The Complexity of the Outer Face in Arrangements of Random Segments
(a modified version of Section 3) *

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1 The phase transition

In this section we prove the following.

Theorem 0.1 [Critically dense arrangements] There exist constants $c_2 > 0$ and $C_2 > C_1 > 0$, such that, for each $1 \geq \ell > 0$ and $n = \lceil \frac{c_2}{\ell^2} \rceil$,

$$C_2 \sqrt{n} \geq f(n, \ell) \geq C_1 \sqrt{n}. \quad (1)$$

Recall that $S$ denotes the unit square. Let $\mathcal{A} = \mathcal{A}(\mathcal{C})$, $\mathcal{C} \in \mathcal{C}_n^\ell$, be an arrangement of $n$ random segments, as defined above. Given a parameter $k$, we let $G = G(k)$ denote the partition of $S$ into a grid of $k \times k$ equal squares.

An $m$-boundary sequence in $G$ is a sequence $(c_1, c_2, \ldots, c_m)$ of $G$-squares with $c_1$ incident to the boundary of $S$ and with each pair of adjacent squares $(c_i, c_{i+1})$ intersecting along a common edge.

An exposed $m$-boundary sequence in $G$, relative to $\mathcal{A}$, is an $m$-boundary sequence in $G$ with all the squares in the sequence intersecting the outer face of $\mathcal{A}$.

A square that belongs to at least one exposed $m$-boundary sequences is called an exposed square. The squares that are incident to the boundary edges of $S$ are the boundary squares of $G$ and the other squares are internal. A $G$-square $c$ is said to be well bounded in an arrangement of segments if the arrangement induced by the segments with sources inside $c$, has an outer face disjoint from $c$. See Figure 1 for an illustration.

Roughly speaking, we argue as follows: Given $\ell > 0$, let $n = \lceil \frac{\mu}{\ell^2} \rceil$, where $\mu > 0$ is a parameter to be determined later, and take $k = \lfloor 1/\ell \rfloor$. Then, for a random arrangement $\mathcal{A} = \mathcal{A}(\mathcal{C})$, $\mathcal{C} \in \mathcal{C}_n^\ell$, and for a fixed square $c$ in $G = G(k)$, the expected number of segments with sources in $c$ is

$$\frac{n}{k^2} \geq \frac{\mu}{\ell^2} \cdot \ell^2 = \mu.$$  

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Figure 1: Example of a well bounded square — the solid segments (drawn as arrows emanating from their source point) completely bound the shaded square.
Therefore, by controlling the value of $\mu$, we can ensure that with high probability, a fixed grid square contains many segment sources, and therefore (using Lemma 1.1 below) is well bounded with high probability. That is, the subset of segments with sources within a fixed grid square induces an arrangement that, with high probability, completely separates the square from the outer face of $A(C)$. This implies that only a small fraction of the squares of $G$ are expected to intersect the outer face. Furthermore, with $\mu$ large enough, we can bound the probability that a fixed $m$-boundary sequence is exposed relative to $A$ by $O(\Delta^m)$, for an appropriate constant $\Delta < 1/3$. Since the overall number of $m$-boundary sequences is $O(k \cdot 3^m)$, it follows that the expected number of exposed $m$-boundary sequences, summed over all $m \geq 1$, is $O(k)$. This in turn provides a trivial upper bound of $O(\sqrt{n})$ on the expected number of exposed squares. Since any segment that meets such a square must have its source in a nearby square, and since the expected number of segments in any square is $O(1)$, one can deduce that the expected number of segments that contribute to the outer face is only $O(\sqrt{n})$.

In more detail, the proof proceeds as follows.

Lemma 1.1 For each $0 < p < 1$ there exists $\mu = \mu(p)$, such that the following holds. Suppose $\ell > 0$, $k = [1/\ell]$ and let $c$ be a $1/k$ by $1/k$ square. Consider a random collection of $\mu$ segments, each of length $\ell$, where each segment is chosen by picking its source uniformly at random in $c$, and by choosing its orientation uniformly at random on the unit circle. Then, the probability that $c$ is well bounded in the arrangement formed by these segments is at least $p$.

Proof. Since each of the segments is of length at least $1/(2k)$ and $c$ is a $1/k \times 1/k$ square, we can arrange a subset of $\mu_0$ segments in some fixed pattern (e.g., $\mu_0 = 25$ as shown in Figure 1) to separate $c$ from the outer face. Since a small perturbation of each segment still yields a configuration that keeps $c$ disjoint from the outer face, we conclude that there is some (small) probability $p_0$ for $c$ to be well bounded. Now a standard amplification argument implies that the probability of $c$ to be well bounded is at least $1 - (1 - p_0)^m$, for $\mu = m \mu_0$. This can be made $> p$ if we choose $m$ (and $\mu$) sufficiently large. ■

For each grid square $c$, define the indicator random variable

$$Z_c = \begin{cases} 1, & \text{if the square } c \text{ is not well bounded}, \\ 0, & \text{otherwise}. \end{cases}$$

By taking $\mu$ sufficiently large, with $n = \lceil \mu / \ell^2 \rceil$, as above, we can ensure, using Lemma 1.1, that $E[Z_c]$ is sufficiently small for each square $c$. Recall that a grid square $c$ is exposed if there exists some $m$-boundary sequence $(c_1, c_2, \ldots, c_m = c)$, for some $m \geq 1$, such that $Z_{c_i} = 1$, for each $1 \leq i \leq m$.

