Bipartite cuts and judicious partitions

DRAFT

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Abstract

Theorem 1 Let $G = (V, E)$ be a graph with $m$ edges whose maximal bipartite cut has cardinality $c(G) = m/2 + \delta$. If $\delta \leq m/30$, then there exists a partition $V = V_1 \cup V_2$ of the vertex set of $G$ such that

$$e(V_i) \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m}, \quad i = 1, 2.$$ 

Theorem 2 Let $G = (V, E)$ be a graph with $m$ edges whose maximal bipartite cut has cardinality $c(G) = m/2 + \delta$. If $\delta \geq m/30$ and $m$ is large enough, then there exists a partition $V = V_1 \cup V_2$ of the vertex set of $G$ such that

$$e(V_i) \leq \frac{m}{4} - \frac{m}{100}, \quad i = 1, 2.$$ 

Proof of Theorem 1. The main ingredient of the proof is the following lemma.

Lemma 3 Let $G = (V, E)$ be a graph with $m$ edges and with a maximal bipartite cut of cardinality $c(G) = m/2 + \delta$, where $\delta \leq \frac{m}{30}$. Suppose $V = V_1 \cup V_2$ is a partition of $V(G)$ for which $d(v, V_1) \leq d(v, V_2)$ for every vertex $v \in V_1$. If $e(V_1) \geq \frac{m}{4} - \frac{\delta}{2}$, then there exists a vertex $v \in V_1$ such that $d(v, V_1) \leq 3\sqrt{m}$ and $d(v, V_2) \leq \left(1 + \frac{10\delta}{m}\right)d(v, V_1)$.

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Proof. We prove the lemma by showing that the total degree of vertices of \( V_1 \) violating any of the required conditions does not reach the total degree of vertices in \( V_1 \).

Let first \( T_1 = \{ v \in V_1 : d(v, V_1) > 3\sqrt{m} \} \). Observe that as \( d(v, V_1) \leq d(v, V_2) \) for every vertex \( v \in V_2 \), if follows that

\[
2e(V_1) = \sum_{v \in V_1} d(v, V_1) \leq \sum_{v \in V_1} d(v, V_2) = e(V_1, V_2) ,
\]

implying \( e(V_1) \leq m/3 \). Thus \( |T_1| \leq 2e(V_1)/(3\sqrt{m}) \leq 2\sqrt{m}/9 \). Therefore the set \( T_1 \) spans at most \( 2m/81 \) edges. As in the summation \( \sum_{v \in T_1} d(v, V_1) \) the edges spanned by \( T_1 \) are counted twice and every other edge inside \( V_1 \) is counted at most once, we get:

\[
\sum_{v \in T_1} d(v, V_1) \leq e(V_1) + e(T_1) \leq e(V_1) + \frac{2m}{81} . \tag{1}
\]

Define now \( T_2 = \{ v \in V_1 : d(v, V_2) > \left( 1 + \frac{10\delta}{m} \right) d(v, V_1) \} \). Then

\[
e(V_1, V_2) = \sum_{v \in T_2} d(v, V_2) + \sum_{v \in V_1 \setminus T_2} d(v, V_2) > \left( 1 + \frac{10\delta}{m} \right) \sum_{v \in T_2} d(v, V_1) + \sum_{v \in V_1 \setminus T_2} d(v, V_1) \]

\[
= \sum_{v \in V_1} d(v, V_1) + \frac{10\delta}{m} \sum_{v \in T_2} d(v, V_1) = 2e(V_1) + \frac{10\delta}{m} \sum_{v \in T_2} d(v, V_1) ,
\]

implying:

\[
\sum_{v \in T_2} d(v, V_1) < \frac{m}{10\delta} (e(V_1, V_2) - 2e(V_1)) .
\]

Observe that \( e(V_1, V_2) \leq c(G) = \frac{m}{2} + \delta \) and that by the lemma assumption \( e(V_1) \geq \frac{m}{4} - \frac{\delta}{2} \). Hence

\[
\sum_{v \in T_2} d(v, V_1) < \frac{m}{10\delta} \left( \frac{m}{2} + \delta - 2 \left( \frac{m}{4} - \frac{\delta}{2} \right) \right) = \frac{m}{5} . \tag{2}
\]

From (1) and (2) we derive:

\[
\sum_{v \in T_1 \cup T_2} d(v, V_1) < e(V_1) + \frac{2m}{81} + \frac{m}{5} < e(V_1) + 0.23m . \tag{3}
\]

On the other hand, recalling our assumption on \( \delta \), we can see that

\[
\sum_{v \in V_1} d(v, V_1) = 2e(V_1) \geq e(V_1) + \frac{m}{4} - \frac{\delta}{2} \geq e(V_1) + \frac{m}{4} - \frac{m}{60} > e(V_1) + 0.23m . \tag{4}
\]

