Let \( c \geq 2 \) and \( n \) be integers, put \( C = \{1, 2, \ldots, c\}, C' = C \cup -C \) and \( N = \{1, 2, \ldots, n\} \). The projection \( P(f) : N \mapsto \{-1, 1\} \) of a function \( f : N \mapsto C' \) is the function defined by \( P(f)(i) = \text{sign}(f(i)) \). Two functions \( f_1 \) and \( f_2 \) from \( N \) to \( C' \) are called separated if \( P(f_1) \neq P(f_2) \).

For a family \( F \) of functions mapping \( N \) to \( C' \), for a subset \( I \subset N \) and for a collection of pairs \( A = (a_i, b_i)_{i \in I} \), where \( a_i \in C \) and \( b_i \in -C \) for all \( i \), we say that \( F \) shatters the pair \((I, A)\) if for every function \( g : I \mapsto \cup_{i \in I} \{a_i, b_i\} \) for which \( g(i) \in \{a_i, b_i\} \) for all \( i \in I \) there is an \( f \in F \) whose restriction to \( I \), \( f|_I \), is \( g \). Note that by definition any nonempty \( F \) shatters the pair \((I = \emptyset, A = \emptyset)\). The dimension of \( F \), \( d(F) \), is the maximum number \( d \) such that there is an \( I \subset N \), \( |I| = d \) and a pair \((I, A)\) so that \( F \) shatters \((I, A)\).

**Theorem 1** There exists an absolute positive constant \( \alpha > 0 \) such that for every family \( F \) of functions \( f : N \mapsto C' \) whose members are pairwise separated that satisfies

\[
|F| > \sum_{i=0}^{k-1} \binom{n}{i},
\]

the dimension \( d(F) \) satisfies

\[
d(F) \geq \frac{k}{(\log c)^2}.
\]

The above theorem is a variant of a lemma of Sauer [4] and Perles and Shelah [5]. See also [1] for some related results. The proof here applies the basic idea in the proof of the combinatorial lemma of [2], together with some additional ideas.

In order to prove the above result we need several simple lemmas. To simplify the presentation, we omit all floor and ceiling signs whenever these are not crucial.

**Lemma 2** Let \( G \) be a family of vectors of length \( n \) with \( \{-1, 1\} \)-coordinates, and let \( p_i \) denote the fraction of members of \( G \) whose \( i \)-th-coordinate is 1. Let \( H(p) = -p \log_2 p - (1-p) \log_2(1-p) \) be the binary entropy function. Then

\[
|G| \leq 2 \sum_{i=1}^{n} H(p_i).
\]

In particular, if \( |G| \geq 2^{n/2} \) then there exists some \( i \) for which \( 1/10 \leq p_i \leq 9/10 \).

**Proof:** This follows from a a standard entropy inequality (cf., e.g., [3]). \( \square \)

Let \( h(m, n, c) \) denote the maximum integer \( h \) for which the following holds. For every family \( F \) of \( |F| = m \) functions from \( N \) to \( C' \) which are pairwise separated, there are at least \( h \) pairs \((I, A)\) shattered by \( F \). Obviously, \( h(m, n, c) = 1 \) for every \( m > 0 \), \( n \) and \( c \), since the pair \((I = \emptyset, A = \emptyset)\) is always shattered.
Claim: If $m \geq 2^{n/2}$ then
\[ h(m, n, c) \geq 2h\left(\frac{m}{10c}, n - 1, c\right). \]

Proof: By Lemma 2 there is a coordinate, say 1, such that the fraction of vectors in the family \( \{P(f) : f \in \mathcal{F}\} \) satisfying \( P(f)(1) = 1 \) is between 1/10 and 9/10. By choosing the most popular positive value \( a_1 \) of \( f(1) \) for \( f \in \mathcal{F} \) and the most popular negative value \( b_1 \) of \( f(1) \) for \( f \in \mathcal{F} \) we conclude that there are two subfamilies \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) of \( \mathcal{F} \) satisfying \( |\mathcal{F}_1| = |\mathcal{F}_2| = \frac{m}{10c} \) such that \( f_1(1) = a_1 \) for all \( f_1 \in \mathcal{F}_1 \) and \( f_2(1) = b_1 \) for all \( f_2 \in \mathcal{F}_2 \). Let \( \mathcal{G}_i \) be the family of all restrictions of the members of \( \mathcal{F}_i \) to \( I - \{1\} \) \( (i = 1, 2) \). It is obvious that all members of \( \mathcal{G}_i \) are pairwise separated and hence, by the definition of \( h \), each \( \mathcal{G}_i \) shatters at least \( h\left(\frac{m}{10c}, n - 1, c\right) \) pairs \((I, A)\). Obviously, any pair \((I, A)\) shattered by one of the families \( \mathcal{G}_i \) is shattered by \( \mathcal{F} \) as well. Moreover, if the same pair \((I, A)\) is shattered both by \( \mathcal{G}_1 \) and by \( \mathcal{G}_2 \), then there is another pair shattered by \( \mathcal{F} \), namely the pair \((I \cup \{1\}, A \cup \{(a_1, b_1)\})\). This completes the proof of the claim. □

Lemma 3
\[ h(2^n, n, c) \geq 2^{\frac{n}{\log_2(10c)}}. \]

Proof. By repeatedly applying the above claim we conclude that for every \( i \) for which \( 2^n/(10c)^i \geq 2^{n/2} \),
\[ h(2^n, n, c) \geq 2^i h\left(\frac{2^n}{(10c)^i}, n - i, c\right), \]
and the result follows as \( h(m, n', c) \geq 1 \) for every positive \( m \). □

Corollary 4 If
\[ 2^{\frac{n}{\log_2(10c)}} > \sum_{i=0}^{d-1} \binom{n}{i} e^{2i} \]
then any family \( \mathcal{F} \) of \( 2^n \) pairwise separated functions from \( N \) to \( C' \) satisfies
\[ d(\mathcal{F}) \geq d. \]

Therefore
\[ d(\mathcal{F}) \geq \alpha \frac{n}{(\log_2 c)^2} \]
for each such \( \mathcal{F} \), where \( \alpha > 0 \) is an absolute constant.

Proof: By Lemma 3, \( \mathcal{F} \) shatters at least \( 2^{\frac{n}{\log_2(10c)}} \) pairs \((I, A)\). Since there are at most \( \binom{n}{i} e^{2i} \) possible pairs \((I, A)\) with \(|I| = i\) the first inequality bounding \( d(\mathcal{F}) \) follows. The second one follows from the first by straightforward calculation. □

Proof of Theorem 1. Let \( \mathcal{F} \) be a family as in the theorem. By the Lemma of [4] there is a set \( K \subset N, |K| = k \) such that
\[ |\{P(f)|_K, f \in \mathcal{F}\}| = 2^k, \]
that is, the restricted projections of the members of \( F \) on \( K \) give all possible sign patterns on \( K \). Let \( G \) be a family of \( 2^k \) pairwise separated functions from \( K \) to \( C' \), each of which is a restriction of an appropriate member of \( F \). The result now follows by applying Corollary 4 to the family \( G \). \( \square \)

References


