.** Consider a good game $G$. In the remaining of the section we compute coalitional best responses in which minimal effort is performed.

**Definition 1.** $(N[v])$ For every network design game on a network $D = (V, E, s, t)$ and set of agents $N$, the set $N[v]$ is the set of agents that have an origin destination path that goes through $v$. 
**Definition 2.** \((N_e)\) For every network design game on a network \(D = (V, E, s, t)\) and set of agents \(N\), the set \(N_e\) is the set of agents that have an origin destination path that goes through \(e\).

To compute \(\{N[v]\}_{v \in V}\) and \(\{N_e\}_{e \in E}\), first iterate over all agents and create for each node \(v\) the set \(D[v]\) of all the agents that \(v\) is their destination, i.e., \(D[v] = \{i : d^i = v\}\). Then reverse the edges of the network, and for each node \(v\) run Breadth First Search (BFS) \(^3\), starting from \(v\), unifying \(D[v]\) with \(N[u]\) for each reachable node \(u\) and with \(N_e\) for each reachable edge. This can be done in \(\tilde{O}(n + |V| \cdot (|V| + |E|))\) (here the \(\tilde{O}\)-notation hides logarithmic factors of set operations such as 'add' and 'union').

**Computing an intersecting vertex.** Let \(p\) be a path of an undominated agent, and \(d^j\) a destination node of agent \(j\) so that there is an intersecting vertex \(y_j^j\) of \(p\) and \(d^j\). For every node \(v\) in the path \(p_{s,y_j^j}\), it holds that \(j \in N[v]\) because there is a path from \(v\) to the \(y_j\) (which in turn has a path to \(d^j\)), and for every node \(v'\) in the path \(p_{y_j,d^j}\), it holds that \(j \notin N[v']\), by definition of an intersecting vertex. Therefore the intersecting vertex of \(p\) and \(d^j\) is the last node \(v\) in the path \(p\) (starting from \(s\)) on which \(j\) appears in \(N[v]\).

**Computing an optimal BCD for an undominated agent.** Given an outcome \(Q\) of a good game \(G\), algorithm \(^1\) returns a best response w.r.t. \(i\) when \(i\) is an undominated agent. Since the returned path \(O\) for agent \(i\) is a path from \(s\) to \(d^i\), it is a feasible deviation. By definition of an intersecting vertex, it is feasible for every agent \(j\) to deviate from \(Q_{s,y_j}^j\) to \(O_{s,y_j}\). Therefore the algorithm returns a feasible coalitional deviation. To see that it is indeed a best response w.r.t. \(i\), observe that the algorithm follows the steps of the proof of Lemma \(^3\).

**Computing dominance based BCDs.** To compute an entire sequence, at each step select an undominated agent \(i\), and compute a best response w.r.t. \(i\) using algorithm \(^1\). Then, update the cost functions of the edges in the same manner of \(^1\) and remove \(i\) from the set of agents. To see that this indeed computes a dominance based BCD sequence, observe that this process follows the steps of the proof of Theorem \(^2\).

**References**

Algorithm 1: Returns a best response w.r.t. agent $j$.

**input**: $G = ((V, E, s, t), N, \{(i, d^i)\}_{i \in N}, \{c_e(.)\}_{e \in E})$, current outcome $Q$, undominated agent $i$.

**output**: Best response w.r.t. an undominated agent $i$.

1. foreach Edge $e \in E$ do Compute $N_e$;
2. Compute a min-cost path $O$ from $s$ to $d_i$ with edge weights $c_e(|N_e|)$;
3. foreach agent $j \in N$ do Compute $y_j$, the intersecting vertex of $O$ and $d_j$;
4. Rename the agents so that if $j < k$ then $y_k$ does not come before $y_j$ in $O$;
5. $L \leftarrow 1$;
6. foreach agent $j$ (by the renaming) from 1 to $n$ do
7. if $C_{Q^{y_j}}(Q) \leq C(O_{s,y_j})$; // $C(O_{s,y_j}) =$ cost of $O_{s,y_j}$ from line 2
8. then
9. $O \leftarrow Q_{s,y_j} \cup O_{y_j,d_i}$;
10. $L \leftarrow j$;
11. end
12. end
13. return The path $O$ for $i$, and for every $j : j > L$ the path $O_{s,y_j} \cup Q^{y_j}_{y_j,d_j}$

---


A Missing proofs

We first restate the intersecting lemma from [4]:

**Lemma 5.** [4] Let $D = (V, E, s, t)$ be a series-parallel network. Given a path $Q$ from $s$ to $t$, and a vertex $t'$, there exists a vertex $y \in Q$, such that for any path $Q'$ from $s$ to $t'$, the path $Q'$ contains $y$ and the paths $Q_{y,t}$ and $Q$ are edge disjoint. (We call the vertex $y$ the intersecting vertex of $Q$ and $t'$.)