Comparing (3) and (4) shows that not all vertices of \( V_1 \) are in the union of \( T_1 \) and \( T_2 \). It follows from the definitions of \( T_1 \) and \( T_2 \) that a vertex in \( V_1 \setminus (T_1 \cup T_2) \) meets the requirements of the lemma. \( \square \)

We now prove Theorem 1. Let \( V = U_1 \cup U_2 \) be a partition of \( V \) satisfying \( e(U_1, U_2) = c(G) = \frac{m}{2} + \delta \) and \( e(U_1) \geq e(U_2) \). Clearly for every vertex \( u \in U_1 \), \( d(u, U_1) \leq d(u, U_2) \), as otherwise
moving \( u \) from \( U_1 \) to \( U_2 \) would create a bipartite cut of size larger than \( e(U_1, U_2) = e(G) \). We will achieve a desired partition by starting from \((U_1, U_2)\) and moving a number of vertices from \( U_1 \) to \( U_2 \) in order to balance the number of edges spanned by those subsets. Lemma 3 will help us to maintain the size of the cut almost unchanged. Formally, we start by assigning \( V_1 = U_1, V_2 = U_2 \).

Then, as long as \( e(V_1) \geq \frac{m}{4} - \frac{\delta}{2} + 3\sqrt{m} \), we find a vertex \( v_i \in V_1 \), for which \( d(v_i, V_1) \leq 3\sqrt{m} \) and \( d(v_i, V_2) \leq \left(1 + \frac{10\delta}{m}\right)d(v_i, V_1) \) and transfer it to \( V_2 \). It is easy to see that the conditions of Lemma 3 still apply and therefore such a vertex indeed can be found. We denote \( d(v_i, V_1) = a_i, d(v_i, V_2) = b_i \). Note that \( b_i \leq \left(1 + \frac{10\delta}{m}\right)a_i \).

Let us look at the final partition \((V_1, V_2)\) after the above described process has terminated. Suppose the vertices moved from \( V_1 \) to \( V_2 \) are \( v_1, \ldots, v_t \). Clearly,

\[
e(V_1) < \frac{m}{4} - \frac{\delta}{2} + 3\sqrt{m}.
\]

We now estimate from above the number of edges in \( V_2 \). To this end, denote \( e(U_1) = m_1 \), then \( e(U_2) = m - e(U_1, U_2) - e(U_1) = \frac{m}{2} - \delta - m_1 \). As \( 2e(U_1) \leq e(U_1, U_2) = \frac{m}{2} + \delta \), we get \( m_1 \leq \frac{m}{4} + \frac{\delta}{2} \).

Notice that while moving a vertex \( v_i \) from \( V_1 \) to \( V_2 \) during the process, we deleted \( a_i \) edges from \( V_1 \) and added \( b_i \) edges to \( V_2 \). Therefore for the final partition \((V_1, V_2)\) we get:

\[
e(V_1) = e(U_1) - \sum_{i=1}^{t} a_i = m_1 - \sum_{i=1}^{t} a_i , \tag{6}
\]

\[
e(V_2) = e(U_2) + \sum_{i=1}^{t} b_i = \frac{m}{2} - \delta - m_1 + \sum_{i=1}^{t} b_i \leq \frac{m}{2} - \delta - m_1 + \left(1 + \frac{10\delta}{m}\right) \sum_{i=1}^{t} a_i . \tag{7}
\]

As each time we moved from \( V_1 \) to \( V_2 \) a vertex \( v_i \) with \( d(v_i, V_1) \leq 3\sqrt{m} \), in the final partition \((V_1, V_2)\), \( e(V_1) \geq \frac{m}{4} - \frac{\delta}{2} \). Hence from (6)

\[
\sum_{i=1}^{t} a_i = m_1 - e(V_1) \leq m_1 - \frac{m}{4} + \frac{\delta}{2} .
\]

Therefore it follows from (7) that

\[
e(V_2) \leq \frac{m}{2} - \delta - m_1 + \left(1 + \frac{10\delta}{m}\right) \left(m_1 - \frac{m}{4} + \frac{\delta}{2}\right) \\
= \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta}{m} \left(m_1 - \frac{m}{4} + \frac{\delta}{2}\right) \\
\leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} .
\]

This together with (5) establishes the theorem. \( \square \)

**Proof of Theorem 2.** The proof here is quite similar to that of Theorem 1, with parameters tuned so as to guarantee the error term \( m/100 \).
We claim that the desired partition can be obtained using the following procedure. Start with an optimal bipartite cut $V = U_1 \cup U_2$, for which $e(U_1, U_2) = c(G) = \frac{m}{4} + \delta$ and $e(U_1) \geq e(U_2)$. Initialize $V_1 = U_1$, $U_2 = V_2$, and then, as long as $V_1$ contains a vertex $v_i$ for which
\[
d(v_i, V_1) \leq m/400
\]
and
\[
d(v_i, V_2) \leq \left(1 + \frac{\delta + \frac{m}{100}}{23m/100}\right) d(v_i, V_1),
\]
move $v_i$ to $V_2$.

