Dominance Products and Faster Algorithms for High-Dimensional Closest Pair under $L_\infty$ *

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Abstract

We give improved algorithmic time bounds for two fundamental problems, and establish a new complexity connection between them. The first is computing dominance product: given a set of $n$ points $p_1, \ldots, p_n$ in $\mathbb{R}^d$, compute a matrix $D$, such that $D_{r,i,j} = |\{k \mid p_i[k] \leq p_j[k]\}|$; this is the number of coordinates at which $p_j$ dominates $p_i$. Dominance product computation has often been applied in algorithm design over the last decade.

The second problem is the $L_\infty$ Closest Pair in high dimensions: given a set $S$ of $n$ points in $\mathbb{R}^d$, find a pair of distinct points in $S$ at minimum distance under the $L_\infty$ metric. When $d$ is constant, there are efficient algorithms that solve this problem, and fast approximate solutions are known for general $d$. However, obtaining an exact solution in very high dimensions seems to be much less understood. We significantly simplify and improve previous results, showing that the problem can be solved by a deterministic strongly-polynomial algorithm that runs in $O(DP(n,d) \log n)$ time, where $DP(n,d)$ is the time bound for computing the dominance product for $n$ points in $\mathbb{R}^d$. For integer coordinates from some interval $[-M,M]$, and for $d = n^r$ for some $r > 0$, we obtain an algorithm that runs in $\tilde{O} \left( \min\{Mn^{\omega(1, r, 1)}, DP(n,d)\} \right)$ time1, where $\omega(1, r, 1)$ is the exponent of multiplying an $n \times n^r$ matrix by an $n^r \times n$ matrix.

1 Introduction

Given a set $S$ of $n$ points $p_1, \ldots, p_n$ in $\mathbb{R}^d$, the well known dominating pairs problem is to find all pairs of points $(p_i, p_j)$ such that $p_i[k] \leq p_j[k]$, for all $k \in [d]$. Connections between computing dominating pairs and other problems have been well established in the last decade. For example, it was used by Chan [8] for developing a subcubic algorithm for computing all pairs shortest paths (APSP) in a real edge weighted directed graph. Dominating pairs computation was also used by Grønlund and Pettie [16] in their recent 3SUM breakthrough result, and in the subsequent 3SUM improvements by Freund [14] and Gold and Sharir [15].

There are several algorithms for computing dominating pairs; the works cited above use the algorithm by Chan [8] that runs in $O(c_d^2 |S|^{1+\varepsilon} + K)$ time, where $K$ is the output size, $\varepsilon \in (0, 1)$.

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1The $\tilde{O}(\cdot)$ notation hides poly-logarithmic factors.

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is arbitrary, and $c_\varepsilon = 2^\varepsilon/(2^\varepsilon - 1)$ is a constant. This algorithm is efficient when $d \leq t \log n$, and $t$ is a sufficiently small constant, as in this case its runtime is linear in the output size, with a strongly-subquadratic overhead. A recent work by Chan [9] gives an efficient (linear in the output size, with a subquadratic overhead, but not strongly subquadratic as before) dominance reporting algorithm for dimensions slightly larger than $\log n$.

For $d = n$, there is a non-trivial algorithm by Matoušek [22], and a slightly improved one by Yuster [31]. For $d \leq n$, there are extensions of Matoušek’s algorithm by Vassilevska-Williams, Williams, and Yuster [28]. All of them use fast matrix multiplication algorithms. These algorithms do more than just determine dominances; they actually compute a matrix $D$ such that for each $i, j \in [n]$, $D[i, j] = \left| \{ k \mid p_i[k] \leq p_j[k] \} \right|$.

This matrix is called the dominance product or dominance matrix for $S$.

Dominance product computations were liberally used to improve some fundamental algorithmic problems. For example, Vassilevska-Williams and Williams [27] gave the first truly subcubic algorithm for finding a maximum weight triangle in a node weighted directed graph. Another application, by Vassilevska-Williams, Williams, and Yuster [28], gives the first truly subcubic algorithm for the all pairs bottleneck paths problem (APBP). Yuster [31] showed that APSP can be solved in truly subcubic time if the number of distinct weights of edges emanating from any fixed vertex is $O(n^{0.338})$. In his algorithm, he uses dominance product computation as a black box.

In the first part of this paper we slightly improve the current bounds for computing the dominance product of $n$ points in $\mathbb{R}^d$. Consequently, these bounds slightly improve many algorithms that use dominance product computation as a bottleneck step, some of them were mentioned above.

To obtain the improvement we follow Matoušek’s original algorithm [22], combine it with the enhancement proposed by Yuster [31], and plug into the analysis Le Gall’s improved exponents for rectangular matrix multiplication, recently obtained in [19,20], combined with an interpolation technique. The entire derivation, while not very deep, is still far from trivial, and requires several tools and some careful implementation. As this problem arises as a basic component in many algorithmic applications, we feel that presenting a coherent spelled-out improved runtime analysis is of major importance, as a service to the community, as well as a major black-box ingredient for the problem studied in the second part of this paper.

In the second part, we show a (what we perceive as surprising) connection between dominance product computation to another well-known problem, finding the closest pair under the $L_\tau$ metric in high dimensions. We develop new algorithms for this problem that are simpler and asymptotically faster than previous ones. Moreover, our new techniques give the first better-than-naive strongly-polynomial algorithm for this problem.

Finding the closest pair among a set of $n$ points in $\mathbb{R}^d$ was among the first studied algorithmic geometric problems, considered at the origins of computational geometry; see [23,26]. The distance between pairs of points is often measured by the $L_\tau$ metric, for some $1 \leq \tau \leq \infty$, under which the distance between the points $p_i = (p_i[1], \ldots, p_i[d])$ and $p_j = (p_j[1], \ldots, p_j[d])$ is

$$\text{dist}_\tau(p_i, p_j) = |p_i - p_j|_\tau = \left( \sum_{k=1}^{d} |p_i[k] - p_j[k]|^\tau \right)^{1/\tau},$$

for $\tau < \infty$, and

$$\text{dist}_\infty(p_i, p_j) = |p_i - p_j|_\infty = \max_k |p_i[k] - p_j[k]|,$$
for $\tau = \infty$. The Closest Pair problem and its corresponding decision variant, under the $L_\tau$-metric, are defined as follows.

