Extremal and Probabilistic Combinatorics

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1 Combinatorics – an introduction

1.1 Examples

Testing friendship relations between children some fifty years ago, the Hungarian Sociologist S. Szalai observed that any group of about twenty children he checked contained a set of four children any two of which were friends, or a set of four no two of which were friends. Despite the temptation to try and draw some behavioral consequences, Szalai realized this may well be a mathematical phenomenon, rather than a sociological one. Indeed, a brief discussion with the mathematicians P. Erdős, P. Turán and V. Sós convinced him this was the case. For every symmetric relation \( R \) on a set \( X \) of size 18 or more, there is a subset \( S \) of size 4 so that the relation \( R \) either contains all \( \binom{4}{2} = 6 \) pairs of distinct members of \( S \) or none of them. Here, the symmetric relation \( R \) consists of all pairs of friends among the group of children \( X \). The above fact is a very special case of the Ramsey Theorem proved by the economist and mathematician F. Ramsey in 1930. This result led to the development of Ramsey Theory, a branch of Extremal Combinatorics.

Motivated by the study of Fermat’s Last Theorem, I. Schur proved in 1916 that for every integer \( k \) and every prime \( p \) bigger than \( 3k \), there are three integers, \( a, b, c \), such that \( p \) divides the difference \( a^k + b^k - c^k \), but does not divide any of the integers \( a, b \) and \( c \). Although this is a result in Number Theory, its (simple) proof is purely combinatorial, and forms another example of the many applications of Graph Theory and Ramsey Theory.

When studying the number of real zeros of random polynomials, Littlewood and Offord investigated in 1943 the following problem. Given \( n \) (not necessarily distinct) complex numbers \( z_1, z_2, \ldots, z_n \) of absolute value at least 1, what is the maximum possible number of sums \( \sum_{i=1}^{n} \epsilon_i z_i \), with \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{ -1, 1 \}^n \), such that the difference between any two sums is of absolute value less than 2? Kleitman and Katona proved that the maximum is \( \binom{n}{\lfloor n/2 \rfloor} \), by applying tools from Extremal Finite Set Theory, another area of Extremal Combinatorics.

Consider a school in which there are \( m \) teachers \( T_1, T_2, \ldots, T_m \), and \( n \) classes \( C_1, C_2, \ldots, C_n \). The teacher \( T_i \) has to teach the class \( C_j \) a specified number of periods \( p_{ij} \) of periods. What is the minimum possible number of periods in a complete timetable? Let \( d_i \) denote the total number of periods the teacher \( T_i \) has to teach, and let \( c_j \) denote the total number of periods the class \( C_j \) has to be taught. Clearly, the number of periods required for a complete schedule is at least \( d \), the maximum of all numbers \( d_i \) and \( c_j \). It turns out that this obvious lower bound is also an upper bound; there is always a complete timetable consisting of that number of periods. This is a consequence of König’s Theorem, a basic result in Graph Theory. Suppose, now, that the situation is not so simple; for every teacher \( T_i \) and class \( C_j \), there is a specified set of \( d \) specific possible periods in which the teaching has to take place. Can we always find a feasible timetable, keeping these constraints? Recent results on a subject known as list coloring of graphs implies this is always possible.

Can the countries of any planar map be colored in at most four colors so that no two countries that share a common boundary have the same color? Here we assume that each country forms a connected region in the plane. Of course, at least four colors may be necessary – think of Belgium, France, Germany and Luxembourg, each having a common border with another. The Four Color Theorem, proved by Appel and Haken in 1976, asserts that indeed four colors always suffice. The study of this problem led to numerous interesting questions and results about Graph Coloring.

Let \( S \) be an arbitrary subset of the lattice \( \mathbb{Z}^2 \). For any two finite subsets \( A, B \subset Z \), let \( d_S(A, B) = \frac{|S \cap A \times B|}{|A||B|} \) denote the density of \( S \) in the combinatorial rectangle \( A \times B \). Define \( d(S) = \limsup_{k \to \infty} \{d_S(A, B) : |A| = |B| = k\} \). What are the possible values of \( d(S) \)? Basic results in Extremal Graph Theory imply that there are only two possibilities: for every \( S \), either \( d(S) = 0 \) or \( d(S) = 1 \). Similarly, we can define a more symmetric quantity. For a finite subset \( A \subset Z \) of car-
dinality \( k \), put

\[
dd_S(A) = \frac{|\{(a, a') \in S : a < a', a, a' \in A\}|}{\binom{k}{2}}
\]

and define \( dd(S) = \limsup_{k \to \infty} \{dd_S(A) : |A| = k\} \). What are the possible values of \( dd(S) \)? Here, as follows from results in Extremal Graph Theory, there are countably many possibilities. For every set \( S \) there is a positive integer \( s \), such that \( dd(S) = 1 - 1/s \).

Suppose that \( n \) basketball teams compete in a tournament and each pair of distinct teams play one game. The organizers wish to award \( k \) prizes at the end of the tournament. They are, however, embarrassed to find out that no matter which \( k \) teams they pick as the winners of these prizes, there is always another team that won its games against each of these \( k \) winners. Is this indeed possible for any \( k \), provided \( n \) is large enough? This problem can be easily solved by the so called Probabilistic Method, a powerful technique in Combinatorics. For any fixed \( k \), and all sufficiently large \( n \), a random tournament on \( n \) teams, in which the result of each game is chosen randomly, uniformly and independently has, with high probability, the property that for each \( k \) teams there is another one that beats all of them. Probabilistic Combinatorics, which is one of the most active areas in modern combinatorics, started with the realization that probabilistic reasoning often provides simple solutions to problems of this type, that may seem very hard.

If \( G \) is a finite group of \( n \) elements, and \( H \) is a subgroup of size \( k \) in \( G \), then there are \( n/k \) left cosets and \( n/k \) right cosets of \( H \). Is there always a set of \( n/k \) elements of \( G \), containing a single representative of each right coset and a single representative of each left coset? Hall’s theorem, a basic result in Graph Theory, implies that this is always the case. In fact, if \( H' \) is another subgroup of size \( k \) in \( G \), then there is always a set of \( n/k \) elements of \( G \), containing a single representative of each right coset of \( H \), and a single representative of each left coset of \( H' \). This may sound like a result in Group Theory, but it really is a (simple) result in Combinatorics.

