Acyclic Edge Colorings of Graphs

Noga Alon ∗ Ayal Zaks †

Abstract

A proper coloring of the edges of a graph $G$ is called acyclic if there is no 2-colored cycle in $G$. The acyclic edge chromatic number of $G$, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of $G$. For certain graphs $G$, $a'(G) \geq \Delta(G) + 2$ where $\Delta(G)$ is the maximum degree in $G$. It is known that $a'(G) \leq 16\Delta(G)$ for any graph $G$ (see [2],[9]). We prove that there exists a constant $c$ such that $a'(G) \leq \Delta(G) + 2$ for any graph $G$ whose girth is at least $c(\Delta(G))^{2 \log(\Delta(G)) \log \log(\Delta(G))}$, and conjecture that this upper bound for $a'(G)$ holds for all graphs $G$.

1 Introduction

All graphs considered here are finite and simple. A coloring of the vertices of a graph is proper if no pair of adjacent vertices are colored with the same color. Similarly, an edge-coloring of a graph is proper if no pair of incident edges are colored with the same color. A proper coloring of the vertices or edges of a graph $G$ is called acyclic if there is no 2-colored cycle in $G$. In other words, if the union of any two color classes induces a subgraph of $G$ which is a forest. The acyclic chromatic number of $G$ introduced in [6] (see also [7, problem 4.11]), denoted by $a(G)$, is the least number of colors in an acyclic vertex coloring of $G$. The acyclic edge chromatic number of $G$, denoted by $a'(G)$, is the least number of colors in an acyclic edge coloring of $G$.

1.1 Lower and Upper Bounds

For a graph $G$, let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in $G$. Any proper edge coloring of $G$ obviously requires at least $\Delta$ colors, and according to Vizing [10] there exists a proper edge coloring with $\Delta + 1$ colors. It is easy to see that any acyclic edge coloring of a $\Delta$-regular graph uses at least $\Delta + 1$ colors. There are cases where more than $\Delta + 1$ colors are needed for coloring the edges acyclically:

$$a'(K_{2n} \setminus F) \geq 2n + 1 = \Delta(K_{2n} \setminus F) + 2,$$

(1)
where $K_{2n}$ is the complete graph on $2n$ vertices and $F \subset E(K_{2n})$ such that $|F| \leq n - 2$. This is because at most one color class can contain $n$ edges (a perfect matching), and all other color classes can contain at most $n - 1$ edges each.

Alon et al. [2] proved that $a'(G) \leq 64\Delta$, and remarked that the constant 64 can be reduced. Molloy and Reed [9] showed that $a'(G) \leq 16\Delta$ using the same proof. The constant 16 can, in fact, be further improved. We conjecture that the lower bound in (1) is an upper bound for all graphs, that is:

**Conjecture 1** $a'(G) \leq \Delta(G) + 2$ for all graphs $G$.

Conjecture 1 is interesting for graphs $G$ with $\Delta(G) \geq 3$. Burnstein [5] showed that $a(G) \leq 5$ if $\Delta(G) = 4$. Since any acyclic vertex coloring of the line graph $L(G)$ is an acyclic edge coloring of $G$ and vice-versa, this implies that $a'(G) = a(L(G)) \leq 5$ if $\Delta(G) = 3$. Hence conjecture 1 is true for $\Delta = 3$. We have found another proof for this case, which also yields a polynomial algorithm for acyclically coloring the edges of a graph of maximum degree 3 using 5 colors.

