Bipartite cuts and judicious partitions

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Abstract

Theorem 1 Let \( G = (V, E) \) be a graph with \( m \) edges whose maximal bipartite cut has cardinality \( c(G) = \frac{m}{2} + \delta \). If \( \delta \leq m/30 \), then there exists a partition \( V = V_1 \cup V_2 \) of the vertex set of \( G \) such that
\[
e(V_i) \leq \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 4\sqrt{m}, \quad i = 1, 2.
\]

Proof of Theorem 1. The main instrument of the proof is the following lemma.

Lemma 2 Let \( G = (V, E) \) be a graph with \( m \) edges and with a maximal bipartite cut of cardinality \( c(G) = \frac{m}{2} + \delta \). Let \( \delta \leq m/30 \). Suppose \( V = V_1 \cup V_2 \) is a partition of \( V(G) \) for which \( d(v, V_1) \leq d(v, V_2) \) for every vertex \( v \in V \). If \( e(V_1) \geq \frac{m}{4} - \frac{\delta}{2} \), then there exists a vertex \( v \in V_1 \) such that \( d(v, V_1) \leq 3\sqrt{m} \) and \( d(v, V_2) \leq \left(1 + \frac{10\delta}{m}\right) d(v, V_1) \).

Proof. We prove the lemma by showing that the total degree of vertices of \( V_1 \) violating any of the required conditions does not reach the total degree of vertices in \( V_1 \).

Let first \( T_1 = \{v \in V_1: d(v, V_1) > 3\sqrt{m}\} \). Observe that as \( d(v, V_1) \leq d(v, V_2) \) for every vertex \( v \in V_2 \), if follows that
\[
2e(V_1) = \sum_{v \in V_1} d(v, V_1) \leq \sum_{v \in V_1} d(v, V_2) = e(V_1, V_2),
\]

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implying $e(V_1) \leq m/3$. Then $|T_1| \leq 2e(V_1)/(3\sqrt{m}) \leq 2\sqrt{m}/9$. Therefore the set $T_1$ spans at most $2m/81$ edges. As in the summation $\sum_{v \in T_1} d(v, V_1)$ the edges spanned by $T_1$ are counted twice and every other edge inside $V_1$ is counted at most once, we get:

$$\sum_{v \in T_1} d(v, V_1) \leq e(V_1) + e(T_1) \leq e(V_1) + \frac{2m}{81} . \quad (1)$$

Let now $T_2 = \{ v \in V_1 : d(v, V_2) > \left(1 + \frac{10\delta}{m}\right)d(v, V_1) \}$. Then

$$e(V_1, V_2) = \sum_{v \in T_2} d(v, V_1) + \sum_{v \in V_1 \setminus T_2} d(v, V_2) > \left(1 + \frac{10\delta}{m}\right)\sum_{v \in T_2} d(v, V_1) + \sum_{v \in V_1 \setminus T_2} d(v, V_1)$$

$$= \sum_{v \in V_1} d(v, V_1) + \frac{10\delta}{m} \sum_{v \in T_2} d(v, V_1) = 2e(V_1) + \frac{10\delta}{m} \sum_{v \in T_2} d(v, V_1) ,$$

implying:

$$\sum_{v \in T_2} d(v, V_1) < \frac{m}{10\delta} (e(V_1, V_2) - 2e(V_1)) .$$

Observe that $e(V_1, V_2) \leq c(G) = \frac{m}{2} + \delta$ and that by the lemma assumption $e(V_1) \geq \frac{m}{4} - \frac{\delta}{2}$. Hence

$$\sum_{v \in T_2} d(v, V_1) < \frac{m}{10\delta} \left(\frac{m}{2} + \delta - 2 \left(\frac{m}{4} - \frac{\delta}{2}\right)\right) = \frac{m^2}{5} . \quad (2)$$

From (1) and (2) we derive:

$$\sum_{v \in T_1 \cup T_2} d(v, V_1) < e(V_1) + \frac{2m}{81} + \frac{m}{5} < e(V_1) + 0.23m . \quad (3)$$

On the other hand, recalling our assumption on $\delta$, we can see that

$$\sum_{v \in V_1} d(v, V_1) = 2e(V_1) \geq e(V_1) + \frac{m}{4} - \frac{\delta}{2} \geq e(V_1) + \frac{m}{4} - \frac{m}{60} > e(V_1) + 0.23m . \quad (4)$$

Comparing (3) and (4) shows that not all vertices of $V_1$ are in the union of $T_1$ and $T_2$. It follows from the definitions of $T_1$ and $T_2$ that a vertex in $V_1 \setminus (T_1 \cup T_2)$ meets the requirements of the lemma. \hfill \Box

