MAX CUT and the smallest eigenvalue
(Extended abstract)

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Abstract

In this paper we determine the precise approximation guarantee of the Max Cut algorithm of Goemans and Williamson for graphs $G = (V, E)$ in which the size of the max-cut is at least $A|E|$, for all $A$ between 0.845 and 1. This extends a result of Karloff.

1 Introduction

The smallest eigenvalue of (the adjacency matrix of) a graph $G$ is closely related to the size of the maximum bipartite subgraph in it. Our main result in this paper is based on this relation.

If $G = (V, E)$ is an undirected graph, and $S$ is a nonempty proper subset of $V$, then $(S, V - S)$ denotes the cut consisting of all edges with one end in $S$ and another one in $V - S$. The size of the cut is the number of edges in it. The MAX CUT problem is the problem of finding a cut of maximum size in $G$. This is a well known NP-hard problem (which is also MAX-SNP hard as shown in [9]- see also [6], [2]), and the best known approximation algorithm for it, due to Goemans and Williamson [5], is based on semidefinite programming and an appropriate (randomized) rounding technique. It is proved in [5] that the approximation guarantee of this algorithm is the minimum of the function $h(t)/t$ in $(0, 1]$, where $h(t) = \frac{1}{2} \arccos(1 - 2t)$. This minimum is attained at $t_0 = 0.844..$ and is roughly 0.878. Karloff [7] showed that this minimum is indeed the correct approximation guarantee of the algorithm, by constructing appropriate graphs. The authors of [5] also proved that their algorithm has a better approximation guarantee for graphs with large cuts. If $A \geq t_0$, with $t_0$ as above, and the maximum cut of $G = (V, E)$ has $A|E|$ edges, then the expected size of the cut

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Theorem 1.1 For any rational \( \eta \) satisfying \(-1 < \eta < 0\), there exists a graph \( H = (V, E) \), \( V = \{1, \ldots, n\} \) and a set of unit vectors \( w_1, \ldots, w_n \) in \( \mathbb{R}^k \), \( 0 \leq k \leq n \) such that \( w_i^t w_j = \eta \) for all \( i \neq j \) with \( ij \in E \) and the size of a maximum cut in \( H \) is equal to

\[
\max_{\|v_i\|^2=1, v_i \in \mathbb{R}^n} \sum_{ij \in E} \frac{1 - v_i^t v_j}{2} = \sum_{ij \in E} \frac{1 - w_i^t w_j}{2}.
\]

The rest of this short paper is organized as follows. In Section 2 we prove Theorem 1.1 by constructing appropriate graphs. Our construction resembles the one in [7], but is more general and its analysis is somewhat simpler. We also discuss the relevance of the construction to the study of the approximation guarantee of the algorithm of [5]. The final Section 3 contains some concluding remarks.

2 Max Cut

2.1 The Goemans-Williamson algorithm and its performance

We first describe the algorithm of Goemans and Williamson. For simplicity, we consider the unweighted case. More details appear in [5].

The MAX CUT problem is that of finding a cut \((S, V - S)\) of maximum size in a given input graph \( G = (V, E) \), \( V = \{1, \ldots, n\} \). By assigning a variable \( x_i = +1 \) to each vertex \( i \) in \( S \) and \( x_j = -1 \) to each vertex \( j \) in \( V - S \) it follows that this is equivalent to maximizing the value of \( \sum_{ij \in E} \frac{1 - x_i x_j}{2} \), over all \( x_i \in \{-1, 1\} \). This problem is well known to be NP-hard, but one can relax it to the polynomialy-solvable problem of finding the maximum

\[
\max_{\|v_i\|^2=1} \sum_{ij \in E} \frac{1 - v_i^t v_j}{2},
\]

where each \( v_i \) ranges over all \( n \)-dimensional unit vectors. Note that all our vectors are considered as column vectors and hence \( v^t u \) is simply the inner product of \( v \) and \( u \). This is a semidefinite programming problem which can be solved (up to an exponentially small additive error) in polynomial
time. The last expression is a relaxation of the max cut problem, since the vectors \( v_i = (x_i, 0, \ldots, 0) \) form a feasible solution of the semidefinite program. Therefore, the optimal value \( z^* \) of this program is at least as large as the size of the max cut of \( G \), which we denote by \( OPT(G) \).