The only graphs $G$ for which we know that $a'(G) > \Delta(G) + 1$ are the subgraphs of $K_{2n}$ that have at least $2n^2 - 2n + 2$ edges (see (1)). Therefore it might even be true that if $G$ is a $\Delta$-regular graph$^1$ then

$$a'(G) = \begin{cases} \Delta + 2 & \text{for } G = K_{2n} \\ \Delta + 1 & \text{otherwise.} \end{cases}$$

### 1.2 Complete Graphs

A conjecture closely related to the problem of determining $a'(G)$ for complete graphs $G = K_n$ is the **perfect 1-factorization conjecture** (see [8],[11],[12]):

**Conjecture 2 (perfect 1-factorization [8])** For any $n \geq 2$, $K_{2n}$ can be decomposed into $2n - 1$ perfect matchings such that the union of any two matchings forms a Hamiltonian cycle of $K_{2n}$.

Apart from proving that the conjecture holds for certain values of $n$, for instance if $n$ is prime [8] (see [11] for a summary of the known cases), this conjecture of Kotzig [8] and others is still open. If such a decomposition of $K_{2n+2}$ (called a perfect one-factorization) exists, then by coloring every perfect matching using a different color and removing one vertex we obtain an acyclic-edge-coloring of $K_{2n+1}$ with $2n + 1 = \Delta(K_{2n+1}) + 1$ colors. Such a coloring is best possible for $K_{2n+1}$ since it is 2n-regular.

A decomposition of $K_{2n+1}$ into $2n + 1$ matchings each having $n$ edges, such that the union of any two matchings forms a Hamiltonian path of $K_{2n+1}$ is called a **perfect near-1-factorization**. As shown above, if $K_{2n+2}$ has a perfect 1-factorization then $K_{2n+1}$ has a perfect near-1-factorization, which in turn implies that $a'(K_{2n+1}) = 2n + 1$. It is easy to see that the converse is also true: if

$^1$There always is a $\Delta$-regular graph $G'$ which satisfies $a'(G') = \max\{a'(G) : \Delta(G) = \Delta\}$.
$K_{2n+1}$ has an acyclic edge coloring with $2n + 1$ colors then this coloring corresponds to a perfect near-1-factorization of $K_{2n+1}$ which implies that $K_{2n+2}$ has a perfect 1-factorization. Therefore the following holds:

**Proposition 3** The following statements are equivalent:

1. $K_{2n+2}$ has a perfect 1-factorization.
2. $K_{2n+1}$ has a perfect near-1-factorization.
3. $a'(K_{2n+1}) = 2n + 1$.

By removing another vertex from the above colored $K_{2n+1}$ we obtain an acyclic-edge-coloring of $K_{2n}$ with $2n + 1 = \Delta(K_{2n}) + 2$ colors, which is best possible for $K_{2n}$. Thus, if the perfect one-factorization conjecture is true, then $a'(K_{2n}) = a'(K_{2n+1}) = 2n + 1$ for every $n$. It may be possible to show the converse, i.e. that if $a'(K_{2n}) = 2n + 1$ then $K_{2n+2}$ has a perfect 1-factorization. It may even be true that any acyclic edge coloring of $K_{2n}$ with $2n + 1$ colors can be completed into an acyclic edge coloring of $K_{2n+1}$ without introducing new colors.

The authors of [2] observed that $a'(K_p) = a'(K_{p-1,p-1}) = p$, where $p > 2$ is prime. The fact that $a'(K_p) = p$ corresponds to the known construction proving that $K_p$ has a perfect near-1-factorization [8]. Note that even finding the exact values of $a'(K_n)$ for every $n$ seems hard, in view of proposition 3 and conjecture 2.

### 1.3 High Girth

Using probabilistic arguments (the Lovász Local Lemma) we can show that conjecture 1 holds for graphs having sufficiently high girth in terms of their maximum degree. Recall that the girth $g(G)$ of a graph $G$ is the minimum length of a cycle in $G$. Let $G$ be a graph of maximum degree $\Delta = \Delta(G)$.

**Theorem 4** There exists $c > 0$ such that if $g(G) \geq \frac{c\Delta^2 \log \Delta}{\log \log \Delta}$ then $a'(G) \leq \Delta + 2$.

By increasing the number of colors, we are able to reduce the condition on the girth as follows.