**Closest Pair:** Given a set $S$ of $n$ points in $\mathbb{R}^d$, find a pair of distinct points $p_i, p_j \in S$ such that

$$\text{dist}_\tau(p_i, p_j) = \min_{e \in S} \{ \text{dist}_\tau(p_i, p_m) \mid p_e, p_m \in S \}.$$

**Closest Pair Decision:** Given a set $S$ of $n$ points in $\mathbb{R}^d$, and a parameter $\delta > 0$, determine whether there is a pair of distinct points $p_i, p_j \in S$ such that $\text{dist}_\tau(p_i, p_j) \leq \delta$.

Throughout the paper, the notation $L_\tau$ Closest Pair refers to the Closest Pair problem under some specific metric $L_\tau$, for $1 \leq \tau \leq \infty$ (and we will mostly consider the case $\tau = \infty$).

In the algebraic computation tree model (see [5]), the Closest Pair problem has a complexity lower bound of $\Omega(n \log n)$ (for any $L_\tau$ metric), even for the one-dimensional case $d = 1$, as implied from a lower bound for the Element-Uniqueness problem [5, 30].

As for upper bounds, Bentley and Shamos [6, 7] were the first who gave a deterministic algorithm for finding the closest pair under the $L_2$ metric that runs in $O(n \log n)$ time for any constant dimension $d \geq 1$, which is optimal in the algebraic computation tree model, for any fixed $d$. Their algorithm uses the divide-and-conquer paradigm, and became since, a classical textbook example for this technique. In 1976 Rabin presented, in a seminal paper [24], a randomized algorithm that finds the closest pair in $O(n)$ expected time, using the floor function (which is not included in the algebraic computation tree model). His algorithm uses random sampling to decompose the problem into smaller subproblems, and uses the floor function in solving them, for a total cost of $O(n)$ expected time. Later, in 1979, based on Rabin’s ideas, Fortune and Hopcroft [12] gave a deterministic algorithm that uses the floor function, and runs in $O(n \log \log n)$ time.

The bounds above hold as long as the dimension $d$ is constant, as they involve factors that are exponential in $d$. Thus, when $d$ is large (e.g., $d = n$), the problem seems to be much less understood. Shamos and Bentley [7] conjectured in 1976 that, for $d = n$, and under the $L_2$ metric, the problem can be solved in $O(n^2 \log n)$ time; so far, their conjectured bound is considerably far from the $O(n^\omega)$ state-of-the-art time bound for this case [18], where $\omega < 2.373$ denotes the exponent for matrix multiplication (see below). If one settles on approximate solutions, many efficient algorithms were developed in the last two decades, mostly based on LSH (locality sensitive hashing) schemes, and dimensionality reduction via the Johnson-Lindenstrauss transform; see [1, 4] for examples of such algorithms. These algorithms are often used for finding approximate nearest neighbors, which itself is of major importance and in massive use in many practical fields of computer science. Nevertheless, finding an exact solution seems to be a much harder task.

We consider the case where $d$ depends on $n$, i.e., $d = n^r$ for some $r > 0$. Note that a naive brute-force algorithm runs in $O(n^2d)$ time and works for any metric $L_\tau$. For some $L_\tau$ metrics, a much faster solution is known; see [18]. Specifically, the $L_2$ Closest Pair problem can be solved by one algebraic matrix multiplication, so for example when $d = n$, the problem can be solved in $O(n^\omega)$ time (as already mentioned above). If $r \geq 2$ is an even integer, then $L_r$ Closest Pair can be solved in $O(r n^\omega)$ time. However, for other $L_\tau$ metrics, such as when $\tau$ is odd (or fractional), or the $L_\infty$ metric, the known solutions are significantly inferior.

For the $L_1$ and $L_\infty$ metrics, Indyk et al. [18] obtained the first (and best known until now) non-naive algorithms for the case $d = n$. For $L_1$, they gave an algorithm that runs in $O\left(n^{\frac{12}{13}}\right) = O(n^{2.687})$ time, and for $L_\infty$, one that runs in $O\left(n^{\frac{12}{13}} \log D\right) = O(n^{2.687} \log D)$ time, where $D$ is the diameter of the given point-set. The bound for $L_\infty$ is weakly polynomial, due to the dependence
on $D$, and, for real data, only yields an approximation. Their paper is perhaps the most related to our work.

1.1 Preliminaries

Before presenting our results, we review some notations that we will use throughout the paper. We denote by $[N] = \{1, \ldots, [N]\}$, the set of the first $[N]$ natural numbers succeeding zero, for any $N \in \mathbb{R}^+$. For a point $p \in \mathbb{R}^d$, we denote by $p[k]$ the $k$-th coordinate of $p$, for $k \in [d]$. For a matrix $A$, we denote the transpose of $A$ by $A^T$. For some of the presented algorithms, we bound their complexity using the $\tilde{O}()$ notation, which hides poly-logarithmic factors.

Most of the algorithms discussed in this paper heavily rely on fast matrix multiplication algorithms. Throughout the paper, $\omega < 2.373$ denotes the exponent of multiplying two $n \times n$ matrices [3,29], and $\omega(1,r,1)$ refers to the exponent of multiplying an $n \times n^r$ matrix by an $n^r \times n$ matrix, for some $r > 0$; see [17,19]. For more details on rectangular matrix multiplication exponents, we refer the reader to the seminal work of Huang and Pan [17], and to a more recent work of Le Gall [19,20].

1.2 Our Results

First, we give an improved time bound analysis for computing the dominance product of $n$ points in $\mathbb{R}^d$. As mentioned above, these bounds will improve the time bounds for algorithms that use dominance product computation as a bottleneck step (see some examples above). Then, we significantly simplify and improve previous results for the $L_8$ Closest Pair problem, and establish a new complexity connection to the dominance product problem. Our novel approach is different and simpler than the ones in [18]. In particular, we give improved time bounds for every dimension $d$ of the form $n^r$, for $r > 0$, and also for the case where the coordinates of the points are integers from some bounded domain. Our algorithms give also the first non-naive bounds which are strongly polynomial.

