Topics in Combinatorics and Graph Theory: Homework Assignment Number 3
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Solutions will be collected in class on Wednesday, May 5, 2010.

1. Show that for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ such that every graph $G = (V, E)$ with $n > n_0$ vertices and at least ϵn^2 edges contains a d-regular (not necessarily spanning or induced) subgraph, where $d \geq \delta n$.

2. Is the following claim correct ? Prove, or describe a counter-example.

   Claim: For every $\epsilon > 0$ there is an $n_0 = n_0(\epsilon)$ such that for every $n > n_0$ and every set $A \subset \{1, 2, \ldots, n\}$ satisfying $|A| \geq \epsilon n$, there are three distinct elements $a, b, c \in A$ satisfying

   $$4a + 6b = 10c.$$

3. Prove that the number of triangle-free graphs on a set V of n labeled vertices is

   $$2^{(\frac{1}{4}+o(1))n^2},$$

   where the $o(1)$-term tends to 0 as n tends to infinity.

4. Prove that for every $\epsilon > 0$ and any integer h there are $\delta = \delta(\epsilon, h) > 0$ and $n_0 = n_0(\epsilon, h)$ such that the following holds. For any graph H on h vertices, any graph G on $n > n_0$ vertices from which one has to delete at least ϵn^2 edges to destroy all copies of H contains at least δn^h copies of H.

5. Is the following claim correct ? Prove, or describe a counter-example.

   Claim: For every $\epsilon > 0$ there are $\delta = \delta(\epsilon) > 0$ and $n_0 = n_0(\epsilon)$ such that for every $n > n_0$ and every set $A \subset Z_n$ from which one has to delete at least ϵn elements to get a set A' with no $a_1, a_2, a_3, a_4, a_5 \in A'$ satisfying

   $$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 \equiv 6 \ (\ mod\ n) \quad (1)$$

   the number of solutions of (1) with $a_1, a_2, a_3, a_4, a_5 \in A$ is at least δn^4.