Given a solution \( v_1, \ldots, v_n \) of the semidefinite program, Goemans and Williamson suggested the following rounding procedure. Choose a random unit vector \( r \) and define \( S = \{ i \mid r^i v_i \leq 0 \} \) and \( V - S = \{ j \mid r^j v_j > 0 \} \). This supplies a cut \((S, V - S)\) of the graph \( G \). Let \( W \) denote the size of the random cut produced in this way and let \( E[W] \) be its expectation. By linearity of expectation, the expected size is the sum, over all \( i j \in E \), of the probabilities that the vertices \( i \) and \( j \) lie in opposite sides of the cut. This last probability is precisely \( \arccos(v_i^t v_j)/\pi \). Thus the expected value of the weight of the random cut is exactly

\[
\sum_{i j \in E} \frac{\arccos(v_i^t v_j)}{\pi}.
\]

However the optimal value \( z^* \) of the semidefinite program is equal to

\[
z^* = \sum_{i j \in E} \frac{1 - v_i^t v_j}{2}.
\]

Therefore the ratio between \( E[W] \) and the optimal value \( z^* \) satisfies

\[
\frac{E[W]}{z^*} = \frac{\sum_{i j \in E} \arccos(v_i^t v_j)/\pi}{\sum_{i j \in E} (1 - v_i^t v_j)/2} \geq \min_{i j \in E} \frac{\arccos(v_i^t v_j)/\pi}{(1 - v_i^t v_j)/2}.
\]

Denote \( \alpha = \frac{2}{\pi} \min_{0 < \theta \leq \pi} \frac{\theta}{1 - \cos \theta} \). An easy computation gives that the minimum \( \alpha \) is attained at \( \theta = 2.3311.. \), the nonzero root of \( \cos \theta + \theta \sin \theta = 1 \), and that \( \alpha \in (0.87856, 0.87857) \). Thus, \( E[W] \geq \alpha \cdot z^* \), and since the value of \( z^* \) is at least as large as the weight \( OPT \) of the maximum cut, we conclude that \( E[W] \geq \alpha \cdot OPT \). It follows that the Goemans-Williamson algorithm supplies an \( \alpha \)-approximation for \( \text{MAX CUT} \). Moreover, by the above discussion, the expected size of the cut produced by the algorithm is not better than \( \alpha \cdot OPT \) in case \( OPT = z^* \) and for an optimal solution \( v_1, \ldots, v_n \) for the semidefinite programming problem, \( \frac{\arccos(v_i^t v_j)/\pi}{(1 - v_i^t v_j)/2} = \alpha \) for all \( i j \in E \).

In case the value of the semidefinite program is a large fraction of the total number of edges of \( G \), the above reasoning together with a simple convexity argument is used in [5] to show that the performance of the algorithm is better. Put \( h(t) = \arccos(1 - 2t)/\pi \) and let \( t_0 \) be the value of \( t \) for which \( h(t)/t \) attains its minimum in the interval \((0, 1)\). Then \( t_0 \) is approximately 0.84458. Define \( A = z^*/|E| \). If \( A \geq t_0 \) then, as shown in [5], \( E[W] \geq \frac{h(A)}{A} z^* \geq \frac{h(A)}{A} OPT \). Here, as before, the actual expected size of the cut produced by the algorithm is not better than \( \frac{h(A)}{A} OPT \) in case \( OPT = z^* \) and for an optimal solution \( v_1, \ldots, v_n \) for the semidefinite programming problem, \( v_i^t v_j = 1 - 2A \) for all \( i j \in E \).

Karloff [7] proved that the approximation ratio of the algorithm is exactly \( \alpha \). To do so, he constructed, for any small \( \delta > 0 \), a graph \( G \) and vectors \( v_1, \ldots, v_n \) which form an optimal solution
of the semidefinite program, such that \( OPT(G) = z^*(G) \) and \( \frac{\arccos(v_i^t v_j)/\pi}{(1-v_i^t v_j)/2} < \alpha + \delta \) for all \( ij \in E \).

Theorem 1.1 is a generalization of his result, and shows that the analysis of Goemans and Williamson is tight not only for the worst case, but also for graphs in which the size of the maximum cut is a larger fraction of the number of edges. Applying the above analysis to the graph \( H \) from Theorem 1.1 together with the vectors \( w_i \) as the solution of the semidefinite program we obtain that in this case \( A = \frac{1-\eta}{\pi} \) and the approximation ratio is precisely

\[
\frac{E[W]}{z^*(H)} = \frac{E[W]}{OPT(H)} = \min_{ij \in E(H)} \frac{\arccos(w_i^t w_j)/\pi}{(1-w_i^t w_j)/2} = \frac{2 \arccos \eta}{\pi} = \frac{h(A)}{A}.
\]