We show that the runtime for computing the dominance product for $n$ points in $\mathbb{R}^d$ is at most proportional to the following function of $n$ and $d$.

$$DP(n, d) = \begin{cases} 
\frac{1.891}{1.891} n^{1.896} + n^{2+\omega(1)} & \text{if } d \leq n^{\frac{2}{1.75}} \\
0.909 n^{1.75} & \text{if } n^{0.687} \leq d \leq n^{0.87} \\
0.921 n^{1.73} & \text{if } n^{0.87} \leq d \leq n^{0.963} \\
0.931 n^{1.73} & \text{if } n^{0.963} \leq d \leq n^{1.056} 
\end{cases}$$

In particular, we obtain that $DP(n, n) = n^{2.6598}$, which improves Yuster’s $O(n^{2.684})$ time bound. In the rest of the paper we will often refer to the function above.

We note that in the case of the Euclidean metric, we may assume that the dimension $d$ of the ambient space containing the $n$ given points is at most $n$, since we can restrict the setup to the space, of dimension at most $n$, spanned by the given points, without affecting the $L_2$-distance between any pair of points. In contrast, this does not work for the $L_\infty$ metric, which can be significantly affected by the change of coordinates that will be required when passing to the spanned subspace. We therefore do not assume that $d \leq n$, and present bounds that also apply to case that $d > n$. Formula (1) covers only the partial range $d \leq n^{1.056}$, and can be extended for larger values of $d$ too (see below for further elaboration of this issue).
We obtain the following results for the $L_\infty$ Closest Pair problem in $\mathbb{R}^d$, where $d = n^r$, for some $r > 0$.

— A deterministic algorithm that runs in $O(DP(n, d) \log n)$ time, and a $2$-linear decision tree\(^2\) with depth $O(nd \log^2 n)$.

— A fairly simple randomized algorithm that runs in $O(DP(n, d) \log n)$ expected time.

— For points with integer coordinates from $[-M, M]$, a deterministic algorithm that runs in $\tilde{O}\left( \min\{ M n^{\omega(1, r, 1)}, DP(n, d) \} \right)$ time.

**Hardness results.** To obtain our bounds, we show some interesting hardness results, implied by reductions to other well studied problems. In particular, we show that the $L_\infty$ Closest Pair Decision problem in $\mathbb{R}^d$ is not harder than computing the dominance product of $n$ points in $\mathbb{R}^d$, and that $L_\infty$ Closest Pair is not harder than computing the $(\min, +)$-product of an $n \times d$ matrix with a $d \times n$ matrix. Computing the $(\min, +)$-product is a well-known, extensively studied problem, which we will review later.

### 2 Dominance Products

We recall the dominance product problem: given $n$ points $p_1, \ldots, p_n$ in $\mathbb{R}^d$, we want to compute a matrix $D$ such that for each $i, j \in [n]$,

$$D[i, j] = \left| \{ k \mid p_i[k] \leq p_j[k] \} \right|.$$  

It is easy to see that the matrix $D$ can be computed naively in $O(dn^2)$ time. Note that, in terms of decision tree complexity, it is straightforward to show that $O(dn \log n)$ pairwise comparisons suffice for computing the dominance product of $n$ points in $\mathbb{R}^d$. This is done by sorting the $n$ points by each coordinate $j \in [d]$, and then by computing the dominance product naively in $O(dn^2)$ time, without using any further comparisons. However, the actual best known time bound to solve this problem is significantly larger than its decision tree complexity bound.

The first who gave a truly subcubic algorithm to compute the dominance product of $n$ points in $\mathbb{R}^n$ is Matoušek [22]. We first outline his algorithm, and then present our extensions and improvements.

**Theorem 2.1** (Matoušek [22]). *Given a set $S$ of $n$ points in $\mathbb{R}^n$, the dominance matrix for $S$ can be computed in $O(n^{3/2}) = O(n^{2.687})$ time.*

**Proof.** For each $j \in [n]$, sort the $n$ points by their $j$-th coordinate. This takes a total of $O(n^2 \log n)$ time. Define the $j$-th rank of point $p_i$, denoted as $r_j(p_i)$, to be the position of $p_i$ in the sorted list for coordinate $j$. Let $s \in [\log n, n]$ be a parameter to be determined later. Define $n/s$ pairs (assuming for simplicity that $n/s$ is an integer) of $n \times n$ Boolean matrices $(A_1, B_1), \ldots, (A_{n/s}, B_{n/s})$ as follows:

$$A_k[i, j] = \begin{cases} 1 & \text{if } r_j(p_i) \in \left[ ks, ks + s \right) \\ 0 & \text{otherwise,} \end{cases} \quad B_k[i, j] = \begin{cases} 1 & \text{if } r_j(p_i) \geq ks + s \\ 0 & \text{otherwise,} \end{cases}$$

\(^2\)A $k$-linear decision tree is one in which each branching is based on a sign test of a linear expression with at most $k$ terms. The complexity of the tree is its depth.
for \( i, j \in [n] \). Put \( C_k = A_k \cdot B^T_k \). Then \( C_k[i, j] \) equals the number of coordinates \( t \) such that \( r_t(p_i) \in [ks, ks + s) \), and \( r_t(p_j) \geq ks + s \).

Thus, by letting
\[
C = \sum_{k=1}^{\lfloor n/s \rfloor} C_k,
\]
we have that \( C[i, j] \) is the number of coordinates \( t \) such that \( p_i[t] \leq p_j[t] \) and \( \lfloor r_t(p_i)/s \rfloor < \lfloor r_t(p_j)/s \rfloor \).

Next, we compute a matrix \( E \) such that \( E[i, j] \) is the number of coordinates \( t \) such that \( p_i[t] \leq p_j[t] \) and \( \lfloor r_t(p_i)/s \rfloor = \lfloor r_t(p_j)/s \rfloor \). Then \( D := C + E \) is the desired dominance matrix.