### 1.2 Topics

Combinatorics, sometimes also called Discrete Mathematics, is a branch of mathematics, studying discrete objects and their properties. Although Combinatorics is probably as old as the human ability to count, the field experienced tremendous growth during the last fifty years and has matured into a thriving area with its own set of problems, approaches and methodology.

The examples above suggest that Combinatorics is a basic mathematical discipline which plays a crucial role in the development of many other mathematical areas. In this essay we discuss some of the main aspects of this modern area, focusing on Extremal and Probabilistic Combinatorics. It is, of course, impossible to cover the area in such a short article. A detailed account of the subject can be found in [3]. Our main intention is to give a glimpse of the topics, methods and applications illustrated by representative examples. The topics we discuss include Extremal Graph Theory, Ramsey Theory, Extremal Finite Set Theory, Combinatorial Number Theory and Combinatorial Geometry, Random graphs and Probabilistic Combinatorics. The methods applied in the area include Combinatorial techniques, Probabilistic methods and tools from Linear Algebra, Spectral techniques and topological methods. We also discuss the algorithmic aspects and some of the many fascinating open problems in the area.

### 2 Extremal Combinatorics

Extremal Combinatorics deals with the problem of determining or estimating the maximum or minimum possible cardinality of a collection of finite objects that satisfies certain requirements. Such problems are often related to other areas including Computer Science, Information Theory, Number Theory and Geometry. This branch of Combinatorics has developed spectacularly over the last few decades, see, e.g., [2], [5], and their many references.

#### 2.1 Extremal Graph Theory

Extremal Graph Theory deals with quantitative connections between various graph parameters such as its numbers of vertices and edges, its
cli"e, independence, matching numbers etc. In many cases a certain optimization problem is to be solved, and its optimal solutions are called extremal graphs for this problem.

2.1.1 Graph coloring

Imagine that a network of radio stations is to be assigned frequencies for radio transmission. Two radio stations can be assigned the same frequency unless they are neighbors, in which case this may lead to undesired interference. What is then the minimum number of frequencies needed to be allocated to the network so that a proper frequency assignment is possible?

We can model this situation as a graph $G$ whose vertices are radio stations. Two vertices are connected by an edge in $G$ if and only if the corresponding stations are neighbors. Then a feasible frequency assignment is a function $f$ from the set of vertices $V$ of $G$ to integers $\{1, \ldots, k\}$, which we call colors. The function $f$ should satisfy: $f(u) \neq f(v)$ for each pair of vertices $u,v \in V(G)$, connected by an edge in $G$. The question is then: what is the minimum $k$ for which such a function exists? We thus arrive at the notion of a graph coloring.

Now we give the relevant formal definitions. Let $G = (V,E)$ be a graph with vertex set $V$ and edge set $E$. A function $f : V(G) \rightarrow \{1, \ldots, k\}$ is called a $k$-coloring of $G$ if no edge of $G$ is monochromatic under $f$, i.e., $f(u) \neq f(v)$ for every $e \in E(G)$. This restriction can be put in the following equivalent form: for every $1 \leq i \leq k$, the set $f^{-1}(i) \subseteq V(G)$ is an independent set (recall that an independent set in $G$ is a set of vertices of $G$ that spans no edges of $G$.) Thus, omitting the names of the colors we can say that a $k$-coloring is a partition of $V(G)$ into $k$ independent sets. A graph is $k$-colorable if it admits a $k$-coloring. Finally, the chromatic number of $G$, denoted by $\chi(G)$, is the minimum $k$ for which $G$ is $k$-colorable.

Two simple examples to illustrate the definition of chromatic number: if $G$ is a complete graph $K_n$ on $n$ vertices (a complete graph is a graph in which every pair of vertices is connected by an edge), then obviously in any coloring of $G$ all vertices get distinct colors, and thus $n$ colors are necessary. Of course, $n$ colors are sufficient, so $\chi(K_n) = n$. If $G$ is a cycle $C_{2n+1}$ on $2n + 1$ vertices, then easy parity arguments show that at least three colors are needed, and three colors are enough – color the vertices along the cycle alternatively by colors 1 and 2, and then color the last vertex by color 3. Thus, $\chi(C_{2n+1}) = 3$.

Clearly, $\chi(G) = 1$ if and only if $G$ has no edges. It is not so hard to prove that $G$ is 2-colorable if and only if $G$ does not contain a cycle of odd length, such graphs are usually called bipartite for obvious reasons. This easy characterization ends here, and no simple criteria equivalent to $k$-colorability is available, for $k \geq 3$.

Coloring is one of the most fundamental notions of Graph Theory, as a huge array of problems in this field and outside it (Computer Science, Operations Research etc.) can be casted in terms of graph coloring. Finding an optimal coloring of a graph is notoriously known to be a very hard task, both theoretically (NP-hard, to be more formal) and practically.

There are two simple yet fundamental lower bounds on the chromatic number. First, as every color in a proper coloring of a graph $G$ forms an independent set, each color class is bounded in size by the size of a maximum independent set. The latter quantity is called the independence number of $G$ and is denoted by $\alpha(G)$. Hence, at least $|V(G)|/\alpha(G)$ colors are necessary for $G$, resulting in $\chi(G) \geq |V(G)|/\alpha(G)$. Also, observe that if $G$ contains a complete graph $K_k$ on $k$ vertices as a subgraph, then $k$ colors are needed to color $K_k$ alone, and thus $\chi(G) \geq k$. Optimizing over the size of a complete subgraph contained in $G$, we derive: $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the maximal size of a complete graph (clique) in $G$, usually called the clique number of $G$.

Now we turn to upper bounds on the chromatic number. One of the simplest approaches to color a graph is to do it greedily: fix an order $\sigma = (v_1, \ldots, v_n)$ of the vertices of $G$, and then color $G$ vertex by vertex in the order prescribed by $\sigma$. When a vertex $v_i$ is to be colored, its color is the smallest color that has not been used on its already colored neighbors. While the greedy algorithm can be sometimes very inefficient (for example it can color bipartite graphs in an unbounded number of colors), usually it works quite well. Observe that when applying the greedy algorithm, a color given to a vertex $v$ is at most one more than the number of the neighbors of $v$ preceding $v$ in the chosen
order, and is thus at most \( d(v) + 1 \), where \( d(v) \) is the degree of \( v \) in \( G \) (the number of vertices \( u \), for which an edge \((u, v)\) is present in the graph).