**Proof of Lemma 5.**

Proof. If there is a path from $d^1$ to $d^2$ then we are in case 1 and we are done. Otherwise, the path $Q$ can be extended to a path $\bar{Q}$ from $s$ to $t$, without going through $d^2$, in which case we can apply Lemma 3 to $\bar{Q}$ and $d^2$.

**Proof of Proposition 2.**

Proof. Consider the network in Figure 2 with two parallel edges $A, B$, where the edge $A$ always costs 2 and the edge $B$ costs 2 if one agent uses it, and costs 1 if both agents use it (i.e., each agent pays 1). Then the outcome $(A, A)$ is a Nash equilibrium because by a unilateral deviation to $B$, an agent doesn’t change his cost, however, $(A, A)$ is not a SE since both agents can jointly deviate to $B$, and each reduces his cost by 1.
Fig. 2. A game with an NE that is not a SE. The left number in the parenthesis is the cost when one agent uses the edge, and the right number is the cost (for each agent) when two agents use the edge.

Proof of Proposition 4.

Proof. Consider two agents, so that the origin and destination of both agents are the terminals of the network in figure 3. The cost vector of the edges \((A, B, C, D)\) is \((12, 10, 5, 14)\). At the beginning, both agents are travelling on the path \((A, B)\) and paying 11 each. A better response of agent 1 is to deviate to \((C, B)\) and by that pay only 10, while the agent 2 now pays 17. In such a case, a better response of agent 2 is to deviate to the path \((D)\), where he pays only 14, while agent 1 now pays 19. Finally, it is beneficial for both to deviate back to the path \((A, B)\). This cycle shows that there exist infinite better-response coalitional dynamics.

Fig. 3. A graph that has cyclic coalitional better response dynamics

Proof of Lemma 2.

Proof. Let \(Q\) be the outcome after the best response w.r.t. \(i\) (therefore \(Q^i\) is \(i\)'s selected path), and let \(J\) be the deviating coalition, i.e., \(\forall j \in J : Q^j = Q^i_{s,y_j} \cup P^j_{y_j,v}\) (where \(y_j\) is the intersecting vertex of \(Q^i\) and \(d^j\)). If \(i\) does not perform minimal effort, then there exists a node \(v\) so that \(P^i_{s,v} \neq Q^i_{s,v}\) and

\[
C_{P^i_{s,v}}(P) \leq C_{Q^i_{s,v}}(Q)
\]  \hspace{1cm} (2)

(consider the pair \(u, v \in P^i \cap Q^i\) that breaks minimal effort and also \(u\) is closest to \(s\)).

Let us see that if \(i\) selects the path \(\tilde{Q}^i = P^i_{s,v} \cup Q^i_{v,d^i}\), then it forms a BCD that is also a best response.
If \( v \) comes after all intersecting vertices of agents from \( J \), then by (2) agent \( i \) can unilaterally deviate (from outcome \( P \)) to \( \tilde{Q}^i \) and achieve a cost that is no higher than the best response from outcome \( P \) to outcome \( Q \). Turning a unilateral deviation to a minimal effort deviation is trivial.

Otherwise, consider the coalition \( \tilde{J} \subseteq J \), where for each \( j \in \tilde{J} \) the intersecting vertex of \( Q^i \) and \( d^j \) is \( y_j \in Q^i_{v,d} \). Clearly, for each \( j \in \tilde{J} \) the intersecting vertex of \( \tilde{Q}^i \) and \( d^j \) is also \( y_j \).

Let us see that the deviation of \( i \) to \( \tilde{Q}^i \) and each agent \( j \in \tilde{J} \) to the path \( \tilde{Q}^i_{s,y_j} \cup P^i_{y_j,d^j} \) is a BCD that minimizes \( i \)'s cost. In other words, let \( \tilde{Q} \) be the outcome after the prescribed deviation, then \( c_i(\tilde{Q}) = c_i(Q) \), and every agent \( j \in \tilde{J} \) benefits from the deviation.