2.2 The proof of Theorem 1.1

Our construction is based on the properties of graphs arising from the Hamming Association Scheme over the binary alphabet. Let \( V = \{v_1, \ldots, v_n\}, n = 2^m \) be the set of all vectors of length \( m \) over the alphabet \( \{-1, +1\} \). For any two vectors \( x, y \in V \) denote by \( d(x, y) \) their Hamming distance, that is, the number of coordinates in which they differ. The Hamming graph \( H = H(m, 2, b) \) is the graph whose vertex set is \( V \) in which two vertices \( x, y \in V \) are adjacent if and only if \( d(x, y) = b \). Here we consider only even values of \( b \) which are greater than \( m/2 \). We show that for any rational \( \eta \) there exists an appropriate Hamming graph which satisfies the assertion of Theorem 1.1. We may and will assume, whenever this is needed, that \( m \) is sufficiently large.

Note that by definition, \( H(m, 2, b) \) is a Cayley graph of the multiplicative group \( Z_2^m = \{-1, +1\}^m \) with respect to the set \( U \) of all vectors with exactly \( b \) coordinates equal to \(-1\). Therefore (see, e.g., [8], Problem 11.8) the eigenvectors of \( H(m, 2, b) \) are the multiplicative characters \( \chi_I \) of \( Z_2^m \), where \( \chi_I(x) = \prod_{i \in I} x_i \), \( I \) ranges over all subsets of \( \{1, \ldots, m\} \), and the corresponding eigenvalues are \( \sum_{x \in U} \chi_I(x) \). The eigenvalues of \( H \) are thus equal to the so called binary Krawtchouk polynomials (see [3]),

\[
P^m_b(k) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{m-k}{b-j}, \quad 1 \leq k \leq n.
\]

The eigenvalue \( P^m_b(k) \) corresponds to the characters \( \chi_I \) with \( |I| = k \) and thus has multiplicity \( \binom{m}{k} \).

Consider any two adjacent vertices of \( H(m, 2, b) \), \( v_i \) and \( v_j \). By the definition of \( H \), the inner product \( v_i^t v_j \) is \( m - 2b \). Choose \( m \) and \( b \) such that \( b > m/2 \) is even and \( \frac{m-2b}{m} = \eta \). This is always possible since \( \eta \) is a rational number \(-1 < \eta < 0\). Let \( w_i = \frac{1}{\sqrt{m}} v_i \) for all \( i \), thus \( \|w_i\|^2 = 1 \) and \( w_i^t w_j = \eta \) for any pair of adjacent vertices. We claim that for such choice of \( m \) and \( b \) the Hamming graph \( H(m, 2, b) \) together with the set of vectors \( w_i, 1 \leq i \leq n \) satisfy the assertion of Theorem 1.1. To prove this we first need to establish a connection between the smallest eigenvalue of a graph and the semidefinite relaxation of the max cut problem.
**Proposition 2.1** Let $G = (V, E)$ be a graph on the set $V = \{1, 2, \ldots, n\}$ of $n$ vertices, with adjacency matrix $A = (a_{ij})$ and let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of $A$. Then

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} \leq \frac{1}{2} |E| - \frac{1}{4} \lambda_n \cdot n,$$

for any $k > 0$ and any set $v_1, \ldots, v_n$ of unit vectors in $R^k$.

**Proof.** Let $B = (b_{ij})$ be the $n \times k$ matrix whose rows are the vectors $v_1, \ldots, v_n$. Denote by $u_1, \ldots, u_k$ the columns of $B$. By definition we have $\sum_{i=1}^k \|u_i\|^2 = \sum_{i<j} b_{ij}^2 = \sum_{i=1}^n \|v_i\|^2 = n$. Therefore

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} = \frac{1}{2} |E| - \frac{1}{2} \sum_{i<j} a_{ij} v_i^t v_j = \frac{1}{2} |E| - \frac{1}{4} \sum_{i=1}^k u_i^t A u_i.$$

By the variational definition of the eigenvalues of $A$, for any vector $z \in \mathbb{R}^n$, $z^t A z \geq \lambda_n \|z\|^2$ and equality holds if and only if $Az = \lambda_n z$. This implies that

$$\sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} \leq \frac{1}{2} |E| - \frac{1}{4} \lambda_n \sum_{i=1}^k \|u_i\|^2 = \frac{1}{2} |E| - \frac{1}{4} \lambda_n \cdot n.$$

Note that in the last expression equality holds if and only if each $u_i$ is an eigenvector of $A$ with eigenvalue $\lambda_n$. This completes the proof. \qed

