Disjoint Systems (Extended Abstract)

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Abstract

A *disjoint system of type* $(\forall, \exists, k, n)$ is a collection $\mathcal{C} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ of pairwise disjoint families of k-subsets of an n-element set satisfying the following condition. For every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ and for every $A \in \mathcal{A}_i$ there exists a $B \in \mathcal{A}_j$ that does not intersect A. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$. It is shown that for every fixed $k \geq 2$,

$$lim_{n\to\infty} D_n(\forall, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

This settles a problem of Ahlswede, Cai and Zhang. Several related problems are considered as well.

1 Introduction

In Extremal Finite Set Theory one is usually interested in determining or estimating the maximum or minimum possible cardinality of a family of subsets of an n element set that satisfies certain properties. See [5], [7]

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and [9] for a comprehensive study of problems of this type. In several recent papers (see [3], [1],[2]), Ahlswede, Cai and Zhang considered various extremal problems that study the maximum or minimum possible cardinality of a collection of families of subsets of an n-set, that satisfies certain properties. They observed that many of the classical extremal problems dealing with families of sets suggest numerous intriguing questions when one replaces the notion of a family of sets by the more complicated one of a collection of families of sets.

In the present note we consider several problems of this type that deal with disjoint systems. Let $N = \{1, 2, \ldots, n\}$ be an n element set, and let $\mathcal{C} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ be a collection of pairwise disjoint families of k-subsets of N. $\mathcal{C}$ is a *disjoint system of type* $(\exists, \forall, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ there exists an $A \in \mathcal{A}_i$ which does not intersect any member of $\mathcal{A}_j$. Similarly, $\mathcal{C}$ is a *disjoint system of type* $(\forall, \exists, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ and for every $A \in \mathcal{A}_i$ there exists a $B \in \mathcal{A}_j$ that does not intersect A. Finally, $\mathcal{C}$ is a *disjoint system of type* $(\exists, \exists, k, n)$ if for every ordered pair $\mathcal{A}_i$ and $\mathcal{A}_j$ of distinct members of $\mathcal{C}$ there exists an $A \in \mathcal{A}_i$ and a $B \in \mathcal{A}_j$ that does not intersect A.

Let $D_n(\exists, \forall, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \forall, k, n)$. Let $D_n(\forall, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\forall, \exists, k, n)$ and let $D_n(\exists, \exists, k)$ denote the maximum possible cardinality of a disjoint system of type $(\exists, \exists, k, n)$. Trivially, for every n,

$$D_n(\exists, \forall, 1) = D_n(\forall, \exists, 1) = D_n(\exists, \exists, 1) = n.$$

It is easy to see that every disjoint system of type $(\exists, \forall, k, n)$ is also a system of type $(\forall, \exists, k, n)$, and every system of type $(\forall, \exists, k, n)$ is also of type $(\exists, \exists, k, n)$. Therefore, for every $n \geq k$

$$D_n(\exists, \forall, k) \leq D_n(\forall, \exists, k) \leq D_n(\exists, \exists, k).$$

In this note we determine the asymptotic behaviour of these three functions for every fixed k, as n tends to infinity.

Theorem 1.1 *For every* $k \geq 2$

$$lim_{n\to\infty} D_n(\exists, \forall, k) \binom{n}{k}^{-1} = \frac{1}{k+1}.$$

Theorem 1.2 *For every* $k \geq 2$

$$lim_{n\to\infty} D_n(\forall, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

Corollary 1.3 *For every* $k \geq 2$

$$lim_{n\to\infty} D_n(\exists, \exists, k) \binom{n}{k}^{-1} = \frac{1}{2}.$$

Theorem 1.1 settles a conjecture of Ahlswede, Cai and Zhang [2], who proved it for $k = 2$ [1]. The main tool in its proof is a result of Frankl and Füredi [8]. The proof of Theorem 1.2, which settles another question raised in [2] and proved for $k = 2$ in [1], is more complicated and combines combinatorial and probabilistic arguments. A sketch of this proof and the simple derivation of Corollary 1.3 from its assertion are presented in Section 2. The proof of Theorem 1.1 and the full proof of Theorem 1.2 will appear in the full version of this paper.

2 Random graphs and disjoint systems

In this section we give a sketch of the proof of Theorem 1.2. We need the following two probabilistic lemmas.

Lemma 2.1 (Chernoff, see e.g. [4], Appendix A) *Let X be a random variable with the binomial distribution $B(n,p)$. Then for every $a > 0$ we have*

$$Pr(|x - np| > a) < 2e^{-2a^2/n}. \ \square$$

Let L be a graph-theoretic function. L satisfies *the Lipschitz condition* if for any two graphs H, H' on the same set of vertices that differ only in one edge we have $|L(H) - L(H')| \leq 1$. Let $G(n,p)$ denote, as usual, the random graph on n labeled vertices in which every pair, randomly and independently, is chosen to be an edge with probability p. (See, e.g., [6].)