To compute \( E \), we use the \( n \) sorted lists we computed earlier. For each pair \((i, j) \in [n] \times [n]\), we retrieve \( q := r_j(p_i) \). By reading off the adjacent points that precede \( p_i \) in the \( j \)-th sorted list in reverse order (i.e., the points at positions \( q - 1, q - 2, \text{ etc.} \)), and stopping as soon as we reach a point \( p_k \) such that \( \lfloor r_j(p_k)/s \rfloor < \lfloor r_j(p_i)/s \rfloor \), we obtain the list \( p_1, \ldots, p_l \) of \( l \leq s \) points such that \( p_k[j] \leq p_i[j] \) and \( \lfloor r_j(p_i)/s \rfloor = \lfloor r_j(p_k)/s \rfloor \). For each \( x = 1, \ldots, l \), we add a 1 to \( E[i_x, i] \). Assuming constant time lookups and constant time probes into a matrix (as standard in the real RAM model), this entire process takes only \( O(n^2 s) \) time. The runtime of the above procedure is therefore \( O(n^2 s + \frac{n}{s} \cdot n^\omega) \). Choosing \( s = n^{-\frac{\omega+2}{2}} \), the time bound becomes \( O(n^{\frac{3\omega+2}{4}}) \).

Yuster [31] has slightly improved this algorithm, by using rectangular matrix multiplication.

**Theorem 2.2** (Yuster [31]). Given a set \( S \) of \( n \) points in \( \mathbb{R}^n \), the dominance matrix for \( S \) can be computed in \( O(n^{2.684}) \) time.

The improved algorithm is based on Matoušek’s technique, but it computes the sum of several matrix products \( C_k \) by performing a single rectangular matrix multiplication instead of computing each (square) matrix product separately (see below for details). Using the rectangular matrix multiplication techniques due to Huang and Pan [17], with a careful choice of parameters, enabled Yuster to obtain the slightly improved bound.

### 2.1 Generalized and Improved Bounds

We extend Yuster’s idea to obtain bounds for dimension \( d = n^r \), for the entire range \( r > 0 \), and, at the same time, give an improved time analysis, using the recent bounds for rectangular matrix multiplications of Le Gall [19, 20] coupled with an interpolation technique. This analysis is not trivial, as Le Gall’s bounds for \( \omega(1, r, 1) \) are obtained by a nonlinear optimization problem, and are only provided for a few selected values of \( r \) (see Table 1 in [20] and [19]). Combining Le Gall’s exponents with an interpolation technique, similar to the one used by Huang and Pan [17], we obtain improved bounds for all values \( d = n^r \), for any \( r > 0 \).

Note that the matrices \( A_k \) and \( B_k \), defined above, are now \( n \times d \) and \( d \times n \) matrices, respectively. Thus, the sum \( C \) defined earlier, can be viewed as a product of block matrices
\[
C = \begin{bmatrix} A_1 & A_2 & \cdots & A_{n/s} \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{n/s} \end{bmatrix}.
\]

Thus, to compute \( C \) we need to multiply an \( n \times (dn/s) \) matrix by a \((dn/s) \times n \) matrix. Computing \( E \) in this case can be done exactly as in Matoušek’s algorithm, in \( O(nds) \) time.
Consider first the case where \( d \) is small; concretely, \( d \leq n^{\frac{\omega - 1}{2}} \). In this case we compute \( C \) using the following result by Huang and Pan.

**Lemma 2.3** (Huang and Pan [17]). Let \( \alpha = \sup\{0 \leq r \leq 1 \mid w(1, r, 1) = 2 + o(1)\} \). Then for all \( n^\alpha \leq m \leq n \), one can multiply an \( n \times m \) matrix with an \( m \times n \) matrix in time

\[
O \left( m^{\frac{\omega - 2}{1 - \alpha}} n^{\frac{2 - \omega \alpha}{1 - \alpha}} \right).
\]

Huang and Pan [17] showed that \( \alpha > 0.294 \). Recently, Le Gall [19, 20] improved the bound on \( \alpha \) to \( \alpha > 0.302 \). By plugging this into Lemma 2.3, we obtain that multiplying an \( n \times m \) matrix with an \( m \times n \) matrix, where \( n^\alpha \leq m \leq n \), can be done in time

\[
O(m^{0.535} n^{1.839}).
\]

From the above, computing \( C \) and \( E \) can be done in \( O \left( (dn/s)^{0.535} n^{1.839} + dns \right) \) time. By choosing \( s = n^{0.896}/d^{0.303} \), the runtime is asymptotically minimized, and we obtain the time bound \( O(d^{0.697} n^{1.896}) \). This time bound holds only when \( n^\alpha < n^{0.302} \leq dn/s \leq n \), which yields the time bound

\[
O(d^{0.697} n^{1.896} + n^{2+o(1)}), \text{ for } d \leq n^{(\omega-1)/2} \leq n^{0.687}.
\]

We now handle the case \( d > n^{(\omega-1)/2} \). Note that in this case, \( dn/s > n \) (for \( s \) as above), thus, we cannot use the bound from Lemma 2.3. Le Gall [19, 20] gives a table (Table 1 in [20] and [19]) of various respective exponents \( \omega(1, r, 1) \). We will confine ourselves to the given bounds for the values \( r_1 = 1.1, r_2 = 1.2, r_3 = 1.3, \) and \( r_4 = 1.4 \). We denote their corresponding exponents \( \omega(1, r_i, 1) \) by \( \omega_1 \leq 2.456151, \omega_2 \leq 2.539392, \omega_3 \leq 2.624703, \) and \( \omega_4 \leq 2.711707 \) respectively. The exponent for \( r_0 = 1 \) is \( \omega_0 = \omega \leq 2.372864 \) due to Vassilevska-Williams [29] and Le Gall [21].

The algorithm consists of two parts. For a parameter \( s \), that we will fix shortly, the cost of computing \( C = A \cdot B^T \) is \( O(n^{\omega_1}) \), where \( \omega_r \) is a shorthand notation for \( \omega(1, r, 1) \), and where \( n^r = dn/s \), and the cost of computing \( E \) is \( O(nds) = O(s^2 n^r) \). Dropping the constants of proportionality, and equating the two expressions, we choose

\[
s = n^{(\omega - r)/2}, \quad \text{that is,} \quad d = sn^r = n^{(\omega - r)/2 - 1} = n^{\zeta_r},
\]

for \( \zeta_r = (\omega_r + r)/2 - 1 \). Put \( \zeta_i = \zeta_{r_i} \), for the values \( r_0, \ldots, r_4 \) mentioned earlier; see Table 1.

