Probabilistic Methods in Combinatorics: Homework Assignment Number 1
Noga Alon

Solutions will be collected in class on Monday, November 12, 2012.

1. Let $p$ be a prime number and let $A \subset \mathbb{Z}_p$ be a set of $|A| < p^{2/3}$ residues modulo $p$. Show that there are elements $x, y \in \mathbb{Z}_p$ such that for $A + x = \{(a + x) \mod p : a \in A\}$ and $A + y = \{(a + y) \mod p : a \in A\}$, the three sets $A, A + x$ and $A + y$ do not have a common intersection, that is $A \cap (A + x) \cap (A + y) = \emptyset$.

2. The (multi-colored) Ramsey number $r_3(k)$ is the smallest integer $r$ so that in any coloring by 3 colors of the edges of the complete graph on $r$ vertices there is a monochromatic copy of $K_k$.
   (i) Prove that if $\binom{n}{k} 3^{1 - \binom{k}{2}} < 1$ then it is possible to color the edges of the complete graph on $n$ vertices by 3 colors without a monochromatic copy of $K_k$ and conclude that for $k > 4$, $r_3(k) > 3^{k/2}$.
   (ii) Prove that for all $k > 4$, $r_8(k) > 4^k$ (which is much bigger than $8^{k/2}$).

3. Prove that every set $A$ of $n$ nonzero integers contains two disjoint subsets $B_1, B_2 \subset A$, so that $|B_1| + |B_2| > 2n/3$ and each set $B_i$ is sum-free (that is, there are no $b_1, b_2, b_3 \in B_i$ so that $b_1 + b_2 = b_3$).

4. Let $G = (V, E)$ be a graph on $n$ vertices and let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be the degrees of the vertices of $G$. Show that $G$ contains two disjoint independent sets of vertices $A$ and $B$ so that
   \[ |A| + |B| \geq \sum_{i=1}^{n} \frac{2}{d_i + 1}. \]

5. (i) Let $F$ be a collection of $n$ aligned closed squares with pairwise disjoint interiors contained in the unit square $[0, 1]^2$. Show that there is a vertical or horizontal line that intersects the unit square and intersects at most $\sqrt{n}$ members of $F$.
   (ii) Is the following statement correct? Prove or provide a counterexample:
   Let $F$ be a collection of $n$ aligned closed rectangles with pairwise disjoint interiors contained in the unit square $[0, 1]^2$. Then there is a vertical or horizontal line that intersects the unit square and intersects at most $n^{0.9}$ members of $F$.

6. (*) Bonus Question
   (i) Let $A_1, A_2, \ldots, A_n$ be $n > 100$ events in a probability space and suppose that the probability of each event $A_i$ is at least $1/3$. Prove that there is a subset $\{A_i : i \in I\}$, $I \subset \{1, 2, \ldots, n\}$ of at least $n/100$ of the events so that the probability of $A_i \cap A_j$ is at least $\frac{1}{3n^{1/3}}$ for all $i, j \in I$. 

(ii) Is the following statement correct? Prove or provide a counterexample:

Let $A_1, A_2, \ldots, A_n$ be $n > 1000$ events in a probability space and suppose that the probability of each event $A_i$ is at least $1/3$. Then there is a subset $\{A_i : i \in I\}, I \subset \{1, 2, \ldots, n\}$ of at least $n/1000$ of the events so that the probability of $A_i \cap A_j$ is at least $\frac{1}{1000}$ for all $i, j \in I$.

Hints:

(i) Take 3 independent random points in the sample space and consider the events that hold in all of them.

(ii) Suppose that $n = \binom{3r}{r}$, let the sample space be $\{1, 2, \ldots, 3r\}$ and let the events be all collections of $r$ points of the sample space.