Probabilistic Methods in Combinatorics: Homework Assignment Number 2
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Solutions will be collected in class on Monday, December 3, 2012.

1. For each $i$, $1 \leq i \leq m = n^{n^2+1}$ let

$$\{A_1^{(i)}, A_2^{(i)}, \ldots, A_n^{(i)}\}$$

be a collection of $n$ pairwise disjoint subsets of the set of integers $\mathbb{Z}$, where $|A_j^{(i)}| = n$ for all $i$ and $j$. Prove that there are $n$ distinct numbers $1 \leq i_1 < i_2 < \ldots < i_n \leq m$ so that the $n$ sets

$$\{A_1^{(i_1)}, A_2^{(i_2)}, \ldots, A_n^{(i_n)}\}$$

are pairwise disjoint.

2. Show that there are infinitely many values of $n$ so that for more than $0.9$ of the graphs $G$ on $n$ labelled vertices the size of the largest clique in $G$ is equal exactly to the size of the largest independent set in it.

3. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be $n$ two dimensional vectors, where each $x_i$ and each $y_i$ is a positive integer that does not exceed $\frac{2n/2}{10\sqrt{n}}$. Show that there are two disjoint nonempty sets $I, J \subset \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

4. Let $A = (a_{ij})$ be an $n$ by $n$ matrix where $a_{ij} \in \{0, 1\}$ for all $i, j$, and suppose further that every column of $A$ contains exactly $n/2$ ones (and hence also exactly $n/2$ zeros). Let $A'$ be a random matrix obtained from $A$ by applying a random permutation to each column of $A$, where all $n$ permutations are chosen independently and uniformly. Prove that if $n$ is large enough then with probability at least 0.9999 the number of rows of the permuted matrix $A'$ in which the number of ones is between $n/2 - \sqrt{n}$ and $n/2 + \sqrt{n}$ is at least $n/4$.

5. (i). Show that for any two integers $k$ and $\ell$ and for any real $p$, $0 < p < 1$, and any integer $n$, the Ramsey number $r(k, \ell)$ is at least

$$n - \binom{n}{k} p^k - \binom{n}{\ell} (1 - p)^\ell.$$

(ii). Apply the above to prove that the Ramsey number $r(4, k)$ satisfies $r(4, k) \geq c(k/\ln k)^\alpha$ for some absolute constant $c > 0$ and for the largest $\alpha > 0$ for which you can derive this inequality from the result in (i).
6. For a tournament $T = (V, E)$ and for two disjoint subsets $A, B$ of $V$, let $e(A, B)$ denote the number of edges of $T$ directed from a vertex of $A$ to a vertex of $B$, that is

$$e(A, B) = |\{(x, y) \in E : x \in A, y \in B\}|.$$

(i) Prove that there exists an absolute constant $c > 0$ so that for any tournament $T = (V, E)$ on $n$ vertices there are two disjoint subsets $A$ and $B$ of $V$ such that $e(A, B) - e(B, A) \geq cn^{3/2}$.

(ii) Prove that there exists an absolute constant $b > 0$ so that for any tournament $T = (V, E)$ on $n$ vertices there is a linear order $L$ of the vertices so that the number of directed edges $(i, j)$ of $T$ for which $i$ precedes $j$ in $L$ is at least

$$\frac{1}{2} \binom{n}{2} + bn^{3/2}.$$