Lemma 2.2 ([4], Chapter 7) *Let L be a graph-theoretic function satisfying the Lipschitz condition and let $\mu = E[L(G)]$ be the expectation of $L(G)$, where $G = G(n,p)$. Then for any $\lambda > 0$*

$$Pr(|L(G) - \mu| > \lambda\sqrt{m}] < 2e^{-\lambda^2/2}$$

where $m = \binom{n}{2}$. $\square$

A Sketch of the proof of Theorem 1.2 Let n_1 be the number of families containing only one element in a disjoint system of type $(\forall, \exists, k, n)$. Since sets in any two one-element families are disjoint we have $n_1 \leq n/k$. This settles the required upper bound for $D_n(\forall, \exists, k)$, since all other families contain at least 2 sets.

We prove the lower bound using probabilistic arguments. We show that for any $\varepsilon > 0$ there are at least $\frac{1}{2}(1-\varepsilon)\binom{n}{k}$ families which form a disjoint system of type $(\forall, \exists, k, n)$, provided n is sufficiently large (as a function of ε and k). Let $G = G(n,p)$ be a random graph, where p is a constant, to be specified later, which is very close to 1. We use this graph to build another random graph G_1, whose vertices are all k-cliques in G. Two vertices of G_1 are adjacent if and only if the induced subgraph on the corresponding k-cliques in G is the union of two vertex disjoint k-cliques with no edges between them. We prove that almost surely (i.e.,with probability that tends to 1 as n tends to infinity) the following two events happen. First, the number of vertices in G_1 is greater than $(1-\varepsilon/2)\binom{n}{k}$. Second, G_1 is almost regular, i.e., for every (small) $\delta > 0$ there exists a (large) number d such that the degree $d(x)$ of any vertex x of G_1 satisfies $(1-\delta)d < d(x) < (1+\delta)d$, provided n is sufficiently large.

Suppose $G_1 = (V, E)$ satisfies these properties. By Vizing's Theorem [10], the chromatic index $\chi'(G_1)$ of G_1 satisfies $\chi'(G_1) \leq (1+\delta)d + 1$. Since for any $x \in G_1$ we have $d(x) \geq (1-\delta)d$, the number of edges $|E|$ of G_1 is at least $\frac{(1-\delta)d|V|}{2}$. Hence there exists a matching in G_1 which contains at least $\frac{(1-\delta)d|V|}{2}/\chi'(G_1) \sim \frac{(1-\delta)|V|}{2(1+\delta)}$ edges. This matching covers almost all vertices of G_1, as δ is small, providing a system of pairs of k-sets covering almost all the $\binom{n}{k}$ k-sets. Taking each pair as a family we have a disjoint system of size at least $\frac{1}{2}(1-\varepsilon)\binom{n}{k}$ and ε can be made arbitrarily small for all n sufficiently large.

We next show that the resulting system is a disjoint system of type $(\forall, \exists, k, n)$. Assume this is false and let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{B} = \{B_1, B_2\}$ be two pairs where $A_1 \cap B_i \neq \emptyset$ for $i = 1, 2$. Choose $x_1 \in A_1 \cap B_1$ and $x_2 \in A_1 \cap B_2$. Since x_1 and x_2 belong to A_1 they are adjacent in $G = G(n, p)$. However, $x_1 \in B_1$, $x_2 \in B_2$ and this contradicts the fact that the subgraph of G induced on $B_1 \cup B_2$ has no edges between B_1 and B_2. Thus the system is indeed of type $(\forall, \exists, k, n)$ and

$$D_n(\forall, \exists, k) > \frac{1}{2}(1-\varepsilon)\binom{n}{k}$$

for every $\varepsilon > 0$, provided $n > n_0(k, \varepsilon_1)$, as needed.

The proof that indeed G_1 has the required properties stated above almost surely can be established using Lemmas 2.1 and 2.2. We omit the details. □

Proof of Corollary 1.3 Let n_1 be the number of one element families in a disjoint system of type $(\exists, \exists, k, n)$. The trivial argument used in the proof of Theorem 1.2 shows that $n_1 \leq n/k$. Since each other family contains at least two elements

$$D_n(\exists, \exists, k) \leq \frac{n}{k} + \frac{1}{2}\binom{n}{k}.$$

As observed in Section 1, $D_n(\forall, \exists, k) \leq D_n(\exists, \exists, k)$ and hence, by Theorem 1.2, the desired result follows. □

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