Let us show first that the algorithm terminates successfully, i.e. reaches the stage where $e(V_1) \leq \frac{m}{4} - \frac{m}{100}$. To do so we need to show that as long as the last condition is not fulfilled a required vertex $v_i \in V_1$, satisfying conditions (8) and (9) exists. Suppose we are at some intermediate stage and the current partition is $(V_1, V_2)$. Define $T_1 = \{v \in V_1 : d(v, V_1) \geq m/400 \}$. Then $|T_1| \leq 2e(V_1)/(m/400) \leq 2m/(3m/400) = 800/3$, and therefore $T$ spans at most $(800/3)^2/2 \leq 36000$ edges. Hence similarly to the proof of Theorem 1,
\[
\sum_{v \in T_1} d(v, V_1) \leq e(V_1) + e(T_1) < e(V_1) + 36000.
\]
Set now
\[
T_2 = \{v \in V_1 : d(v, V_2) > \left(1 + \frac{\delta + \frac{m}{100}}{23m/100}\right) d(v, V_1)\}.
\]
Then again as in the proof of Theorem 1 we get:
\[
\sum_{v \in T_2} d(v, V_1) < \frac{23m/100}{\delta + \frac{m}{50}} (e(V_1, V_2) - 2e(V_1))
\leq \frac{23m/100}{\delta + \frac{m}{50}} \left(\frac{m}{2} + \delta - 2 \left(\frac{m}{4} - \frac{m}{100}\right)\right) = \frac{23m}{100}.
\]
Therefore from (10) and (11) we get
\[
\sum_{v \in T_1 \cup T_2} d(v, V_1) < e(V_1) + 36000 + \frac{23m}{100} < e(V_1) + 0.24m < 2e(V_1)
\]
for sufficiently large $m$, and hence $V_1 \setminus (T_1 \cup T_2) \neq \emptyset$, implying the existence of a vertex with the required properties.

Let us now estimate the number of edges spanned by the final sets $V_1$ and $V_2$. Obviously,
\[
e(V_1) \leq \frac{m}{4} - \frac{m}{100}.
\]
Denote $e(U_1) = m_1$, then $m_1 \leq e(U_1, U_2)/2 = \frac{m}{4} + \frac{\delta}{2}$. Suppose we transferred from $V_1$ to $V_2$ vertices $v_1, \ldots, v_t$, whose degrees (at the time of movement) were $a_i = d(v_i, V_1)$ and $b_i = d(v_i, V_2)$. 

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As in the end $e(V_1) \geq \frac{m}{4} - \frac{m}{100} - \frac{m}{400} = \frac{19m}{80}$, we get:

$$
\sum_{i=1}^{t} a_i \leq m_1 - \frac{19m}{80},
$$

implying:

$$
\sum_{i=1}^{t} b_i \leq \left(1 + \frac{\delta + \frac{m}{50}}{\frac{23m}{100}}\right) \left(m_1 - \frac{19m}{80}\right).
$$

Therefore:

$$
e(V_2) = \frac{m}{2} - \delta - m_1 + \sum_{i=1}^{t} b_i < \frac{m}{2} - \delta - m_1 + \left(1 + \frac{\delta + \frac{m}{50}}{\frac{23m}{100}}\right) \left(m_1 - \frac{19m}{80}\right)
$$

$$
= \frac{21m}{80} - \delta + \frac{(\delta + \frac{m}{50}) \left(m_1 - \frac{19m}{80}\right)}{\frac{23m}{80}}
$$

$$
\leq \frac{21m}{80} - \delta + \frac{(\delta + \frac{m}{50}) \left(\frac{\delta}{2} + \frac{m}{80}\right)}{\frac{23m}{80}}.
$$

We may assume that $\delta \leq \frac{13m}{50}$, as otherwise the initial partition $(U_1, U_2)$ satisfies the theorem requirements. An easy check shows that for every $\delta$ in the interval $[\frac{m}{30}, \frac{13m}{50}]$ the expression in the last display, viewed as a quadratic function of the parameter $\delta$, is strictly less than $0.24m$. This together with (12) completes the proof of Theorem 2. □