As we show later, the smallest eigenvalue of the adjacency matrix $A_H = (a_{ij})$ of the graph $H(m, 2, b)$ is $P_b^m(1)$. By the above discussion it has multiplicity $m$ and eigenvectors $u_1, \ldots, u_m$ with $\pm 1$ coordinates, where for each vertex $v_j = (v_{j1}, \ldots, v_{jm})$, $u_k(v_j) = v_{ji}$. Therefore, the columns of the matrix $B$, whose rows are the vectors $w_i$, are the eigenvectors $\frac{1}{\sqrt{m}} u_i$ of $A_H$ corresponding to the eigenvalue $P_b^m(1)$. By the proof of Proposition 2.1 it follows that

$$\max_{\|v_i\|^2} \sum_{i<j} a_{ij} \frac{1 - v_i^t v_j}{2} = \frac{1}{2} |E(H)| - \frac{1}{4} P_b^m(1) \cdot n = \sum_{i<j} a_{ij} \frac{1 - u_i^t u_j}{2}.$$

On the other hand, $u_i$ is a vector with $\pm 1$ coordinates. Thus the coordinates of $u_i$ correspond to a cut in $H(m, 2, b)$ of size equal to

$$\sum_{k<j} a_{kj} \frac{1 - u_i(v_k) u_i(v_j)}{2} = \frac{1}{2} |E(H)| - \frac{1}{4} u_i^t A_H u_i = \frac{1}{2} |E(H)| - \frac{1}{4} P_b^m(1) \|u_i\|^2 = \frac{1}{2} |E(H)| - \frac{1}{4} P_b^m(1) \cdot n.$$

Thus the size of a maximum cut in $H(m, 2, b)$ is equal to the optimal value of the semidefinite program. To complete the proof of Theorem 1.1 it remains to prove the following statement.

**Proposition 2.2** Let $P_b^m(k), 1 \leq k \leq m$ be the binary Krawtchouk polynomials and let $b$ be an even integer satisfying $b = \frac{1-\eta}{2} m$ for some fixed $-1 < \eta < 0$. Then $P_b^m(1) \leq P_b^m(k)$ for all $0 \leq k \leq n, n > n_0(\eta)$.  

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Proof. We need the following well known properties of the Krawtchouk polynomials (see, e.g., [3]).

\[(m - k)P^m_b(k + 1) = (m - 2b)P^m_b(k) - kP^m_b(k - 1), \tag{1}\]

\[P^m_b(k) = (-1)^b P^m_b(m - k), \quad P^m_b(k) = \binom{m}{b} P^m_k(b) \binom{m}{k}^{-1}. \tag{2}\]

By definition we have that \(P^m_b(1) = \frac{m - 2b}{m - k} < 0 < P^m_b(0) = \binom{m}{b}\) and \(P^m_b(k) = P^m_b(m - k)\) since \(b\) is even. Therefore it is enough to prove the statement of the proposition only for \(1 \leq k \leq m/2\).

First assume that \(k\) is at most \(\frac{1 + \eta}{2} m\). Then the equality (1) implies that

\[|P^m_b(k + 1)| = \frac{1}{m - k} |(m - 2b)P^m_b(k) - kP^m_b(k - 1)| \leq \frac{2b - m}{m - k} |P^m_b(k)| + \frac{k}{m - k} |P^m_b(k - 1)| \leq \frac{2b - m + k}{m - k} \max(|P^m_b(k)|, |P^m_b(k - 1)|).\]

Since \(b\) is equal to \(\frac{1 + \eta}{2} m\) and \(k \leq \frac{1 + \eta}{2} m\), it follows that \(\frac{2b - m + k}{m - k} = 2 - \frac{b}{m - k} - 1 \leq 1\). Therefore, arguing by induction on \(k\) we obtain that \(|P^m_b(k + 1)| \leq \max(|P^m_b(1)|, |P^m_b(2)|)\). By definition \(|P^m_b(2)| = |(m - 2b(2 - m)\binom{m}{b_k}| < |P^m_b(1)|\). This proves that \(|P^m_b(1)| \geq |P^m_b(k)|\) for all \(k \leq \frac{1 + \eta}{2} m\).