Now if we are lucky and \( d = n^{\zeta_i} \), for \( i = 0, 1, 2, 3, 4 \), then the overall cost of the algorithm is \( O(n^{\omega_i}) \). For in-between values of \( d \), we need to interpolate, using the following bound, which is

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \omega )</th>
<th>( \zeta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_0 = 1.0 )</td>
<td>( \omega_0 = 2.372864 )</td>
<td>( \zeta_0 = 0.6865 )</td>
</tr>
<tr>
<td>( r_1 = 1.1 )</td>
<td>( \omega_1 = 2.456151 )</td>
<td>( \zeta_1 = 0.7781 )</td>
</tr>
<tr>
<td>( r_2 = 1.2 )</td>
<td>( \omega_2 = 2.539392 )</td>
<td>( \zeta_2 = 0.8697 )</td>
</tr>
<tr>
<td>( r_3 = 1.3 )</td>
<td>( \omega_3 = 2.624703 )</td>
<td>( \zeta_3 = 0.9624 )</td>
</tr>
<tr>
<td>( r_4 = 1.4 )</td>
<td>( \omega_4 = 2.711707 )</td>
<td>( \zeta_4 = 1.0559 )</td>
</tr>
</tbody>
</table>

Table 1: The relevant entries from Le Gall’s table. The dominance product can be computed in \( O(n^{\omega_1}) \) time, for dimension \( d_i = n^{\zeta_i} \).
<table>
<thead>
<tr>
<th>$\zeta_{\min}$</th>
<th>$\zeta_{\max}$</th>
<th>$u$</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.687</td>
<td>0.87</td>
<td>0.909</td>
<td>1.75</td>
</tr>
<tr>
<td>0.87</td>
<td>0.963</td>
<td>0.921</td>
<td>1.739</td>
</tr>
<tr>
<td>0.963</td>
<td>1.056</td>
<td>0.931</td>
<td>1.73</td>
</tr>
</tbody>
</table>

Table 2: The time bound for computing dominance product for $n$ points in dimension $n^{\zeta_{\min}} \leq d \leq n^{\zeta_{\max}}$ is $O(d^un^v)$.

derived in the earlier studies (see, e.g., Huang and Pan [17]), and which asserts that, for $a \leq r \leq b$, we have

$$\omega_r \leq \frac{(b - r)\omega_a + (r - a)\omega_b}{b - a}. \quad (2)$$

That is, given $d = n^\zeta$, where $\zeta_i \leq \zeta \leq \zeta_{i+1}$, for some $i \in \{0, 1, 2, 3\}$, the cost of the algorithm will be $O(n^{\omega_r})$, where $r$ satisfies

$$\zeta = \zeta_r = \frac{\omega_r + r}{2} - 1.$$ 

Substituting the bound for $\omega_r$ from (2), with $a = r_i$ and $b = r_{i+1}$, we have

$$\frac{(r_{i+1} - r)\omega_i + (r - r_i)\omega_{i+1}}{r_{i+1} - r_i} + r = 2(\zeta + 1).$$

Eliminating $r$, we get

$$r = \frac{2(\zeta + 1)(r_{i+1} - r_i) - r_{i+1}w_i + r_iw_{i+1}}{w_{i+1} + r_{i+1} - w_i - r_i}, \quad (3)$$

and the cost of the algorithm will be $O(n^{\omega_r})$, where

$$\omega_r \leq \frac{(r_{i+1} - r)\omega_i + (r - r_i)\omega_{i+1}}{r_{i+1} - r_i}. \quad (4)$$

Note that $r$ is a linear function of $\zeta$, and so is $\omega_r$. Writing $\omega_r = u\zeta + v$, the cost is

$$O(n^{u\zeta + v}) = O(n^{u\zeta + v}) = O(d^un^v).$$

The values of $u$ and $v$ for each of our intervals are given in Table 2. (The first row covers the two intervals $1.0 \leq r \leq 1.1$ and $1.1 \leq r \leq 1.2$, as the bounds happen to coincide there.) See also (1) in Section 1.2. We have provided explicit expressions for $DP(n, d)$ only for $d \leq n^{\zeta_4} = n^{1.056}$, which includes the range $d \leq n$, which is the range one expects in practice. Nevertheless, the recipe that we provide can also be applied to larger values of $d$, using larger entries from Le Gall’s table [19,20].

Dropping constant factors, we denote the time bound for computing the dominance product of $n$ points in $\mathbb{R}^d$ by $DP(n, d)$; see (1) in Section 1.2.

We note that this approach can also be used to obtain slightly improved time bounds (on top of our already improved ones) for the case $d \leq n^{(\omega-1)/2}$, by plugging the corresponding values of $0.302 < r < 1$ from Le Gall’s Table 1 in [20]. We also note that, for $d = n$, the time bound is $O(n^{2.6598})$, which improves Yuster’s $O(n^{2.684})$ time bound mentioned above.
3 \( L_\infty \) Closest Pair

Recall that, given a set \( S \) of \( n \) points \( p_1, \ldots, p_n \) in \( \mathbb{R}^d \), the \( L_\infty \) Closest Pair problem is to find a pair of points \((p_i, p_j)\), such that \( i \neq j \) and \( |p_i - p_j|_\infty = \min_{\ell \in \{1, \ldots, d\}} |p_{i,\ell} - p_{j,\ell}|_\infty \). The corresponding decision version of this problem is to determine whether there is a pair of distinct points \((p_i, p_j)\) such that \(|p_i - p_j|_\infty \leq \delta\), for a given \( \delta > 0 \).

Naively, we can compute all the distances between every pair of points in \( O(n^2d) \) time, and choose the smallest one. However, as we see next, a significant improvement can be achieved, for any \( d = n^r \), for any \( r > 0 \).

Specifically, we first obtain the following theorem.

**Theorem 3.1.** Given a parameter \( \delta > 0 \), and a set \( S \) of \( n \) points \( p_1, \ldots, p_n \) in \( \mathbb{R}^d \), the set of all pairs \((p_i, p_j)\) with \(|p_i - p_j|_\infty \leq \delta\), can be computed in \( O(DP(n,d)) \) time.