Since the cost functions are non-increasing, the cost of the path \( P^i_{s,v} \) doesn’t increase when agents from \( \tilde{J} \) join it, i.e.

\[
C_{P^i_{s,v}}(\tilde{Q}) \leq C_{P^i_{s,v}}(P)
\]  

(3)

Furthermore, since \( \tilde{Q}^i_{v,d^i} = Q^i_{v,d^i} \), and the intersecting vertex of every agent \( j \in \tilde{J} \) is the same for \( Q^i \) and \( \tilde{Q}^i \), agent \( j \) uses the path \( Q^i_{v,y_j} \) in \( \tilde{Q} \). As a result, the costs of the edges in \( Q^i_{v,d^i} \) are the same in \( Q \) and \( \tilde{Q} \), i.e.,

\[
C_{Q^i_{v,d^i}}(\tilde{Q}) = C_{Q^i_{v,d^i}}(Q)
\]  

(4)

Putting (2), (3) and (4) together, we get that for every agent \( j \in \tilde{J} \) it holds that

\[
C_{\tilde{Q}^i_{s,y_j}}(\tilde{Q}) \leq C_{\tilde{Q}^i_{s,y_j}}(Q)
\]

Since it was beneficial for each \( j \in \tilde{J} \) to deviate from \( P \) to \( Q \), then it is beneficial to deviate from \( P \) to \( \tilde{Q} \). Furthermore, (2), (3) and (4) directly imply that \( c_i(\tilde{Q}) \leq c_i(Q) \), and since the deviation to \( Q \) was a best response w.r.t. \( i \), then so is the deviation to \( \tilde{Q} \).

An agent \( k \notin \tilde{J} \) with an intersecting vertex \( y_k \) of \( \tilde{Q}^i \) and \( d^k \), will join the BCD by deviating to \( \tilde{Q}^i_{k,y_k} \cup P^i_{y_k,d^k} \) if \( k \) strictly benefits from it. This will not harm the costs of agents in \( \tilde{J} \) since the edge costs are non-increasing. The result is a BCD that minimizes \( i \)'s cost, where \( \tilde{Q}^i_{s,v} = P^i_{s,v} \), i.e., the minimal effort property is not broken until node \( v \). Therefore, we can repeat this argument until we reach a best response w.r.t. \( i \) in which \( i \) performs minimal effort.

**Proof of Lemma 3.**

**Proof.** Let \( P \) be the current outcome of the game \( G \). If \( c_i(P) = \text{opt}_i(G) \) then we are done. Otherwise, let \( R \) be an outcome so that \( c_i(R) = \text{opt}_i(G) \). Therefore, the lowest possible cost of \( R^i \) is the optimal cost for \( i \). Since \( i \) is an undominated agent, by Lemma 1 for every agent \( j \), there is an intersecting vertex \( y_j \) of \( R^i \) and \( d^j \).
Consider the coalitional deviation in which $i$ selects $R^i$, and every agent $j$ deviates to $R^i_{s,y_j}$ (and from $y_j$ continues as before). Note that this is not necessarily a BCD.

Let $Q$ be the outcome after that coalitional deviation. Let us see that $c_i(Q) = \text{opt}_i(\mathcal{G})$. When $i$ uses the strategy $R^i$, the strategy profile $R^{-i}$ results in an optimal cost for $i$. By the intersecting vertex lemma, in outcome $Q$ no agent $j$ can reduce the cost of the path $R^i$ because every path from $y_j$ to $d^i$ and $R^i$ are edge disjoint. Therefore, the cost of the path $R^i$ cannot be less than its cost in $Q$.

Since we assumed $i$ is not at an optimal cost in $P$, the deviation is beneficial for $i$. If the deviation is beneficial to all agents, then this is a best response w.r.t. $i$ and we are done. Otherwise, the deviation is not beneficial to all agents. W.l.o.g. assume the agents are named so that if $j < k$, then $y_j$ does not come after $y_k$ in the path $R^i$ (e.g., $y_1$ is the intersecting vertex that is closest to $s$ in the path $R^i$, and $i = n$). Let $j$ be the agent with the largest index that does not benefit from the deviation, i.e., the agents $K = \{k: k > j\}$ benefit from the deviation. Since $j$ does not benefit, the cost of $P^j_{s,y_j}$ in outcome $P$ is no greater than the cost of $R^i_{s,y_j}$ in outcome $Q$, i.e.,

$$C_{P^j_{s,y_j}}(P) \leq C_{R^i_{s,y_j}}(Q) \quad \quad (5)$$

Therefore, consider the following coalitional deviation: Every member $k \in K$ deviates to $\tilde{P}^k = P^j_{s,y_j} \cup R^i_{y_j,y_k}$. Let $\tilde{P}$ be the outcome after that coalitional deviation. The cost of the path $P^j_{s,y_j}$ in outcome $\tilde{P}$ is no greater than its cost in outcome $P$ because the load on each edge of $P^j_{s,y_j}$ will not decrease by that deviation, and by (5) the cost of $P^j_{s,y_j}$ in outcome $\tilde{P}$ is no greater than the cost of $R^i_{s,y_j}$ in outcome $Q$. Furthermore, for every $k \in K$, the cost of the path $R^i_{y_j,y_k}$ in $\tilde{P}$ is the same as in $Q$, therefore, the deviation of the coalition $K$ that results in $\tilde{P}$ is still beneficial for every member in coalition $K$, i.e., this is a BCD. Finally, the cost of the path $P^j_{s,y_j} \cup R^i_{y_j,d^i}$ in $\tilde{P}$ is not higher than the cost of $R^i$ in $Q$, which is optimal, which concludes that $c_i(\tilde{P}) = \text{opt}_i(\mathcal{G})$.