We thus conclude that the greedy algorithm uses at most \( \Delta(G) + 1 \) colors, where \( \Delta(G) \) is the maximum degree of \( G \). Therefore \( \chi(G) \leq \Delta(G) + 1 \). This bound is tight for complete graphs and odd cycles, and as shown by Brooks in 1941 those are the only cases: if \( G \) is a graph of maximum degree \( \Delta \), then \( \chi(G) \leq \Delta \) unless \( G \) contains a clique \( K_{\Delta+1} \), or \( \Delta = 2 \) and \( G \) contains an odd cycle.

There is a very important class of graphs for which the chromatic number is always small, those are planar graphs. A graph \( G \) is planar if it can be drawn in the plane in such a way that no two edges intersect. (We intentionally omit formalities related to the Jordan curve theorem here, they can be found in almost any textbook on Graph Theory.) Such a drawing is called a plane graph. A plane graph plays a special role here, it is easy to see that every graph can be embedded in the three-dimensional space without intersecting edges. A way (and in a sense the way) to obtain a planar graph is from a planar map by taking the regions of the map to be the vertices of the graph, and connecting two regions by an edge if they have a common boundary. Francis Guthrie conjectured in 1852 that every planar graph can be colored in four colors. After a long series of efforts and several flawed proofs, this was finally proven by Appel and Haken in 1976 in a computer-assisted proof. This result is the famous Four Color Theorem, arguably the most famous graph theoretic result.

We would like to mention a different kind of coloring, where the colored objects are edges rather than vertices. Given a graph \( G = (V, E) \), a function \( f : E \to \{1, \ldots, k\} \) is called a \( k \)-edge coloring of \( G \), if no two incident edges get the same color, i.e., \( f(e) \neq f(e') \) for every pair \( e \neq e' \in E(G) \) for which \( e \cap e' \neq \emptyset \). The chromatic index of \( G \), denoted by \( \chi'(G) \), is the minimum \( k \) for which \( G \) admits a \( k \)-edge coloring. For example, one can prove that \( \chi'(K_{2n}) = 2n - 1 \) (ask the manager of your soccer league how to organize a round robin tournament of \( 2n \) teams in \( 2n - 1 \) rounds – this is exactly the problem of edge coloring \( K_{2n}^1 \)), while \( \chi'(K_{2n-1}) = 2n - 1 \). Observe that in any proper edge coloring of \( G \) all edges of \( G \) containing \( v \) get distinct colors, and thus \( \chi'(G) \geq \Delta(G) \).

This bound is tight for bipartite graphs, as proven by Kőnig in ???. For general graphs the fundamental theorem of Vizing from 1964 states that \( \chi'(G) \leq \Delta(G) + 1 \). So in some sense the chromatic index of \( G \) is a much easier quantity than the chromatic number of \( G \), as it is always squeezed between \( \Delta(G) \) and \( \Delta(G) + 1 \).

### 2.1.2 Excluded subgraphs

Let \( n \geq 2 \) be an integer and let \( H \) be a graph on at most \( n \) vertices. Denote by \( \text{ex}(n, H) \) the maximal number of edges in a graph on \( n \) vertices not containing \( H \) as a subgraph ("\( \text{ex} \)" stands for "excluded"). The empty graph on \( n \) vertices does not contain \( H \) (unless \( E(H) = \emptyset \)), while the complete graph \( K_n \) does, so the answer lies somewhere between 0 and \( \binom{n}{2} \). The function \( \text{ex}(n, H) \) is usually called the Turán number of \( H \).

In many cases it is quite easy to give a decent lower bound for \( \text{ex}(n, H) \). Assume that the chromatic number \( \chi(H) \) of \( H \) is \( r \geq 3 \). Partition the \( n \) vertices into \( r - 1 \) groups \( V_1, \ldots, V_{r-1} \) and create a graph \( G \) by connecting, for all \( 1 \leq i \neq j \leq r - 1 \), every vertex \( u \in V_i \) to every vertex \( v \in V_j \). Such a graph is called a complete \((r-1)\)-partite graph. Since \( \chi(G) = r - 1 \), the target graph \( H \) cannot be embedded into \( G \). This implies: \( \text{ex}(n, H) \geq |E(G)| \). The graph \( G \) has the largest number of edges, when all of its parts are of nearly equal size, i.e., \(||V_i| - |V_j|| \leq 1 \). The graph satisfying this condition is called the Turán graph \( T_{r-1}(n) \) and its number of edges is denoted by \( t_{r-1}(n) \). It follows that \( \text{ex}(n, H) \geq t_{r-1}(n) \). An easy computation gives \( \text{ex}(n, H) \geq t_{r-1}(n) = (1 - o(1)) \left( 1 - \frac{1}{r-1} \right) \binom{n}{2} \).

The most important case, where \( H \) is a complete graph \( K_r \) on \( r \) vertices, was resolved by Turán in 1941, who proved that in fact \( \text{ex}(n, K_r) = t_{r-1}(n) \), and the only \( K_r \)-free graph on \( n \) vertices with \( \text{ex}(n, K_r) \) edges is (up to isomorphisms) the Turán graph \( T_{r-1}(n) \). Turán’s paper is generally considered a starting point of Extremal Graph Theory.

Later, Erdős, Stone and Simonovits extended Turán’s theorem by proving that the above simple lower bound for \( \text{ex}(n, H) \) is asymptotically tight for any \( H \) with \( \chi(H) \geq 3 \). Formally, they proved...
that
\[ ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right) \left(\frac{n}{2}\right) + o(n^2). \]

The above formula appears to give the asymptotic value of \( ex(n, H) \) for all graphs \( H \). This is not quite true, as for bipartite graphs \( H \) (i.e., whenever \( \chi(H) = 2 \)) it gives only \( ex(n, H) = o(n^2) \). The determination of Turán numbers for bipartite graphs remains a challenging open problem with quite many unsettled questions. Partial results obtained so far use a variety of techniques from different fields, sometimes as apparently distant as Algebraic Geometry.