**Proof.** First, we note the following trivial but useful observation.

**Observation 3.2.** For a pair of points \( p_i, p_j \in \mathbb{R}^d \), \(|p_i - p_j|_\infty \leq \delta \iff p_i[k] \leq p_j[k] + \delta \) and \( p_j[k] \leq p_i[k] + \delta \), for every coordinate \( k \in [d] \).

Indeed, a pair of points \((p_i, p_j)\) satisfies \(|p_i - p_j|_\infty = \max_{k \in [d]} |p_i[k] - p_j[k]| \leq \delta \iff \) for every coordinate \( k \in [d] \), \(|p_i[k] - p_j[k]| \leq \delta \). The last inequalities hold iff \( p_i[k] - p_j[k] \leq \delta \) and \( p_j[k] - p_i[k] \leq \delta \), or, equivalently, iff \( p_i[k] \leq p_j[k] + \delta \) and \( p_j[k] \leq p_i[k] + \delta \), for each \( k \in [d] \). Although the rephrasing in the observation is trivial, it is crucial for our next step. It can be regarded as a (simple) variant of what is usually referred to as “Fredman’s trick” (see [13]).

For every \( i \in [n] \) we create a new point \( p_{n+i} = p_i + (\delta, \delta, \ldots, \delta) \). Thus in total, we now have \( 2n \) points. Concretely, for every \( i \in [n] \), we have the points

\[
p_i = (p_i[1], p_i[2], \ldots, p_i[d]),
p_{n+i} = (p_i[1] + \delta, p_i[2] + \delta, \ldots, p_i[d] + \delta).\]

We compute the dominance matrix \( D_\delta \) for these \( 2n \) points, using the algorithm from Section 2.1. By Observation 3.2, a pair of points \((p_i, p_j)\) satisfies

\[
|p_i - p_j|_\infty \leq \delta \iff (D_\delta[i,n+j] = d) \land (D_\delta[j,n+i] = d),
\]

so we can find all these pairs in \( O(n^2) \) additional time. Clearly, the runtime is determined by the time bound of computing the dominance matrix \( D_\delta \), that is, \( O(DP(n,d)) \).

The proof of Theorem 3.1 shows that solving the \( L_\infty \) Closest Pair Decision is not harder than computing the dominance matrix for \( n \) points in \( \mathbb{R}^d \). In particular, by the decision tree complexity bound for computing dominance matrices, as discussed in Section 2, the following result is straightforward.

**Corollary 3.3.** Given a parameter \( \delta > 0 \), and a set \( S \) of \( n \) points \( p_1, \ldots, p_n \) in \( \mathbb{R}^d \), determining all pairs \( i \neq j \) such that \(|p_i - p_j|_\infty \leq \delta\) can be done using \( O(dn \log n) \) pairwise comparisons (of real numbers).

By Corollary 3.3, we obtain that the 2-linear decision tree complexity for the \( L_\infty \) Closest Pair Decision problem is \( O(dn \log n) \). This bound matches a special case of an old conjectured algorithmic complexity bound by Shamos and Bentley (see Section 1, and [7]).
3.1 Solving the Optimization Problem

The algorithm from Theorem 3.1 solves the \( L_\infty \) Closest Pair Decision problem. It actually gives a stronger result, as it finds all pairs of points \( (p_i, p_j) \) such that \( |p_i - p_j|_\infty \leq \delta \). We use this algorithm in order to solve the optimization problem \( L_\infty \) Closest Pair, i.e., to find a pair of points at minimum \( L_\infty \) distance.

As a “quick and dirty” solution, one can solve the optimization problem by using the algorithm from Theorem 3.1 to guide a binary search over the diameter \( W \) of the input point set, which is at most twice the largest absolute value of the coordinates of the input points. If the coordinates are integers then we need to invoke the algorithm from Theorem 3.1 \( O(\log W) \) times. If the coordinates are reals, we invoke it \( O(B) \) times for \( B \) bits of precision. However, the dependence on \( W \) makes this method weakly polynomial, and, for real data, only yields an approximation. As we show next, this naive approach can be replaced by strongly-polynomial algorithms, one deterministic and one randomized, with \( O(DP(n, d) \log n) \) running time.

Deterministic strongly-polynomial algorithm.

**Theorem 3.4.** Given a set \( S \) of \( n \) points \( p_1, \ldots, p_n \) in \( \mathbb{R}^d \), the \( L_\infty \) Closest Pair problem can be solved for \( S \) in \( O(DP(n, d) \log n) \) time.

**Proof.** Since the distance between the closest pair of points, say \( p_i, p_j \), is

\[
\delta_0 = |p_i - p_j|_\infty = \max_{k \in [d]} |p_i[k] - p_j[k]|
\]

it is one of the \( O(n^2d) \) values \( p_\ell[k] - p_m[k], \ell, m \in [n], k \in [d] \). Our goal is to somehow search through these values, using the decision procedure (i.e., the algorithm from Theorem 3.1). However, enumerating all these values takes \( \Omega(n^2d) \) time, which is too expensive, and pointless anyway, since by having them, the closest pair can be found immediately. Instead, we proceed in the following more efficient manner.

For each \( k \in [d] \), we take the \( k \)-th coordinates of the points of \( S \) and sort them into a sequence \( \sigma_k \). This takes \( O(nd \log n) \) time in total. We now conduct the binary search for the optimum \( \delta_0 \). At each stage of the search we have an interval \((\delta^-, \delta^+)\) that contains \( \delta_0 \), and we want to narrow it down using the decision procedure (i.e., the algorithm from Theorem 3.1). At the initial stage, we set \( \delta^- = 0 \), and for \( \delta^+ \) we choose an arbitrary pair of points from \( S \), compute their distance, and add 1. Clearly, \( \delta_0 \in (\delta^-, \delta^+) \).

For a fixed coordinate \( k \in [d] \), let \( q_1, \ldots, q_n \) denote the points of \( S \) sorted (in \( \sigma_k \)) by their \( k \)-th coordinate. For each \( i = 1, \ldots, n - 1 \), let \( q_{i-} \) and \( q_{i+} \) be the two points in the sequence, such that

\[
\begin{align*}
& i) \quad q_i[k] < q_{i-}[k] \leq q_{i+}[k]; \\
& ii) \quad q_{i-}[k] - q_i[k] < \delta^- \leq q_{i-}[k] - q_i[k]; \\
& iii) \quad q_{i+}[k] - q_i[k] \leq \delta^+ < q_{i+1}[k] - q_i[k];
\end{align*}
\]

note that these points need not always exist.