**Proof of Lemma**

The lemma: In a dominance based BCD sequence, after the best response with respect to the first agent $i$, no agent deviates away from an edge in $i$’s path throughout the entire sequence.

**Proof.** Let $P_1, P_2, P_3, \ldots$ be a sequence of outcomes that are a result of a dominance based BCD sequence. Assume the first best response is w.r.t. 1, and let $p^* = P^*_1$ be $i$’s path in $P_1$. Assume by contradiction that there exists an outcome in the sequence, for which an agent deviated away from some edge of $p^*$. W.l.o.g. let $P_j$ be the first such outcome. Therefore, in outcomes $P_1 \ldots P_j$ the load on every edge of $p^*$ weakly increases, i.e., the cost of the path $p^*$ remains optimal for $i$. Furthermore, the agent $i$ was not part of any deviation in the sequence $P_1 \ldots P_j$ because it had an optimal cost at every outcome in the sequence (therefore $i$ couldn’t benefit by deviating). As a result, $i$’s strategy in $P_j$ is $p^*$. Let $J$ be the coalition of agents that deviates between the outcomes
Let \( P(z) \) and \( P(z-1) \), and let \( j \in J \) be the agent that corresponds with this best response between \( P(z-1) \) and \( P(z) \). Let \( p^{**} = P_j^* \) be the path to which agents from \( J \) deviate to (i.e., each agent \( k \in J \) up to its intersecting vertex \( y_k^{**} \) of \( p^{**} \) and \( d^j \)). Since \( i \) is an undominated agent, by Lemma 1, agent \( j \) has an intersecting vertex \( y_j \) of \( p^{*} \) and \( d^j \).

First we claim that the edges of the path \( p^{*}, y_j, d^i \) do not incur any change in their loads between \( P(z-1) \) and \( P(z) \), i.e.,

\[
C_{p^{*}, y_j, d^i}(P(z)) = C_{p^{*}, y_j, d^i}(P(z-1)) \tag{6}
\]

If by contradiction you assume that an agent \( k \in J \) causes a change in the load of an edge \( u \rightarrow v \in p^{*}, y_j, d^i \), then \( u \rightarrow v \) must also be on a path to \( d^j \) because there is a path from \( u \rightarrow v \) to the intersecting vertex \( y_k^{**} \), which in turn has a path to \( d^j \). This contradicts Lemma 1.

Therefore, since we assumed by contradiction that an agent deviated away from an edge of \( p^{*} \) it must be that \( p^{**} \neq p^{**}, y_j \). By the minimal effort property, it must be that

\[
C_{p^{**}, y_j}(P(z)) < C_{p^{**}, y_j, d^i}(P(z-1)) \leq C_{p^{**}, y_j, d^i}(P(1)) \tag{7}
\]

Where the last inequality is implied by the way we selected \( z \). Let \( \tilde{p} = p^{**}, y_j \cup p^{*}, y_j, d^i \). Clearly \( \tilde{p} \) is a strategy of \( i \), and since \( i \) is undominated, every agent \( t \) has an intersecting vertex \( y_t^{**} \) of \( \tilde{p} \) and \( d^t \). Therefore, the deviation from outcome \( P(z) \) of every agent \( t \) to \( \tilde{p}, y_j, d^t \) results in an outcome \( \tilde{P} \) in which the cost of \( \tilde{p} \) is minimal. Note that this is not necessarily a BCD, however it is a feasible deviation, i.e., it results in a feasible outcome \( \tilde{P} \) in which the cost of \( \tilde{p} \) does not increase. Therefore:

\[
C_{p^{*}}(P(1)) > opt_i(G), \text{ which contradicts Lemma 3.}
\]

Proof of Corollary 1.

Proof. Clearly, the existence of a BCD in \( G^i \) from the outcome \((S^j)_{j \in N^i}\) implies the existence of a BCD in \( G \) from the outcome \( S \). Since agent \( i \) is undominated in the game \( G^i \), by Lemma 3, if \( i \) not at an optimal cost in outcome \((S^j)_{j \in N^i}\) in game \( G^i \), then there exists a BCD that brings \( i \) to an optimal cost, which contradicts with \( S \) being an SE.