A question closely related to the latter problem is the so called Zarankiewicz problem, asking to determine \( z(m, n; s, t) \), the maximal number of edges in a bipartite graph \( G \) with the first part of size \( m \) and the second of size \( n \), not containing a copy of a complete bipartite graph with \( s \) vertices in the first part and \( t \) vertices in the second. In 1956 check! spelling! Kövári, Sós and Turán used a double counting argument to prove

\[ m \left( \frac{z/m}{t} \right) \leq (s-1) \left( \frac{n}{t} \right), \]

where \( z = z(m, n; s, t) \). In particular, for \( m = n \) and fixed \( s = t \) the above bound shows that \( z(n, n; t, t) = O(n^{2-1/1}) \); this is known to be tight for few values of \( t \).

In a perhaps surprising twist one can prove that the Kövári-Sós-Turán bound for the first non-trivial case \( z(n, n; 2, 2) \) (where the cycle \( C_4 = K_{2,2} \) is excluded) is tight if and only if there exists a finite projective plane with \( n \) points.

### 2.1.3 Matchings and cycles

Given a graph \( G \), a matching \( M \) in \( G \) is a collection of pairwise disjoint edges of \( G \). A matching is called perfect if it covers all the vertices of \( G \). Of course, in order to have a perfect matching the number of vertices of \( G \) should be even.

One of the most well known theorems about graphs is Hall’s theorem providing a necessary and sufficient condition for the existence of a matching in a bipartite graph. Let \( G = (A \cup B, E) \) be a bipartite graph with sides \( A \) and \( B \). For a set \( S \subseteq A \), we denote by \( N(S) \) its neighborhood in \( B \), i.e., \( N(S) = \{ b \in B : b \text{ is connected by an edge to } A \} \). We say that \( G \) satisfies the Hall condition for side \( A \) if

\[ |N(S)| \geq |S| \]

for every subset \( S \) of \( A \). Hall’s theorem (1935) asserts that the Hall condition is both necessary and sufficient for the existence of a matching \( M \) in \( G \) covering \( A \). Observe that in the case \( |A| = |B| \) (which is the only case where a bipartite graph can have a perfect matching) a matching covering \( A \) is perfect.

Hall’s theorem can be reformulated in the equally popular form of a system of distinct representatives (SDR). Let \( \{S_i\}_{i \in I} \) be a finite collection of sets. Denote \( U = \bigcup_{i \in I} S_i \). An ordered set \( \{s_i\}_{i \in I} \) of distinct elements of \( U \) is called a system of distinct representatives if \( s_i \in S_i \) for every \( i \in I \). Then Hall’s Theorem postulates that the family \( \{S_i\} \) has an SDR if only if \( |\bigcup_{i \in I} S_i| \geq |I| \) for every subset of indices \( J \subseteq I \). To see the equivalence of those two forms, define a bipartite graph \( G \) with sides \( I \) and \( U \), where \( (i, u) \) is an edge if \( u \in S_i \). Then a perfect matching in \( G \) translates to an SDR for the family \( \{S_i\} \).

Let us see how Hall’s theorem can be applied to solve the problem of finding a system of representatives for right and left cosets of a subgroup \( H \), mentioned in Section 1.1. We can set a bipartite graph \( F \), whose two sides (of size \( n/k \)) are the left and the right cosets of \( H \). A left coset \( g_1 H \) is connected by an edge of \( F \) to a right coset \( H g_2 \) if they share a common element. The graph \( F \) is easily seen to satisfy the Hall condition, and hence is has a perfect matching \( M \). Choosing for each edge \( (g_1 H, H g_2) \) of \( M \) a common element of \( g_1 H \) and \( H g_2 \), we obtain the required family of representatives.

There is also a necessary and sufficient condition for the existence of a perfect matching in a general (i.e., non-bipartite) graph \( G \), it is called Tutte’s theorem. We will not state it here.

A cycle of length \( k \) in a graph \( G = (V, E) \) is a sequence \( x_0, x_1, \ldots, x_k \), where \( x_i \in V \), \( x_0 = x_k \), for \( 1 \leq i \leq k - 1 \) the vertex \( x_i \) is distinct from all \( x_j \), \( j < i \), and all edges \( (x_0, x_1), \ldots, (x_{k-1}, x_k) \) are in \( G \). We denote by \( C_k \) a cycle of length \( k \) and call it odd if it has an odd number of vertices, and even otherwise. A cycle is a very basic graph structure, and – as one can expect – there are many extremal
results concerning cycles.

A graph without cycles is called a forest. A connected graph without cycles is a tree. (A graph \( G \) is called connected if every vertex of \( G \) can be reached from every other vertex by a path in \( G \).) Each tree on \( n \) vertices has exactly \( n - 1 \) edges. It follows that every graph \( G \) on \( n \) vertices with at least \( n \) edges has a cycle. In order to satisfy more elaborate requirements on cycles, more edges may be required. For example, the Turán theorem, applied for \( r = 3 \), asserts that a graph \( G \) with \( n \) vertices and more than \( n^2/4 \) edges contains a triangle \( C_3 = K_3 \). Also, one can prove that a graph \( G = (V,E) \) with \( |V| = n \) and \( |E| > \frac{n}{2}(n-1) \) has a cycle of length longer than \( k \), and this is tight.

A Hamilton cycle in a graph \( G \) is a cycle containing all of its vertices. This term originates in a game, invented by W. R. Hamilton in 1857 whose contents was to find a complete graph in the graph of the dodecahedron. A graph containing a Hamilton cycle is called Hamiltonian. This concept is pretty much related to the well known Traveling Salesperson problem (TSP) asking to find a Hamilton cycle of a shortest total weight in a weighted graph. There are many sufficient criteria for a graph to be Hamiltonian, quite a few of them are based of the sequence of graph degrees. For example, Dirac proved in 1952 check that a graph on \( n \) vertices all of whose degrees are at least \( n/2 \) is Hamiltonian.

2.1.4 Tools

- Regularity Lemma, possibly Blow Up Lemma;
- ?? Minors, Robertson-Seymour theory ??;
- ? Dimension arguments+some concrete example (Oddtown?)?;
- Eigenvalues, \(|e(B)| - |B^2|d/(2n)| \leq \lambda |B| + \) describe one application (constructive bound for \( R(3,k) \))?;
- Topological methods+describe/mention one concrete example.

2.2 Ramsey Theory

Ramsey theory studies quantitatively the following very general phenomenon: every large structure, chaotic as it can be, contains a substructure, perhaps of a much smaller size, that has much more order. As succinctly put by the mathematician T. S. Motzkin, "Complete disorder is impossible". One can expect that due to the simple and very general form of this paradigm, it has many diverse manifestations in different mathematical areas. Probably the first Ramsey-type statement one usually encounters is the pigeon hole principle, which we formulate in the following form: every coloring of \( n \) objects in \( s \) colors contains a subset of at least \( n/s \) objects, all having the same color.