Lemma 1.2 For any constants $0 \leq a < A < \frac{1}{3}$, the following statement holds:
If for every $m$-boundary sequence $(c_1, \ldots, c_m)$

$$\Pr \left[ \bigwedge_{1 \leq i \leq m} Z_{c_i} = 1 \right] \leq a^m,$$ \hspace{1cm} (1)

then the expected number of exposed squares is $O(k)$, where the (hidden) constant of proportionality depends only on $A$. 

Proof. Denote by $E_m$ the expected number of exposed squares that have a witness boundary sequence of length $m$. Let $Y_m = O(k \cdot 3^m)$ denote the total number of boundary sequences of length $m$. By (1), the expected number of exposed squares is at most
\[
\sum_{m=1}^{\infty} E_m \leq \sum_{m=1}^{\infty} Y_m a^m \leq O(k) \sum_{m=1}^{\infty} 3^m a^m \leq O(k),
\]
using the assumption $a < A < 1/3$ for the last inequality.

Lemma 1.3 There exists a constant $b > 0$ so that if $n > b/\ell^2$ then for every $m$-boundary sequence $(c_1, c_2, \ldots, c_m)$, the inequality (1) holds with $a = 1/4$.

Proof. By Lemma 1.1 there exists some $\mu$ such that for a fixed grid square with $\mu$ random segments whose sources are chosen uniformly in it, the probability that the square is not well bounded is at most, say, $1/256$.

Suppose $b > 2\mu$ and $n \geq \lceil b/\ell^2 \rceil$. Each square $c$ that satisfies $Z_c = 1$ is classified as being either of type $A$, if $c$ contains fewer than $\mu$ segment sources, or of type $B$, if $c$ contains at least $\mu$ segment sources while not being well bounded.

Clearly, for any $m \geq 1$, if some fixed $m$-boundary sequence $(c_1, \ldots, c_m)$ is such that all its squares are exposed, then it contains either (at least) $m' = \lceil m/2 \rceil$ squares of type $A$, or (at least) $m'$ squares of type $B$. We proceed to show that the probability of each of these two events is small. To this end, it is convenient to consider the following procedure for generating the random segments in the collection $C$. Each segment $s \in C$ is chosen, randomly and independently, in two steps. In the first step, select the $G(k)$-square containing the source of $s$, where all $k^2$ choices are equally likely. In the second step, choose the precise location of the source inside the selected grid-square, as well as the direction of the segment. Obviously this is equivalent to the original way of generating our random collection. This equivalent description is, however, more convenient for what follows.

The probability that there are at least $m'$ squares of type $A$ in our fixed $m$-boundary sequence can be bounded by examining the results of the random choices in the first step for all segments. Indeed, there are $\binom{m}{m'} < 2^m$ possible ways to choose $m'$ squares among those of the sequence, and the probability that each of them contains less than $\mu$ sources of segments is at most the probability that the value of a binomial random variable with parameters $n$ and $P = m'/k^2$ is less than half its expectation. By Chernoff’s Inequality (see, e.g., AlonSpencer, Appendix A, Theorem A.1.13), this is bounded by
\[
e^{-nP/8} \leq e^{-b m/16}
\]
which is less than $1/16^m$ provided, say, $b > 50$. Multiplying this estimate by the number of possible choices for the $m'$ uncrowded squares of the sequence we conclude that the probability of this event is smaller than $1/8^m$.

We next claim that the probability that there are at least $m'$ squares of type $B$ in our fixed $m$-boundary sequence is at most
\[
\binom{m}{m'} \left( \frac{1}{256} \right)^{m/2} < \frac{1}{8^m}.
\]
Indeed, there are $\binom{m}{m'}$ ways to select $m'$ squares in the sequence. Fixing $m'$ squares consider our two-step choice of the segments in the collection $C$. The probability that all these squares are of
type B is at most the conditional probability that this happens, assuming that each of them contains at least \( \mu \) sources of segments by the end of the first step. But this conditional probability depends only on the choices in the second step, and in the second step this is the intersection of \( m' \) mutually independent events, each having probability at most \( \frac{1}{256} \), by Lemma 1.1 and the choice of \( \mu \). This proves the claim and completes the proof of the lemma, as \( \frac{1}{8m} + \frac{1}{8m} \leq \frac{1}{4m} \) (with room to spare for all \( m > 1 \)). ■

**Proof [of Theorem]** Take \( k = \lfloor \frac{1}{\ell} \rfloor \), and construct the \( k \times k \) grid partition \( G = G(k) \) of \( S \).