**Theorem 5** There exists $c > 0$ such that if $g(G) \geq c\Delta$ then $a'(G) \leq 2\Delta + 2$. In general, $c = 4 + o(1)$ will do.

In section 2 we present the proof of theorem 4, and in section 3 we present the proof of theorem 5. Section 4 contains some concluding remarks.

## 2 Proof of Theorem 4

Let $G$ be a graph with maximum degree $d$. Let $k$ be such that $8(48k)^{k-3} > d$, and put $x = 48kd^2$. The absolute constants can easily be improved. We do not attempt to optimize the constants here.
and in what follows. In this section we show that if \( g(G) \geq 3x \) where \( g(G) \) is the girth of \( G \) (the minimum length of a cycle in \( G \)) then there exists an acyclic-edge-coloring of \( G \) with \( d + 2 \) colors.

The proof is probabilistic, and consists of two steps. The edges of \( G \) are first colored properly using \( d + 1 \) colors (by Vizing [10]). Let \( c : E \mapsto \{1, \ldots, d + 1\} \) denote the coloring. Next, one edge is selected from every bichromatic cycle and recolored with a new color \( d + 2 \). It remains to show that there exists a set of edges \( C \) such that by recoloring all the edges of \( C \) with color \( d + 2 \),

(A) the coloring remains proper — \( C \) does not contain any pair of incident edges, and

(B) the coloring becomes acyclic — every cycle of \( G \) contains at least three different colors.

This is proved using the Lovász local lemma. Before continuing with the proof, we state the asymmetric form of the Lovász local lemma we use (cf., e.g. [3]).

The Lovász local lemma. Let \( A_1, \ldots, A_n \) be events in a probability space \( \Omega \), and let \( G = (V, E) \) be a graph on \( V = [1, n] \) such that for all \( i \), the event \( A_i \) is mutually independent of \( \{A_j : (i, j) \notin E\} \). Suppose that there exist \( x_1, \ldots, x_n, 0 < x_i < 1 \), so that, for all \( i \), \( \text{Prob}[A_i] < x_i \prod_{(i,j) \notin E} (1 - x_j) \). Then \( \text{Prob}[\bigwedge A_i] > 0 \).

Let \( B \) denote the set of all cycles that are bichromatic after the first coloring. We restrict our attention to an arbitrarily chosen path \( c(p) \) of \( x \) edges from every bichromatic cycle \( c \in B \). Denote these paths by \( P = \{p(c)\}_{c \in B} \). To form the set \( C \) we now select from each path \( p \in P \) one edge, denoted by \( e(p) \), randomly and independently. We show that with positive probability, the random set of edges \( C = \{e(p)\}_{p \in P} \) satisfies both (A) and (B) above. To this end we define the following two types of events:

**Type I:** For each pair of (distinct) intersecting paths \( p, p' \in P \) let \( E_{p,p'} \) be the event that \( e(p) \) is incident to \( e(p') \).

Recall that \( k \) satisfies \( 8(48k)^{k-3} > d \). Let \( S = \{p_1, \ldots, p_k\} \subseteq P \) (if \( |P| \geq k \)). A simple path of \( 2k - 1 \) edges \( q = e_1, e_2, \ldots, e_{2k-1} \) is called an alternating connecting path of \( S \) if \( c(e_2) = c(e_4) = \cdots = c(e_{2k-2}) \) and \( e_{2i-1} \in p_i \) for each \( 1 \leq i \leq k \). The set \( S \) is called a special set if it has an alternating connecting path (see example in figure 1).

**Type II:** For each special set \( S \) let \( E_S \) denote the event “the edges \( \{e(p)\}_{p \in S} \) belong to an alternating connecting path of \( S \)”.