Although several Ramsey-type theorems had appeared before, the origin of Ramsey theory is usually credited to the English mathematician Frank Ramsey, who in 1930 proved the following theorem. Let \( k, l \geq 2 \) be integers. Then there exists a minimal integer \( n \), which we denote by \( R(k,l) \), such that every Red-Blue coloring of the edges of the complete graph \( K_n \) on \( n \) vertices contains a Red complete graph on \( k \) vertices of a Blue complete graph on \( l \) vertices. In this language, the observation of Szai1, mentioned in the introduction, states that \( R(4,4) \leq 20 \) (in fact, \( R(4,4) = 18 \)). Of course, one cannot guarantee a large complete graph of a specific color, but one of the two colors will do. Actually, Ramsey proved a much more general theorem, allowing for an arbitrary but fixed number of colors and for coloring \( r \)-tuples of the set of size \( n \), and not just 2-tuples (pairs). As a side remark we note here that the exact computation of small Ramsey numbers is a notoriously difficult task, and already the value of \( R(5,5) \) is unknown at present.

The second cornerstone of Ramsey theory was laid by Paul Erdős and George Szekeres, who in 1935 wrote a paper, containing several important Ramsey-type results. Apparently unaware of Ramsey’s paper, they redefined the Ramsey numbers \( R(k,l) \) and proved a recursion \( R(k,l) \leq R(k-1,l) + R(k,l-1) \). Here is their argument, quite typical for many proofs of Ramsey-type theorems. Set \( n = R(k-1,l) + R(k,l-1) \). Consider a Red-Blue coloring \( f \) of the edges of \( K_n \). Let \( v \) be a vertex of \( K_n \). Then by the pigeon hole principle \( v \) should have at least \( R(k-1,l) \) Red neighbors or at least \( R(k,l-1) \) Blue neighbors under \( f \). Consider, say, the first case, and let \( U \) be the Red neighborhood of \( v \). Then, by the definition of Ramsey numbers, \( U \) should span a Red \( K_{k-1} \) or
2. EXTREMAL COMBINATORICS

exists an points in a plane contains a convex
substructure – it is NOT true that for
Seidenberg + 1. A particularly elegant proof was found by
n of
n
of
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\( R(2, l) = l, R(k, 2) = k, \) the recursion leads to the estimate \( R(k, l) \leq \binom{k+l-2}{k-1} \). In particular,
for the so called diagonal case \( k = l \) we obtain \( R(k, k) \leq 4^{k-o(k)} \). No improvement in the exponent
of the latter estimate has been found so far.
The best known lower bound, discussed in Section
3.2, is \( R(k, k) \geq 2^{k/2+o(k)} \), so the gap is quite sub-
stantial.

Another Ramsey-type statement, proven by
Erdős and Szekeres, is of geometric nature. They
showed that for every \( n \geq 3 \) there exists a mini-
mal \( N = N(n) \), such that for any configuration of \( N \)
points in a plane in a general position (i.e. no
tree on a line), there are \( n \) that form a convex n-
gone. (Prove \( N(4) = 5 \) as an easy exercise.) There
are several proofs of this theorem, some using the
general Ramsey theorem. The conjectured value of
\( N \) is \( N(n) = 2^n-2 + 1 \). It is probably a good
point to warn the reader who may get the mislead-
ing feeling that every large structure contains any
substructure – it is NOT true that for \( n \geq 7 \) there
exists an \( N' \) such that every configuration of \( N' \)
points in a plane contains a convex n-gone whose interior
does not contain another point from the
configuration. This was proved by Horton in ???.

(Check, complete!)

The classic Erdős-Szekeres paper contains also
the following Ramsey-type result: any sequence
of \( n^2 + 1 \) distinct numbers contains a monotone
(increasing or decreasing) subsequence of length
\( n + 1 \). A particularly elegant proof was found by
Seidenberg Check!: let \((x_1, x_2, \ldots, x_{n^2+1})\) be the
sequence. For each \( 1 \leq i \leq n^2 + 1 \) let \( a_i \), resp. \( b_i \),
be the length of a longest increasing, resp. decreas-
ing, subsequence ending at \( x_i \).
Observe that for
\( i < j \) we have \( a_i < a_j \) or \( b_i < b_j \), and thus all pairs
\((a_i, b_i)\) are distinct. As altogether we have \( n^2 + 1 \)
pairs, due to the pigeon hole principle (again!),
there should be an \( i \) for which \( a_i > n \) or \( b_i > n \),
indicating that there is a monotone subsequence of
desired length. This result of Erdős and Szekeres
immediately provides a lower bound of \( \sqrt{n} \) for the
famous Ulam problem, asking for a typical length
of a longest increasing subsequence of a random se-
quence of length \( n \) and resolved recently by Baik,
Deift and Johansson.

In 1927 van der Waerden proved what became
known as van der Waerden’s theorem: for all posi-
tive integers \( k \) and \( r \) there exists an integer \( W(k, r) \)
so that for every coloring of the set of integers
\( \{1, \ldots, W(k, r)\} \) in \( r \) colors, one of the colors con-
tains an arithmetic progression of length \( k \). Van
der Waerden’s bounds for \( W(k, r) \) are enormous –
they grow like an Ackermann-type function. A new
proof of van der Waerden’s theorem was found by
the Israeli logician Shelah in 1987. Shelah’s bound
for \( W(k, r) \) is much more “modest” – it grows like
the fourth function in the Ackermann hierarchy.
Known lower bounds for \( W(k, r) \) are significantly
smaller – they are just exponential in \( k \), for a fixed
\( r \geq 2 \). The (much deeper) density version of van
der Waerden’s theorem is discussed in Section 2.4.