By Lemma 1.2 and Lemma 1.3 there are positive constants \( b, B \) such that if \( n \geq \frac{b}{\ell^2} \) then the expected number of exposed grid squares is at most \( Bk \). It is convenient to prove the theorem with \( c_2 = b + 1 \). We first prove the upper bound. Note that \( n = \lceil \frac{c_2}{\ell^2} \rceil \) is at least \( \lceil \frac{b}{\ell^2} \rceil + 1 \). For a fixed random segment \( s \) in our collection \( C \), the probability that \( s \) intersects the outer face is at most the probability that its source lies within distance \( \ell \) from a grid square which is exposed in the random arrangement of all segments in \( C \) besides \( s \). Therefore, the probability that \( s \) intersects the outer face is at most

\[
\frac{1}{k^2} \sum_{c \in G(k)} \sum_{c' \in G(k), c \cap c' \neq \emptyset} Pr[c' \text{ is exposed in } C - \{s\}] \leq \frac{9}{k^2} \sum_{c \in G(k)} Pr[c \text{ is exposed in } C - \{s\}].
\]

The last sum is precisely the expected number of exposed squares in \( C - \{s\} \) which, by Lemmas 1.2 and 1.3 is at most \( Bk \). By linearity of expectation, the expected number of segments that intersect the outer face is at most

\[
\frac{n9B}{k} = \lceil \frac{6}{\ell^2} \rceil \frac{9B}{k} \leq O(k) = O(\sqrt{n}).
\]

This establishes the upper bound.

The proof of the lower bound is simpler. The probability that the source of a fixed segment lies in a boundary grid square \( c \), and no source of any other segment lies in any square intersecting \( c \), is at least

\[
\frac{4k - 2}{k^2} (1 - \frac{6}{k^2})^{n-1} = \Omega(\frac{1}{k}).
\]

By linearity of expectation, the expected number of such segments is \( \Omega(n \frac{1}{k}) = \Omega(k) = \Omega(\sqrt{n}) \). The lower bound follows, since any such segment intersects the outer face. ■

Note that the constants in the proof can be easily improved, and we make no attempts to optimize them here.

**Corollary 1.4** There exists \( \mu > 0 \) such that

\[
\lim_{\ell \to 0} \frac{f(n(\ell), \ell)}{n(\ell)} = 0,
\]

where \( n(\ell) = \lceil \mu/\ell^2 \rceil \).

**Remark.** Note that for small \( \ell \), the combinatorial complexity of the outer face undergoes a rapid phase transition quite a while before all grid squares in \( S \) are expected to be well bounded.
Specifically, with high probability, $S$ still contains many grid squares that are not well bounded in $A(C)$, for $C \in \mathcal{C}_n^\ell$, when $n = \mu(1/\ell^2 \ln(1/\ell^2))$, $\ell$ is sufficiently small, and $\mu > 0$ is smaller than $\frac{1}{12}$. Here is a proof of this fact. For each boundary grid square $c$ besides the 4 corner squares, let $Y_c$ denote the indicator random variable whose value is 1 iff no source of a segment of the random collection lies in a grid square intersecting $c$. Let $Y = \sum_c Y_c$ be the total number of such non-corner grid squares. Note that each such square is exposed in the arrangement; in fact, no segment intersects it. The expected value of each fixed $Y_c$ is precisely $(1 - \frac{12}{k^2})^n$. Thus, the expectation of $Y$ is $E(Y) = 4(k - 1)(1 - \frac{6}{k^2})^n$.

The variance of $Y$ satisfies

$$Var[Y] = \sum_c Var[Y_c] + 2 \sum_{c \neq c'} Cov(Y_c, Y_{c'}) = \sum_c Var[Y_c] + 2 \sum_{c \neq c'} (E[Y_c Y_{c'}] - E[Y_c]E[Y_{c'}]),$$

where in the second sum $c, c'$ run over all unordered pairs of distinct non-corner boundary grid squares.

If $c$ and $c'$ do not have any grid square intersecting both of them, then

$$E[Y_c Y_{c'}] = (1 - \frac{12}{k^2})^n < (1 - \frac{6}{k^2})^{2n} = E[Y_c]E[Y_{c'}],$$

implying that for such pairs, the covariance $Cov(Y_c, Y_{c'})$ is negative. For every other pair, $Cov(Y_c, Y_{c'}) \leq E[Y_c Y_{c'}] \leq E(Y_c)$. This, together with the easy fact that for every $c$, $Var(Y_c) \leq E(Y_c)$, and the fact that there are less than $(4k - 2)2$ unordered pairs of boundary non-corner squares that do have a grid square intersecting both of them, imply that

$$Var(Y) \leq 5E(Y).$$

It thus follows, by Chebyschev’s Inequality, that the probability that $Y$ is 0 is at most $\frac{Var(Y)}{E(Y)^2} \leq \frac{5}{E(Y)}$. Note that $E(Y)$ tends to infinity when $n = (1 - \epsilon)\frac{k^2}{6} \ln k$ for any fixed positive $\epsilon$, as $k$ tends to infinity. Therefore, for large $k$ and $n$ as above, there will indeed be exposed boundary squares with high probability, a claimed.