Now suppose that no event of type I or type II holds. We claim that \( C \) satisfies both (A) and (B). To see that \( C \) satisfies (A), notice that two incident edges can belong to \( C \) only if each edge is selected from a different path and the two paths intersect. Therefore, since no event of type I holds, \( C \) satisfies (A). Suppose that \( C \) does not satisfy (B), i.e. there exists a cycle \( c \) which is
Figure 1: A special set $S$ of 8 paths \{p_1, p_2, \ldots, p_8\} \subseteq P$, where each $p_i$ belongs to a bichromatic cycle, having an alternating connecting path $q = e_1, \ldots, e_{15}$. The number near each edge corresponds to its color. Note that in this example paths $p_1, p_2$ do not intersect, paths $p_3, p_4$ intersect in a vertex and paths $p_6, p_7$ intersect in an edge.
bichromatic after the recoloring. Clearly $c$ contains edges of color $d + 2$; indeed half the edges of $c$ (every other edge) are colored $d + 2$. The length of cycle $c$ is obviously more than $2k - 1$, since the girth of $G$ is assumed to be at least $144kd^2$. Let $q = e_1, e_2, \ldots, e_{2k-1}$ be a path of $c$ where $c(e_1) = c(e_3) = \cdots = c(e_{2k-1}) = d + 2$. Since each edge $e_{2i-1}$ belongs to $C$ ($1 \leq i \leq k$) it also belongs to a path $p_i \in P$ from which it was selected (if it belongs to several such paths, let $p_i$ denote one of them). Thus $S = \{p_i\}_{i=1}^k$ is a special set with alternating connecting path $q$, a contradiction to the assumption that no event of type II holds. Therefore, if none of these events hold, $C$ satisfies both (A) and (B).

It remains to show that with positive probability none of these events happen. To prove this we apply the local lemma. Let us construct a graph $H$ whose nodes are all the events of both types, in which two nodes $E_X$ and $E_Y$ (where each of $X, Y$ is either a pair of intersecting paths $p, p' \in P$ or a special set) are adjacent if and only if $X$ and $Y$ contain a common path. Since the occurrence of each event $E_X$ depends only on the edges selected from the paths in $X$, $H$ is a dependency graph for our events. In order to apply the local lemma we need estimates for the probability of each event, and for the number of nodes of each type in $H$ which are adjacent to any given node. These estimates are given in the two lemmas below. We first prove the following simple fact, which is needed for the next section as well.

**Proposition 6** A special set of $k$ paths each of length $x$, has at most $36$ different alternating connecting paths, provided $g(G) > 2x + 2k \geq 4k - 2$.

**Proof of Proposition 6.** Let $S = \{p_1, \ldots, p_k\}$ be a special set and let $q = e_1, \ldots, e_{2k-1}$ be any alternating connecting path of $S$. The distance between any pair of consecutive paths $p_i, p_{i+1} \in S$ is at most 1 since $e_{2i-1} \in p_i$ and $e_{2i+1} \in p_{i+1}$. One the other hand the distance between any pair of (distinct) non-consecutive paths $p_i, p_j \in S$ is more than 1, since otherwise there would be a simple cycle on $p_i, p_j$ and $q$ of length $\leq 2x + 2k < g(G)$. Therefore any alternating connecting path of $S$ refers to the paths $\{p_i\}$ of $S$ in the same order.

Let $F$ denote the vertices of $p_1$ which are at distance at most 2 from $p_2$, and $L$ be the vertices of $p_k$ which are at distance at most 2 from $p_{k-1}$. The first vertex of any alternating path $q$ of $S$ must belong to $F$, and the last vertex of $q$ must belong to $L$. Now $|F|, |L| \leq 6$ since $g(G) > 2x$ and $p_1, p_2$ can share either one vertex or one edge (they belong to two different bichromatic cycles), and the same applies to $p_{k-1}, p_k$. The proof is completed by observing that there exists at most one alternating connecting path of $S$ starting from any given vertex of $F$ and ending at any given vertex of $L$, since there is no cycle of length $4k - 2$ or less. \hfill \Box