Informally, only the points \( q_j \) between \( q_{i-} \) and \( q_{i+} \), inclusive, are such that \( q_j[k] - q_i[k] \) can be the distance of the closest pair; see Figure 3.1 for an illustration. Finding \( q_{i-} \) and \( q_{i+} \) is done by a binary search over \( \sigma_k \). (We do these searches only once, upon initialization; accessing these points is
In order to control the process, we need to give a weight to each interval $r_i$. We take a random pair of points within the interval halve the size of half of the intervals. This allows us, for each $i$, that we still need to consider.

We keep running this search, for $O(\log n)$ steps, until the number of surviving triplets becomes sufficiently small, in which case we simply report all these triplets, generate the distances $p_{ij}[k] - p_{ij}[k]$, that still need to consider.

We now take the midpoint $i_m$ of the interval $[i^-, i^+]$ (these notations depend on $k$, but we omit here this dependence, for the sake of simplicity). Repeating this over all $k \in [d]$ and $i \in [n]$ takes $O(dn \log n)$ time, and yields $O(dn)$ distances $q_{im}[k] - q_{im}[k]$, $k \in [d]$, $i \in [n]$. We run a binary search over these distances, using the decision procedure from Theorem 3.1, and find two consecutive distances so that the interval that they delimit contains $\delta_0$. This allows, for each $i \in [n]$ and $k \in [d]$, to discard either the interval $[i^-, i_m]$ or $[i_m, i^+]$. That is, we reduce by a factor of 2 the number of triplets $(p_i, p_j, k)$ that generate a distance $p_{ij}[k] - p_{ij}[k]$ that we still need to consider.

The runtime can be improved to $O(DP(n, d) \log n)$, using a variant of Cole’s trick for parametric search [11]. In brief, rather than running a complete binary search through the sequence of $O(dn)$ values, we take their median and call the decision procedure with this value only. The output allows us to halve the size of half of the intervals $[i^-, i^+]$, while the others remain large (i.e., unchanged). In order to control the process, we need to give a weight to each interval $[i^-, i^+]$, which depends on its size, and use a weighted median for the decision procedure. A careful implementation of this approach, resembling that in [11], yields the improved runtime, with a single $\log n$ factor. We omit the rather routine details.

**Randomized strongly-polynomial algorithm.** While the preceding deterministic algorithm runs in $O(DP(n, d) \log n)$ time, the actual implementation, including Cole’s trick, is somewhat involved. Using randomization, we present a simpler strongly-polynomial algorithm that runs in $O(DP(n, d) \log n)$ expected time.

Our goal is to somehow run binary search over distances between pairs of points and find the minimum. Ideally, we would like to pick the median distance in each remaining range, but, if implemented naively, this would be too costly. Instead, we take a random pair of points within the remaining range of pairs and use its distance as an approximate median. What helps us in this process is the fact that the decision procedure produces all pairs at distance below the threshold.

We start by enumerating the $\binom{n}{2}$ pairs of indices of points from $S$ to a list (note that we only store the indices and not the actual points), in $O(n^2)$ time. We sample a pair of indices $(i, j)$ uniformly at random and compute the distance $\rho = |p_i - p_j|$. Then, we compute the set $\mathcal{X}_\rho$ of all pairs of indices $(\ell, m)$ such that $|p_\ell - p_m| < \rho$. By Theorem 3.1 this can be done in $O(DP(n, d))$ time. (Note that the strong inequality can be achieved in a straightforward way by disallowing equalities in the dominance matrix computation.)

If $\mathcal{X}_\rho$ is empty, we return $(i, j)$; if $\mathcal{X}_\rho$ contains exactly one pair $(\ell, m)$, we return $(\ell, m)$. Other-
wise, we sample a pair of indices \((i', j')\) uniformly at random from \(X_p\), compute \(\rho' = |p_{i'} - p_{j'}|_\infty\), and repeat the search with \(\rho'\). (Note that the cost of enumerating \(X_p\), which is \(O(n^2)\), is subsumed asymptotically by \(DP(n, d)\).)

For the runtime analysis, let \(T(k)\) denote the expected time for the procedure to run on \(n\) points in \(\mathbb{R}^d\), where \(k\) is the number of pairs with distance smaller than the current value of \(\rho\). Then clearly \(T(0) = T(1) = O(1)\), and in general,

\[
T(k) \leq \frac{1}{k} \sum_{i=1}^{k-1} T(k-i) + c \cdot DP(n, d),
\]

for some constant \(c \geq 1.3\). It is an easy exercise to show, by induction on \(k\), that the solution of this recurrence is \(T(k) = O(DP(n, d) \log k)\). Hence, the original problem takes \(T\left(\binom{n}{2}\right) = O(DP(n, d) \log n)\) expected time.

### 4 \(L_\infty\) Closest Pair with Integer Coordinates

A considerable part of the algorithm from the previous section is the reduction to computing a suitable dominance matrix. The algorithms for computing dominance matrices given in Section 2 do not make any assumptions on the coordinates of the points, and support real numbers. When the coordinates are bounded integers, we can improve the algorithms. In particular, for \(n\) points in \(\mathbb{R}^n\) with small integer coordinates we can solve the optimization problem in \(O(n^\omega)\) time, which is a significant improvement compared to the \(O(n^{2.6598})\) time bound of our previous algorithm for this case\(^4\). Our improvement is based on techniques for computing \((\min, +)\)-matrix multiplication over integer-valued matrices.

**Theorem 4.1.** Let \(S\) be a set of \(n\) points \(p_1, \ldots, p_n\) in \(\mathbb{R}^d\) such that \(d = n^r\) for some \(r > 0\), and for all \(i \in [n]\), \(k \in [d]\), \(p_i[k]\) is an integer in \([-M, M]\). Then the \(L_\infty\) closest pair can be computed in

\[
\tilde{O}\left(\min\left\{Mn^{\omega(1, 1)}, DP(n, d)\right\}\right)\text{ time.}
\]

We first define \((\max, +)\)-product and \((\min, +)\)-product over matrices.