Next we prove that for \(\frac{1 + \eta}{2} m \leq k \leq m/2\) the value of \(|P^m_b(k)|\) is at most \(o(m^k)\). This would imply that \(|P^m_b(1)| = \frac{2b - m - \eta}{m} \geq |P^m_b(k)|\), since by our assumptions about \(b\), \(\frac{2b - m}{m} = -\eta\) is a constant, bounded away from zero. By the equality (2)

\[\frac{P^m_b(k)}{\binom{m}{b}} = \frac{P^m_k(b)}{\binom{m}{k}} = \sum_{j=0}^{k} (-1)^j \binom{b}{j} \binom{m - b}{k - j} \binom{m}{k}^{-1} = S_1 + S_2,\]

where

\[S_1 = \sum_{r \leq j < q} (-1)^j \binom{b}{j} \binom{m - b}{k - j} \binom{m}{k}^{-1},\]

\(S_2\) contains all the remaining summands, and \(r = (b/m - m^{-1/3})k + O(1)\) as well as \(q = (b/m + m^{-1/3})k + O(1)\) are chosen so that \(S_1\) contains an even number of terms. Note that \(S_1\) is a sum of at most \(O(m^{2/3})\) summands of the form \(t_j = (-1)^j (\binom{m}{k})^{-1} (\binom{b}{j}) (\binom{m-b}{k-j}) - (\binom{b}{j+1}) (\binom{m-b}{k-j-1})\). Therefore to bound \(|S_1|\) it is enough to bound \(|t_j|\) for \(r \leq j < q\). A simple calculation shows that

\[|t_j| = \binom{m}{k}^{-1} \binom{b}{j+1} \binom{m-b}{k-j-1} \frac{(jm-bk) + (m-b-k+2j+1)}{(b-j)(k-j)}.\]

From the assumption about \(j\) we have that \(jm - bk = O(m^{2/3})\) and that \(b - j > k - j \geq k(1 - b/m - m^{-1/3}) - O(1) = \Omega(k) = \Omega(m)\). Thus \(|t_j| = O(m^{-4/3}) \binom{m}{k}^{-1} \binom{b}{j+1} \binom{m-b}{k-j-1} = O(m^{-4}) = O(1)\). Hence we obtain the following inequality

\[|S_1| \leq \sum_{j=r}^{q} |t_j| \leq O(m^{-4}) \binom{m}{k}^{-1} \sum_{j} \binom{b}{j+1} \binom{m-b}{k-j-1} = O(m^{-4}) = o(1).\]
Next we obtain an upper bound on $|S_2|$. Let $t = 2m^{-1/3}$ and $p = b/m$, by definition

$$|S_2| \leq \sum_{j=1}^{(p-t)k} \binom{m}{k}^{-1} \binom{b}{j} \binom{m-b}{k-j} + \sum_{j=(p+t)k}^{k} \binom{m}{k}^{-1} \binom{b}{j} \binom{m-b}{k-j}.$$ 

Note that the right hand side in the last inequality is exactly the probability that a hypergeometric distribution with parameters $(m, b, k)$ deviates by $tk$ from its expectation. By the result of [1], the probability of this event is bounded by $2e^{-2t^2k} = 2e^{-\Omega(m^{1/3})}$. This implies that $|S_2| = o(1)$. Therefore

$$\left| \frac{P^m_b(k)}{\binom{b}{k}} \right| \leq |S_1| + |S_2| = o(1) \leq \frac{2b - m}{m} = \left| \frac{P^m_b(1)}{\binom{b}{1}} \right|.$$ 

It follows that $|P^m_b(1)| \geq |P^m_b(k)|$ for all $1 \leq k \leq m/2$. Since the value of $P^m_b(1)$ is negative, $P^m_b(1) = -|P^m_b(1)| \leq -|P^m_b(k)| \leq P^m_b(k)$. This completes the proof of the proposition. 

\[\Box\]

3 Concluding remarks

- Let $a_{ij}, 1 \leq i < j \leq n$, and $b$ be reals. We call a constraint

$$\sum_{i<j} a_{ij}v_i^tv_j \geq b$$

valid if it is satisfied whenever each $v_i$ is an integer in $\{-1, 1\}$. Feige, Goemans and Williamson (see [4],[5]) proposed adding to the semidefinite program a family of valid constrains, in the hope of narrowing the gap between the optimal value of the semidefinite program and the weight of the max cut. It is easy to see that, as observed by Karloff [7], since the vectors $w_1, \ldots, w_n$ from Section 2 have all their coordinates equal to $\pm 1/\sqrt{m}$ they satisfy any valid constraint. Therefore the proof of Theorem 1.1 shows that the addition of any family of valid constrains cannot improve the performance ratio of the Goemans-Williamson algorithm even for graphs containing large cuts.

- It would be interesting to determine the precise approximation guarantee of the Goemans Williamson algorithm for graphs $G = (V, E)$ in which the size of the max cut is $A|E|$ for all values of $A \geq 1/2$. The discussion in Section 2 determines this value for all $A > 0.845$.

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