**Lemma 7**

1. For each event $E_{p, p'}$ of type I, $\text{Prob}[E_{p, p'}] \leq \frac{6}{2x}$.

2. For each event $E_S$ of type II, $\text{Prob}[E_S] \leq \frac{36}{2x}$.
Proof of Lemma 7. To prove the first claim, note that any two distinct paths \( p, p' \in P \) can intersect at most once, sharing one vertex or one edge. For any pair of intersecting paths \( p, p' \in P \) there are at most 6 pairs of incident edges, one from each path, and the probability of selecting any such pair of edges is exactly \( \frac{1}{x^2} \).

To prove the second claim, let \( S = \{p_i\}_{i=1}^k \) be a special set of paths. Since \( 4k - 2 < 2x + 2k < 3x < g(G) \) proposition 6 holds, implying that there are at most 36 alternating connecting paths of \( S \), and the probability that \( \{e(p)\}_{p \in S} \) belongs to any such path is \( \frac{1}{x^2} \).

Lemma 8 For any given path \( p \in P \),

1. there are less than \( xd^2 \) paths \( p' \in P \) that intersect \( p \), and
2. there are at most \( kxd^{2k-1} \) special sets that contain \( p \).

Proof of Lemma 8. Let \( p \in P \). To prove the first claim, recall that two intersecting paths \( p, p' \in P \) share either one vertex or one edge. There are at most \( (x + 1)\binom{d}{2} < \frac{1}{2}xd^2 \) paths \( p' \in P \) that intersect \( p \) in one vertex, since there are \( x + 1 \) possibilities for the intersection vertex and at most \( \binom{d}{2} \) pairs of colors to determine \( p' \). (Determining a vertex of \( p' \) and the two colors of \( p' \) defines the bichromatic cycle that \( p' \) belongs to, which in turn identifies \( p' \) itself.) In addition there are at most \( xd(d - 1) < \frac{1}{2}xd^2 \) paths \( p' \in P \) that intersect \( p \) in an edge, since there are \( x \) possibilities for the intersection edge and \( d - 1 \) possibilities for the second color of \( p' \).

To prove the second claim, let \( p \in P \). A special set \( S = \{p_1, \ldots, p_k\} \) that contains \( p \) can be determined by the following five steps. First an alternating connecting path \( q = e_1, \ldots, e_{2k - 1} \) is determined by fixing one edge of \( q \) (steps 1,2 below) and choosing all the colors of \( q \) (steps 3,4). Now \( q \) determines one edge \( (e_{2i-1}) \) of each \( p_i \), hence the special set \( S \) is uniquely determined by selecting the second color of each \( p_i \) (step 5).

Step 1: Determine the index \( i \) such that \( p_i = p \) (\( k \) possibilities).

Step 2: Determine the edge \( e_{2i-1} \) of \( p_i \cap q \) (\( x \) possibilities).

Step 3: Determine the color \( c_0 \) of edges \( \{e_2, e_4, \ldots, e_{2k-2}\} \) (\( d \) possibilities — all colors but \( c(e_{2i-1}) \)).

Step 4: Determine the color of each edge \( e_{2j-1} \) for \( j \neq i \) (\( d^{k-1} \) possibilities — all colors but \( c_0 \)).

Step 5: Determine the second color of each \( e_{2j-1} \) for \( j \neq i \) (\( d^{k-1} \) possibilities — all colors but \( c(e_{2j-1}) \)).

Therefore the number of special sets \( S \) that contain \( p \) is at most \( kxd^{2k-1} \).

It follows from lemma 8 that

1. each event of type I is independent of all but at most \( 2xd^2 \) events of type I,
2. each event of type I is independent of all but at most $2kxd^{2k-1}$ events of type II,

3. each event of type II is independent of all but at most $kxd^2$ events of type I, and

4. each event of type II is independent of all but at most $k^2xd^{2k-1}$ events of type II.