**Definition 4.2** (Distance products of matrices). Let \(A\) be an \(n \times m\) matrix and \(B\) be an \(m \times n\) matrix. The \((\max, +)\)-product of \(A\) and \(B\), denoted by \(A \ast B\), is the \(n \times n\) matrix \(C\) whose elements are given by

\[
c_{ij} = \max_{1 \leq k \leq m} \{a_{ik} + b_{kj}\}, \quad \text{for } i, j \in [n].
\]

Similarly, the \((\min, +)\)-product of \(A\) and \(B\) denoted by \(A \ast B\) is the \(n \times n\) matrix \(C'\) whose elements are given by

\[
c'_{ij} = \min_{1 \leq k \leq m} \{a_{ik} + b_{kj}\}, \quad \text{for } i, j \in [n].
\]

We refer to either of the \((\min, +)\)-product or the \((\max, +)\)-product as a *distance product*.

\(^3\)Since in each step of the search we can remove points that are not participating in a pair from \(X_p\), the second term of the recurrence is bounded by \(c \cdot DP(2k, d)\), if \(2k < n\), and by \(DP(n, d)\) otherwise. For some values of \(k\), this could lead to a better time bound, however, for our choice of \(k = \binom{n}{2}\), it does not change the asymptotic bound.

\(^4\)For integer coordinates that are bounded by a constant, the \(L_\infty\)-diameter of the points is also a constant (bounded by twice the largest coordinate), hence, one can use the decision procedure to (naively) guide a binary search over the diameter in constant time.
The distance product of an $n \times m$ matrix by an $m \times n$ matrix can be computed naively in $O(n^2m)$ time. When $m = n$, the problem is equivalent to APSP (all pairs shortest paths) problem in a directed graph with real edge weights, and the fastest algorithm known is a recent one by Chan and Williams [10] that runs in $O\left(\frac{n^3}{2\sqrt{\ln(n)}}\right)$ time. It is a prominent long-standing open problem whether a truly subcubic algorithm for this problem exists. However, when the entries of the matrices are integers, we can convert distance products of matrices into standard algebraic products. We use a technique by Alon, Galil, and Margalit [2] and Zwick [32].

**Lemma 4.3** (Zwick [32]). Given an $n \times m$ matrix $A = \{a_{ij}\}$ and an $m \times n$ matrix $B = \{b_{ij}\}$ such that $m = n^r$ for some $r > 0$, and all the elements of both matrices are integers from $[-M, M]$, their $(\min, +)$-product $C = A \ast B$ can be computed in $\tilde{O}(Mn^{\omega(1,r,1)})$ time.

**Proof.** We create new matrices $A' = \{a'_{ij}\}$ and $B' = \{b'_{ij}\}$, such that, for each $i, j$,

$$d'_{ij} = (m + 1)^{M-a_{ij}}, \quad b'_{ij} = (m + 1)^{M-b_{ij}}.$$ 

The algebraic integer matrix multiplication $C' = A' \cdot B'$ yields

$$c'_{ij} = \sum_{k=1}^{m} (m + 1)^{2M-(a_{ik}+b_{kj})},$$

for every $i, j \in [n]$. We can then recover the desired elements $c_{ij}$ by

$$c_{ij} = 2M - \lceil \log(m+1) \rceil c'_{ij} = \min_{1 \leq k \leq m} \{a_{ik} + b_{kj}\}.$$ 

The fast algebraic matrix multiplication performs here $O(n^{\omega(1,r,1)})$ arithmetical operations on $O(M \log n)$-bit integers. The Schönhage-Strassen integer multiplication algorithm [25] multiplies two $k$-bit integers using $O(k \log k \log \log k)$ bit operations. Thus, when $k = O(M \log n)$, we get that the complexity of each arithmetical operation is $O(M)$. The logarithms used in the computation of $c_{ij}$ can be easily implemented using binary search. Hence, the complexity of the algorithm is $\tilde{O}(Mn^{\omega(1,r,1)})$. \hfill \Box

With minor appropriate modifications, the $(\max, +)$-product of matrices $A$ and $B$ can be computed within the same time as in Lemma 4.3. We now give an algorithm for computing all-pairs $L_\infty$ distances, by using the fast algorithm for computing $(\max, +)$-product over bounded integers.

**Lemma 4.4.** Let $S$ be a set of $n$ points $p_1, \ldots, p_n$ in $\mathbb{R}^d$ such that $d = n^r$ for some $r > 0$, and for all $i \in [n]$, $p_i[k]$ is an integer from the interval $[-M, M]$, for all $k \in [d]$. Then the $L_\infty$ distances between all pairs of points $(p_i, p_j)$ from $S$ can be computed in $\tilde{O}(Mn^{\omega(1,r,1)})$ time.

**Proof.** We create the $n \times d$ matrix $A = \{a_{ik}\}$ and the $d \times n$ matrix $B = (-A)^T = \{b_{ki}\}$, where

$$a_{ik} = p_i[k], \quad \text{for } i \in [n], k \in [d]$$

$$b_{ki} = -p_i[k], \quad \text{for } i \in [n], k \in [d].$$

Now we compute the $(\max, +)$-product $C = A \ast B$. The matrix $L$ of all-pairs $L_\infty$-distances is then easily seen to be

$$L[i,j] = \max\{C[i,j], C[j,i]\} = |p_i - p_j|_\infty,$$

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for every pair $i, j \in [n]$.

Clearly, the runtime is determined by computing the $(\max, +)$-product $C = A \cdot B$. This is done as explained earlier, and achieves the required running time.

Consequently, by taking the minimum from the algorithm above, and the (say, deterministic) algorithm from Section 3, we obtain that for points in $\mathbb{R}^d$ with integer coordinates from $[-M, M]$, where $d = n^r$ for some $r > 0$, we can find the $L_\infty$ closest pair in

$$\tilde{O}\left(\min\left\{Mn^{\omega(1, r, 1)}, DP(n, d)\right\}\right)$$

time, as stated in Theorem 4.1.

References