Even earlier, in 1916, Schur showed the follow-
ing result: for every positive integer \( r \) there exists
an integer \( S(r) \) such that for every \( r \)-coloring
of \( \{1, \ldots, S(r)\} \) one of the colors contains a solu-
tion of the equation \( x + y = z \). The proof can
be derived rather easily from the general Ramsey
theorem. Schur was motivated by Fermat’s Last
Theorem and proved the following result, men-
tioned in Section 1.1: for every \( k \) and all suffi-
ciently large primes \( p \), the equation \( a^k + b^k = c^k \)
has a non-zero solution in the integers modulo \( p \).
To prove this result, assume \( p \geq S(k) \) and consider
the field \( Z_p \). Let \( H = \{ x^k : x \in Z_p^* \} \). Then \( H \)
is a subgroup of the multiplicative group \( Z_p^* \) of index
\( r = \gcd(k, p - 1) \leq k \). The partition of \( Z_p^* \) into
the cosets of \( H \) induces an \( r \)-coloring \( \chi \) of \( Z_p^* \). By
Schur’s theorem there exist \( x, y, z \in \{1, \ldots, p-1\} \)
with \( \chi(x) = \chi(y) = \chi(z) \) and \( x + y = z \). Then
there exists a residue \( d \in Z_p^* \) such that \( x = da^k, \)
\( y = db^k, z = dc^k \) and \( da^k + db^k = dc^k \) modulo \( p \).
The theorem follows.

The reason we survey in such a detail all these
fundamental results here is not only their historical
value, but the influence they have had on the field,
and quite a typical character of their statements
and proofs. Of course, many Ramsey-type results
have been proven since then, the reader can consult
Chapter ?? of [3] or a monograph of Graham,
Rothschild and Spencer on Ramsey Theory.
2.3 Extremal Finite Set Theory

The generic problem in Extremal Finite Set Theory is the problem of determining or estimating the maximum possible cardinality of a family $F$ of distinct subsets of an $n$-element set that satisfies some given conditions. The first result of this form was proved by Sperner in 1928. He showed that the maximum possible number of subsets of an $n$-element set in which no subset contains another one is $\binom{n}{\lfloor n/2 \rfloor}$. The lower bound is given by the family of all subsets of cardinality $\lfloor n/2 \rfloor$. This result supplies a quick solution to the real analog of the problem of Littlewood and Offord described in subsection 1.1. If $x_1, x_2, \ldots, x_n$ are $n$ not necessarily distinct real numbers of absolute value at least 1, then the number of sums $\sum_{i=1}^{n} \epsilon_i x_i$ with $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-1, 1\}^n$, so that no two sums differ by at least 2, is at most $\binom{n}{\lfloor n/2 \rfloor}$. This bound is tight as shown by taking $x_i = 1$ for all $i$. To prove the bound observe, first, that we may assume that all $x_i$ are positive, as we can always replace an element $x_i$ by $-x_i$ and $\epsilon_i$ by $-\epsilon_i$ in all the sums. Given a set of sums as above, associate each sum $\sum_{i=1}^{n} \epsilon_i x_i$ with the subset of all indices $i$ for which $\epsilon_i = 1$. Note that if the subset corresponding to one sum contains the subset corresponding to another one, then the two sums differ by at least twice the value of some $a_i$, that is, by at least 2, contradicting the assumption. The upper bound thus follows from Sperner’s Theorem.

A family of sets is intersecting if any two members of the family intersect. What is the maximum possible size of an intersecting family of subsets of cardinality $k$ of an $n$ element set? We may assume that $n \geq 2k$ as otherwise the solution is trivial. Erdős, Ko and Rado proved that the maximum is $\binom{n - 1}{k - 1}$. Their proof uses a shifting technique, which proved to be very useful in tackling similar problems.

Let $n > 2k$ be two positive integers. What is the minimum possible number of colors $t$ such that there is a coloring of all $k$-subsets of an $n$-element set $\{1, 2, \ldots, n\}$ by $t$ colors, such that each color class forms an intersecting family? It is not difficult to see that $n - 2k + 2$ colors suffice. Indeed, one color class can be the family of all subsets of $\{1, 2, \ldots, 2k - 1\}$, which is clearly intersecting. In addition, for each $i$ satisfying $2k \leq i \leq n$, the family of all subsets whose largest element is $i$ is also intersecting. These $n - 2k + 1$ families can thus form the remaining color classes.

Kneser conjectured in 1955, and Lovász proved in 1978 that this is tight; in any coloring of all $k$-subsets of an $n$-element set by less than $n - 2k + 2$ colors, there are two disjoint sets having the same color. Lovász’ proof is topological, and relies on the Borsuk-Ulam Theorem. Several simpler proofs have been found during the years; all of them are based on the topological idea in the first proof, which played a crucial role in the development of topological tools in Combinatorics.

2.4 Combinatorial Number Theory

Combinatorial Number Theory deals with the arithmetic properties of “dense” sets of integers, with extremal problems dealing with addition of sets of integers or subsets of groups, as well as with some classical problems of number theory in which combinatorial methods proved useful. We describe below a few examples. Many more can be found in Chapter 20 of [3] and in [7].

The Cauchy-Davenport Theorem, which has numerous applications in Additive Number Theory, is the statement that if $p$ is a prime, and $A, B$ are two nonempty subsets of $\mathbb{Z}_p$, then

$$|A + B| = |\{a + b : a \in A, b \in B\}| \geq \min\{p, |A| + |B| - 1\}.$$ 

Cauchy proved this theorem in 1813, and applied it to give a new proof to a lemma of Lagrange in his well known 1770 paper that shows that every positive integer is a sum of four squares. Davenport formulated the theorem as a discrete analogue of a conjecture of Khintchine about the Schnirelman density of the sum of two sequences of integers. The proofs given by Cauchy and by Davenport are combinatorial, but there is also a more recent algebraic proof, based on some properties of roots of polynomials. Its advantage is that it provides many variants that do not seem to follow from the combinatorial approach. One variant is the fact that if $p$ is a prime, $A, B$ are two nonempty subsets of $\mathbb{Z}_p$ and $|A| \neq |B|$, then

$$|A \oplus B| = |\{a + b : a \in A, b \in B, a \neq b\}| \geq \min\{p, |A| + |B| - 2\}.$$
Additional extensions can be found in [7].