Taking twice the bounds from lemma 7 for the real constants in order to apply the Lovász local lemma, we conclude that there exists a selection of edges in which no event of type I or II occurs, provided that:

$$\frac{6}{x^2} < \frac{12}{x^2} \left(1 - \frac{12}{x^2}\right)^{2xd^2} \left(1 - \frac{72}{x^k}\right)^{2kxd^{2k-1}}, \quad \text{and} \quad (2)$$

$$\frac{36}{x^k} < \frac{72}{x^k} \left(1 - \frac{12}{x^2}\right)^{kxd^2} \left(1 - \frac{72}{x^k}\right)^{k^2xd^{2k-1}}. \quad \text{(3)}$$

Now

$$\left(1 - \frac{12}{x^2}\right)^{2xd^2} \geq \left(1 - \frac{12}{x^2}\right)^{kxd^2} \geq 1 - \frac{12kd^2}{x} = \frac{3}{4}$$

since $x = 48kd^2$, and

$$\left(1 - \frac{72}{x^k}\right)^{kxd^{2k-1}} \geq \left(1 - \frac{72}{x^k}\right)^{k^2xd^{2k-1}} \geq 1 - \frac{72k^2d^{2k-1}}{x^{k-1}}.$$ 

Inequalities (2) and (3) are satisfied if $1 - \frac{72k^2d^{2k-1}}{x^{k-1}} > \frac{3}{4}$, which is equivalent to $x^{k-1} > 288k^2d^{2k-1}$. Since $x = 48kd^2$ this is equivalent to having $(48k)^{k-1}d^{2k-2} > 288k^2d^{2k-1}$ or $8(48k)^{k-3}d = d$ which holds by the choice of $k$.

## 3 Proof of Theorem 5

Let $G$ be a graph with maximum degree $d$ and girth $g$. We show that if $g \geq 4d(1 + o(1))$ there exists an acyclic-edge-coloring of $G$ with $2d + 2$ colors.

The proof is probabilistic, and consists of two steps. The edges of $G$ are first colored properly using $d + 1$ colors (by Vizing [10]). Let $c : E \mapsto \{1, \ldots, d + 1\}$ denote the coloring. Next, one edge $e$ is selected from every bichromatic cycle and recolored with the opposite color: $-c(e)$. Notice that the coloring will remain proper after this recoloring. It remains to show that there exists a set of edges $C$ such that by recoloring each edge of $C$ with its opposite color, every cycle of $G$ will contain at least three different colors. This is proved using the symmetric form of the Lovász local lemma, which is stated below (cf., e.g. [3]).

**The Lovász local lemma (symmetric case).** Let $A_1, \ldots, A_n$ be events in a probability space $\Omega$. Suppose that each event $A_i$ is mutually independent of a set of all the other events $A_j$ but at most $d$, and that $\Pr[A_i] \leq p$ for all $i$. If $ep(d + 1) \leq 1$, then $\Pr[\bigwedge A_i] > 0$. 

As in the proof of theorem 4, let $B$ denote the set of all cycles that are bichromatic after the first coloring. We restrict our attention to an arbitrarily chosen path $p(c)$ of $x = \lfloor g/4 \rfloor - 1$ edges from every bichromatic cycle $c \in B$. Denote these paths by $P = \{p(c)\}_{c \in B}$. To form the set $C$ we now select from each path $p \in P$ one edge, denoted by $e(p)$, randomly and independently. We show that with positive probability, the coloring obtained after recoloring each edge $e(p)$ with its opposite color $-c(e(p))$ is acyclic.