We now prove Theorem 1. Let $V = U_1 \cup U_2$ be a partition of $V$ satisfying $e(U_1, U_2) = c(G) = \frac{m}{2} + \delta$ and $e(U_1) \geq e(U_2)$. Clearly for every vertex $u \in U_1$, $d(u, U_1) \leq d(u, U_2)$, as otherwise moving $u$ from $U_1$ to $U_2$ would create a bipartite cut of size larger than $e(U_1, U_2) = c(G)$. We will achieve a desired partition by starting from $(U_1, U_2)$ and moving a number of vertices from $U_1$ to $U_2$ in order to balance the number of edges spanned by those subsets. Lemma 2 will help us to maintain the size of the cut almost unchanged. Formally, we start by assigning $V_1 = U_1$, $V_2 = U_2$. Then, as long as $e(V_1) \geq \frac{m}{4} - \frac{\delta}{2}$, we find a vertex $v_i \in V_1$, for which $d(v_i, V_1) \leq 3\sqrt{m}$.
and \( d(v_i, V_2) \leq \left( 1 + \frac{10\delta}{m} \right) d(v_i, V_1) \) and transfer it to \( V_2 \). It is easy to see that the conditions of Lemma 2 still apply and therefore such a vertex indeed can be found. We denote \( d(v_i, V_1) = a_i \), \( d(v_i, V_2) = b_i \). Note that \( b_i \leq \left( 1 + \frac{10\delta}{m} \right) a_i \).

Let us look at the final partition \((V_1, V_2)\) after the above described process has terminated. Suppose the vertices moved from \( V_1 \) to \( V_2 \) are \( v_1, \ldots, v_t \). Clearly,

\[
e(V_1) < \frac{m}{4} - \frac{\delta}{2}.
\]

We now estimate from above the number of edges in \( V_2 \). To this end, denote \( e(U_1, U_2) = m \), then

\[
e(U_2) = m - e(U_1, U_2) - e(U_1) = \frac{m}{2} - \delta - m_1.
\]

As \( 2e(U_1) \leq e(U_1, U_2) = \frac{m}{2} + \delta \), we get \( m_1 \leq \frac{m}{4} + \frac{\delta}{2} \).

Notice that while moving a vertex \( v_i \) from \( V_1 \) to \( V_2 \) during the process, we deleted \( a_i \) edges from \( V_1 \) and added \( b_i \) edges to \( V_2 \). Therefore for the final partition \((V_1, V_2)\) we get:

\[
e(V_1) = e(U_1) - \sum_{i=1}^{t} a_i = m_1 - \sum_{i=1}^{t} a_i,
\]

\[
e(V_2) = e(U_2) - \sum_{i=1}^{t} b_i = m_1 - \delta - m_1 + \sum_{i=1}^{t} b_i \leq \frac{m}{2} - \delta - m_1 + \left( 1 + \frac{10\delta}{m} \right) \sum_{i=1}^{t} a_i.
\]

As each time we moved from \( V_1 \) to \( V_2 \) a vertex \( v_i \) with \( d(v_i, V_1) \leq 3\sqrt{m} \), in the final partition \((V_1, V_2)\), \(|V_1| \geq \frac{m}{4} - \frac{\delta}{2} - 3\sqrt{m}\). Hence from (6)

\[
\sum_{i=1}^{t} a_i = m_1 - e(V_1) \leq m_1 - \frac{m}{4} + \frac{\delta}{2} + 3\sqrt{m}.
\]

Therefore it follows from (7) that

\[
e(V_2) \leq \frac{m}{2} - \delta - m_1 + \left( 1 + \frac{10\delta}{m} \right) \left( \frac{m}{4} - \frac{m}{4} + \frac{\delta}{2} + 3\sqrt{m} \right)
\]

\[
= \frac{m}{4} - \frac{\delta}{2} + 3\sqrt{m} + \frac{10\delta}{m} \left( \frac{m}{4} - \frac{\delta}{2} + 3\sqrt{m} \right)
\]

\[
\leq \frac{m}{4} - \frac{\delta}{2} + 3\sqrt{m} + \frac{10\delta}{m} \left( \delta + 3\sqrt{m} \right)
\]

\[
= \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 3\sqrt{m} + \frac{30\delta}{\sqrt{m}}
\]

\[
= \frac{m}{4} - \frac{\delta}{2} + \frac{10\delta^2}{m} + 4\sqrt{m}.
\]

This together with (5) establishes the theorem. \( \square \)