The theorem of van der Waerden mentioned in subsection 2.2 asserts that for any positive integer \( r \), in any coloring of the integers by \( r \) colors there is a color class that contains arbitrarily long arithmetic progressions. Erdős and Turán conjectured in 1936 that this always holds for the “most popular” color class. More precisely, they conjectured that any set of positive integers with positive upper density contains arbitrarily long arithmetic progressions. This is equivalent, by compactness, to the following statement about finite sets of integers. For any positive integer \( k \) and for any real \( \epsilon > 0 \), there is an \( n_0 = n_0(k, \epsilon) \) such that if \( n > n_0 \), then any set of at least \( cn \) positive integers between \( 1 \) and \( n \) contains a \( k \)-term arithmetic progression. After several partial results, this conjecture was proved by Szemerédi in 1975. His deep proof is combinatorial, and applies techniques from Ramsey Theory and Extremal Graph Theory. Furstenberg gave another proof in 1977, based on techniques of Ergodic Theory. In 2000 Gowers has given a new proof, combining combinatorial arguments with tools from analytic number theory. This proof supplied a much better quantitative estimate. A related very recent spectacular result of Green and Tao asserts that there are arbitrarily long arithmetic progressions of prime numbers. Their proof combines number theoretic techniques with the ergodic theory approach. Erdős conjectured that any infinite sequence \( n_i \) for which the sum \( \sum_i \frac{1}{n_i} \) diverges, contains arbitrarily long arithmetic progressions. This conjecture, which implies the assertion of the Green-Tao result, is still open.

### 2.5 Discrete Geometry

Discrete Geometry investigates combinatorial properties of configurations of geometric objects. Many of the questions considered can be explained to a layman. Here are a few examples. What is the maximum possible number of incidences between \( m \) points and \( n \) lines in the plane? What is the maximum possible number of unit distances determined by a set of \( n \) points in the plane? What is the minimum number \( h = h(p, q) \) so that for any family \( \mathcal{F} \) of compact, convex sets in the plane in which for any \( p \) of the sets there is a point lying in \( q \) of them, there is a set of \( h \) points that intersects each member of the family? Problems and results of this type have been applied extensively in Computational Geometry and in Combinatorial Optimization during the last decades; two recent books on the subject are [8] and [6].

Let \( P \) be a set of points, and \( L \) a set of lines in the plane. The number of incidences between \( P \) and \( L \), denoted by \( I(P, L) \), is the number of ordered pairs \((p, \ell)\) with \( p \in P, \ell \in L \) and \( p \in \ell \). Let \( I(m, n) \) denote the maximum possible value of \( I(P, L) \), where the maximum is taken over all sets \( P \) of \( m \) distinct points and all sets \( L \) of \( n \) distinct lines. Szemerédi and Trotter determined the asymptotic behavior of this quantity, up to a constant factor, for all possible values of \( m \) and \( n \). There are two absolute positive constants \( c_1, c_2 \) such that for all \( m, n \),

\[
c_1(m^{2/3}n^{2/3} + m + n) \leq I(m, n) \leq c_2(m^{2/3}n^{2/3} + m + n).
\]

The lower bound, in the non-trivial cases that \( m \) and \( n \) are not too far from each other, follows by taking the points to contain all points of a \( \lfloor \sqrt{m} \rfloor \) by \( \lfloor \sqrt{m} \rfloor \) grid, and by taking the \( n \) most popular lines determined by these grid points. The upper bound is more difficult. The most elegant proof of it is due to Székely, and is based on a result about the minimum number of crossings between edges in an embedding of a graph with a given number of vertices and edges in the plane.

Similar ideas can be used to bound the maximum possible number of unit distances between pairs of points in a set of \( n \) points in the plane. It is not surprising that the two problems are related; the number of these unit distances is simply the number of incidences between the given \( n \) points, and the \( n \) unit circles centered at these points. Here, however, there is a large gap between the resulting upper bound, which is \( cn^{4/3} \) for some absolute constant \( c \), and the best known lower bound, which is only \( n^{1+c'/\log \log n} \), for an appropriate constant \( c' > 0 \).

A fundamental theorem of Helly asserts that if in a collection \( \mathcal{F} \) of compact convex sets in \( \mathbb{R}^d \) every \( d+1 \) sets or less intersect, then all sets have a common point. Suppose we only know that out of every \( p \) sets some \( d+1 \) intersect, for some \( p > d+1 \). Is there, in this case, a set of at most \( C \) points that intersects each member of \( \mathcal{F} \), where \( C \) is bounded...
by a function of $p$, independent of the size of $F$? This question was raised by Hadwiger and Debrunner in 1957, and solved by Kleitman and the first author in 1992. The proof combines a fractional version of Helly’s Theorem with the duality of linear programming and various additional geometric results. Unfortunately, it gives a very poor estimate for the number of points required $C$, and even in the case $p = 4$ in dimension 2 it is not known what the best possible value of $C$ is.

3 Probabilistic Combinatorics

The discovery, demonstrated in the early work of Paley, Zygmund, Erdős, Turán, Shannon and others, that deterministic statements can be proved by probabilistic reasoning, led already in the first half of the century to several striking results in Analysis, Number Theory, Combinatorics and Information Theory. It soon became clear that the method, which is now called the probabilistic method, is a very powerful tool for proving results in Discrete Mathematics. The early results combined combinatorial arguments with fairly elementary probabilistic techniques, whereas the development of the method in recent years required the application of more sophisticated tools from probability. The book [1] is a recent text dealing with the subject.

The applications of probabilistic techniques in Discrete Mathematics, initiated by Paul Erdős who contributed to the development of the method more than anyone else, can be classified into three groups. The first one deals with the study of certain classes of random combinatorial objects, like random graphs or random matrices. The results here are essentially results in Probability Theory, although most of them are motivated by problems in Combinatorics. The second group consists of applications of probabilistic arguments in order to prove the existence of combinatorial structures which satisfy a list of prescribed properties. Existence proofs of this type often supply extremal examples to various questions in Discrete Mathematics. The third group, which contains some of the most striking examples, focuses on the application of probabilistic reasoning in the proofs of deterministic statements whose formulation does not give any indication that randomness may be helpful in their study.

This section contains a brief description of several typical results in each of these three groups.

3.1 Random structures

The systematic study of Random Graphs was initiated by Erdős and Rényi in 1960. The most common model for a random graph is a slight variation of their model, and is denoted by $G(n,p)$. The term “the random graph $G(n,p)$” means a graph on a fixed set of $n$ labelled vertices, where each pair of vertices forms an edge, randomly and independently, with probability $p$. Each graph property $A$ is an event in this probability space, and one may study its probability $Pr[A]$, that is, the probability that the random graph $G(n,p)$ satisfies A.