Let $S = \{p_1, \ldots, p_x\} \subseteq P$ (if $|P| \geq x$). A simple path of $2x - 1$ edges $q = e_1, e_2, \ldots, e_{2x-1}$ is called a bichromatic alternating connecting path of $S$ if it is bichromatic and $e_{2i-1} \in p_i$ for each $1 \leq i \leq x$. The set $S$ is called a very special set if it has a bichromatic alternating connecting path. Notice that a bichromatic alternating connecting path is simply a bichromatic alternating connecting path, as defined in section 2. Therefore a very special set is also a special set. For each very special set $S$ let $E_S$ denote the event "the edges $\{e(p)\}_{p \in S}$ belong to a bichromatic alternating connecting path of $S$".

It is easy to see that if no event $E_S$ occurs, the recoloring is acyclic. Indeed suppose that a cycle $c$ is bichromatic after the recoloring. Then $c$ was bichromatic in the original coloring as well ($c \in B$). In addition, all or exactly half (every other edge) of the edges of $c$ had to be recolored. Any path of $c$ having $2x - 1$ edges starting (and ending) with a recolored edge, is a bichromatic alternating connecting path of some very special set, thereby causing some event $E_S$ to occur.

Setting $k = x$ we can apply proposition 6 (recall that $g(G) > 4x$) and conclude that a very special set has at most $36$ (bichromatic) alternating connecting paths. Therefore $\text{Prob}[E_S] \leq \frac{36}{x^2}$.

Each event $E_S$ is mutually independent of all events $E_T$ such that $S$ and $T$ do not have a common path. By slightly modifying the proof of lemma 8 (2) one can show that the number of very-special sets that contain a given path $p$ is less than $x^2d^{x+1}$. Indeed only step 4 requires special attention, since there are now at most $d$ (instead of $d^{k-1}$) possibilities for choosing the single color of $q$’s odd edges. Also note that in step 5 there are now strictly less than $d^{k-1}$ possibilities for choosing the second color of the $p_j$’s, since at most one $p_j$ can use $c_0$ as its second color.

Thus $E_S$ is independent of all but at most $x^2d^{x+1} - 1$ other events $E_T$. Using the Lovász local lemma (symmetric case), it suffices to have $36e^{x^2d^{x+1}} < 1$ or equivalently $x > d(36ed^3)^{1/(x-2)}$. This requires $g > 4x \geq 4d(1 + o(1))$. \(\square\)

4 Concluding Remarks

1. The following weaker version of theorem 4 can be proved in a similar but much simpler way using the symmetric Lovász local lemma:

**Proposition 9** There exists a constant $c > 0$ such that $a(G) \leq \Delta(G) + 2$ if $g(G) > c\Delta(G)^3$.

This can be achieved by recoloring one edge from each bichromatic cycle using one additional color (as in the proof of theorem 4), while avoiding recoloring two edges which are incident or
2. For graphs $G$ of class 1 Vizing (that is: graphs whose edges can be properly colored using $\Delta(G)$ colors), the bounds for $a(G)$ presented in theorems 4 and 5 can be slightly improved. Indeed the proofs of these theorems show that for graphs $G$ of class 1 there exist constants $c_1, c_2$ such

$$a'(G) \leq \Delta(G) + 1 \quad \text{if} \quad g(G) \geq \frac{c_1 \Delta^2 \log \Delta}{\log \log \Delta},$$

$$a'(G) \leq 2\Delta \quad \text{if} \quad g(G) \geq c_2 \Delta.$$

Note that this shows that $a'(G) = \Delta + 1$ for any $\Delta$-regular graph $G$ of class 1 whose girth is sufficiently large as a function of $\Delta$.

3. Molloy and Reed [9] presented a polynomial-time algorithm that produces an acyclic coloring with $20\Delta$ colors for any given input graph with maximum degree $\Delta$. The known results about the algorithmic version of the local lemma, initiated by Beck ([4], see also [1],[9]), can be combined with our method here to design a polynomial algorithm that produces an acyclic $\Delta + 2$ coloring for any given input graph with maximum degree $\Delta$ whose girth is sufficiently large as a function of $\Delta$.

References