One of the most important discoveries of Erdős and Rényi was the discovery of threshold functions. A graph property is monotone if it is closed under the addition of edges. Many interesting graph properties are monotone. For each of these three properties, when $n$ is fixed and large, the probability of the random graph $G(n,p)$ to satisfy it changes very rapidly from nearly 0 to nearly 1 as $p$ increases. This is related to the phase transition phenomenon in mathematical physics, which is treated in another chapter of this Companion. A recent result of Friedgut shows that the threshold for a graph property which is, in a sense that can be made precise, “global”, is sharper than the one for a “local” property.

Another interesting early discovery in the study of Random Graphs was that of the fact that many interesting graph invariants are highly concentrated. A striking result of this type is the fact that for fixed values of $p$ almost all graphs $G(n,p)$ have the same clique number. The clique number of a graph is the maximum number of vertices in a clique of it, that is, in a subgraph in which any two vertices are adjacent. It turns out that for every fixed positive value of $p < 1$ and every $n$, there is a real number $r_0 = r_0(n,p)$ which is roughly $2 \log n/ \log(1/p)$, such that the clique number of $G(n,p)$ is either $\lfloor r_0 \rfloor$ or $\lceil r_0 \rceil$ almost surely. Moreover, $r_0(n,p)$ can be chosen to be an integer for most values of $n$ and $p$. The proof of this result is based on the second moment method. One estimates the expectation and the variance of the number of cliques of a given size contained in $G(n,p)$,
4. ALGORITHMIC ASPECTS

and applies the inequalities of Markov and Cheby- 
shhev.

The chromatic number of the random graph 
$G(n, p)$ is also highly concentrated. This was 
proved by Bollobás for values of $p$ bounded away 
from 0, and by Shamir, Spencer, Luczak and the 
authors of the present chapter for smaller values 
of $p$. In particular, it can be shown that for every 
$\alpha < 1/2$ and every integer valued function 
$r(n) < n^\alpha$, there exists a function $p(n)$ such that 
the chromatic number of $G(n, p(n))$ is precisely 
$r(n)$ almost surely. Still, the determination of the 
concentration of the chromatic number of $G(n, p)$ 
for the most important case $p = 0.5$ (for which 
all labeled graphs on $n$ vertices are equiprobable) 
remains an intriguing open problem.

Many additional results on random graphs can 
be found in [4].

3.2 Probabilistic constructions

The definition of the Ramsey number $R(k, k)$ is 
given in subsection 2.2. In one of the first applica-
tions of the probabilistic method in Combinato-
rics, Erdős proved that if \( \binom{n}{k} 2^{1 - \frac{k}{2}} < 1 \) then 
$R(k, k) > n$, that is, there exists a graph on $n$ 
vertices containing neither a clique of size $k$ nor an 
independent set of size $k$. Note that $n = \lfloor 2^{k/2} \rfloor$ 
satisfies the above inequality for all $k \geq 3$, supplying 
an exponential lower bound for $R(k, k)$. The proof 
is simple; Every fixed set of $k$ vertices in the ran-
dom graph $G(n, 0.5)$ is a clique or an independent 
set with probability $2^{1 - \frac{k}{2}}$. Thus \( \binom{n}{k} 2^{1 - \frac{k}{2}} < 1 \) 
is an upper bound for the probability that the ran-
dom graph $G(n, 0.5)$ contains a clique or an 
independent set of size $k$, and hence there is a graph 
without any such clique or independent set.

A similar computation yields a solution for the 
tournament problem mentioned in subsection 1.1. 
Let $k$ and $n$ be two integers, and suppose that 
\[
\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1.
\]

Then there is a tournament on $n$ teams, in which 
for every set of $k$ teams, there is another one who 
beats them all. If $n$ is larger than about $k^22^k \ln 2$ 
the above inequality holds.

Probabilistic constructions proved to be very 
powerful in supplying lower bounds for Ramsey 
numbers. Besides the bound for $R(k, k)$ mentioned 
above, there is a subtle probabilistic proof, due to 
Kim, that $R(3, k) \geq ck^2/\log k$, for some $c > 0$. 
This is known to be tight up to a constant factor, 
as proved by Ajtai, Komlós and Szemerédi, also by 
probabilistic methods.

3.3 Proving deterministic theorems

4 Algorithmic aspects

The rapid development of theoretical Computer 
Science and its tight connection to Discrete Math-
ematics motivated the study of the algorithmic as-
pects of combinatorial results.

The study of the algorithmic problems corre-
sponding to probabilistic proofs is related to the 
investigation of randomized algorithms, a topic 
which has been developed tremendously during the 
last decade. In particular, it is interesting to find 
explicit constructions of combinatorial structures 
whose existence is proved by probabilistic arguments. “Explicit” here means that there is an effi-
cient algorithm that constructs the desired struc-
ture in time polynomial in its size. Constructions 
of this type, besides being interesting in their own, 
have applications in other areas. Thus, for exam-
ple, explicit constructions of error correcting codes 
that are as good as the random ones are of ma-
jor interest in coding and information theory, and 
explicit constructions of certain Ramsey type col-
orings may have applications in derandomization – 
the process of converting randomized algorithms 
into deterministic ones.

It turns out, however, that the problem of find-
ing a good explicit construction is often very dif-
ficult. Even the simple proof of Erdős, described 
in subsection 3.2, that there are graphs on $[2^{k/2}]$ 
vertices containing neither a clique nor an inde-
pendent set of size $k$, leads to an open problem which 
seems very difficult. Can we construct, explicitly, 
such a graph on $n \geq (1+\epsilon)^k$ vertices, in time which 
is polynomial in $n$, where $\epsilon > 0$ is any positive 
absolute constant? This problem is still wide open, 
despite a considerable amount of efforts.

The application of other advanced tools such as 
algebraic and analytic techniques, spectral meth-
ods and topological proofs, also tend to lead in 
many cases to non-constructive proofs. The con-
version of these to algorithmic ones may well be
one of the main future challenges of the area. Another interesting recent development is the increased appearance of computer aided proofs in Combinatorics, starting with the proof of the Four Color Theorem. A successful incorporation of such proofs in the area, without losing its special beauty and appeal, is another challenge.

5 Summary

These challenges, the fundamental nature of the area, its tight connection to other disciplines, and the many fascinating specific open problems studied in it, ensure that Combinatorics will keep playing an essential role in the general development of Science in the future as well.

Bibliography


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