Disjoint Simplices and Geometric Hypergraphs

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INTRODUCTION

Let \( A \) be a set of \( 2n \) points in general position in the Euclidean plane \( \mathbb{R}^2 \), and suppose \( n \) of the points are colored red and the remaining \( n \) are colored blue. A celebrated Putnam problem (see [6]) asserts that there are \( n \) pairwise disjoint straight line segments matching the red points to the blue points. To show this, consider the set of all \( n! \) possible matchings and choose one, \( M \), that minimizes the sum of lengths \( l(M) \) of its line segments. It is easy to show that these line segments cannot intersect. Indeed, if the two segments \( v_1, b_1 \) and \( v_2, b_2 \) intersect, where \( v_1, v_2 \) are two red points and \( b_1, b_2 \) are two blue points, the matching \( M' \) obtained from \( M \) by replacing \( v_1, b_1 \) and \( v_2, b_2 \) by \( v_1, b_2 \) and \( v_2, b_1 \) satisfies \( l(M') < l(M) \), contradicting the choice of \( M \). Our first result in this paper is a generalization of this result to higher dimensions.

THEOREM 1. Let \( A \) be a set of \( d \cdot n \) points in general position in \( \mathbb{R}^d \), and let \( A = A_1 \cup A_2 \cup \ldots \cup A_d \) be a partition of \( A \) into \( d \) pairwise disjoint sets, each consisting of \( n \) points. Then there are \( n \) pairwise disjoint \((d - 1)\)-dimensional simplices, each containing precisely one vertex from each \( A_i \), \( 1 \leq i \leq d \).

We prove this theorem in the next section. The proof is short but uses a non-elementary tool: the well-known Borsuk-Ulam theorem.

Combining Theorem 1 with an old result of Erdős from extremal graph theory we obtain a corollary dealing with geometric hypergraphs. A geometric \( d \)-hypergraph is a pair \( G = (V, E) \), where \( V \) is a set of points called vertices, in general position in \( \mathbb{R}^d \), and \( E \) is a set of (closed) \((d - 1)\)-dimensional simplices called edges, whose vertices are points of \( V \). If \( d = 2 \), \( G \) is called a geometric graph. It is well known (see [3], [5]) that every geometric graph with \( n \) vertices and \( n + 1 \) edges contains two disjoint edges, two nonintersecting edges, and this result is the best possible. The number of edges that guarantees \( l \) pairwise disjoint edges is not known for \( l > 2 \), although Perles [7] determined the exact number for the case that the set of vertices
$V$ is the set of vertices of a convex polygon. The situation seems much more difficult for geometric $d$-hypergraphs, when $d > 2$. Even the number of edges that guarantees two disjoint simplices is not known in this case. Clearly this number is greater than $n(d) = \binom{n-1}{d-1}$ (simply take all edges containing a given point) and is at most $\binom{n}{d}$. In the final section we prove the following theorem, that implies that for every fixed $d$, $l \geq 2$, every geometric $d$-hypergraph on $n$ vertices that contains no $l$ pairwise nonintersecting edges has $O(n^l)$ edges.

**Theorem 2:** Every geometric $d$-hypergraph with $n$ vertices and at least $n^{d-1+(d-1)}$ edges contains $1$ pairwise nonintersecting edges.

It is worth noting that the following, much stronger conjecture seems plausible.

**Conjecture:** For every $l, d \geq 2$ there exists a constant $c = c(l, d)$ such that every geometric $d$-hypergraph with $n$ vertices and at least $c \cdot n^{d-1}$ edges contains $l$ pairwise nonintersecting edges.

We do not know how to prove this conjecture, even for $d = 2, l = 3$.

**Proof of Theorem 1**

We need the following lemma, sometimes called the "Ham-Sandwich theorem," which is a well-known consequence of the Borsuk-Ulam theorem (see [1], [2]).

**Lemma 1:** Let $\mu_1, \mu_2, \ldots, \mu_d$ be $d$ continuous probability measures in $\mathbb{R}^d$. Then there exists a hyperplane $H$ in $\mathbb{R}^d$ that bisects each of the $d$ measures, that is, $\mu_i(H^+) = \mu_i(H^-)$ for all $1 \leq i \leq d$, where $H^+$ and $H^-$ denote, respectively, the open positive side and the open negative side of $H$.

Theorem 1 will be derived from the following lemma.

**Lemma 2:** Let $A_1, A_2, \ldots, A_d$ be as in Theorem 1. Then there exists a hyperplane $H$ in $\mathbb{R}^d$ such that

$$|H^+ \cap A_i| = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad |H^- \cap A_i| = \left\lceil \frac{n}{2} \right\rceil$$

for all $1 \leq i \leq d$. \hfill (1)

(Notice that if $n$ is odd (1) implies that $H$ contains precisely one point from each $A_i$.)

**Proof:** Replace each point $p \in A_i$ by a ball of radius $\epsilon$ centered in $p$, where $\epsilon$ is small enough to guarantee that no hyperplane intersects more than $d$ balls. Associate each ball with a uniformly distributed measure of $1/n$. For $1 \leq i \leq d$ and a (Lebesgue)-measurable subset $T$ of $\mathbb{R}^d$, define $\mu_i(T)$ as the total measure of balls centered at a point of $A_i$ captured by $T$. Clearly $\mu_1, \mu_2, \ldots, \mu_d$ are continuous probability measures. By Lemma 1 there exists a hyperplane $H$ in $\mathbb{R}^d$ such that $\mu_i(H^+) = \mu_i(H^-)$ for all $1 \leq i \leq d$. If $n$ is odd, this implies that $H$ intersects at least one ball centered at a point of $A_i$. However, $H$ cannot intersect more than $d$ balls altogether, and thus it intersects precisely one ball centered at a point of $A_i$, and it must bisect these $d$ balls. Hence, for odd $n$, $H$ satisfies (1). If $n$ is even, $H$ intersects at most $d$ balls, and by slightly rotating $H$ we can divide the centers of these balls between $H^+$ and $H^-$ as we wish, without respect to $H$. One can easily satisfy (1). \hfill $\Box$

We can now prove Theorem 1 assuming the result for the $d-1$-dimensional case. Let $H$ be a hyperplane, guaranteeing that $C_i \cap A_i$ for $1 \leq i \leq n$ contains precisely one vertex from each $C_i$. Clearly it lies in $H$.

We thus obtained $2 \cdot \lceil n/2 \rceil$ with the simplex spanned by $A_1, \ldots, A_d$, and we have proved Theorem 1. \hfill $\Box$

**Proof of Theorem 2**

We need the following result.

**Lemma 3 [4]:** Every $d$-uniform hypergraph on $n$ vertices contains a complete $d$-uniform subhypergraph on at least $n \frac{n^{d-1}}{d-1}$ edges.

Now suppose that $G$ is a hypergraph on $n$ vertices. Then $G$ contains a complete $d$-uniform subhypergraph on at least $n \frac{n^{d-1}}{d-1}$ edges. \hfill $\Box$

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constant \( c = c(l, d) \) such that
at least \( c \cdot n^{d-1} \) edges contains \( l \)

for \( d = 2, l = 3 \).

1. Then there exists a hyper-

2. for all \( 1 \leq i \leq d \), \( \beta_i \) is

3. \( 1/n \). For \( 1 \leq i \leq d \) and \( A_i \)

equidistant from every point \( p \) in \( \epsilon \)

4. in \( R^d \) such that \( \mu(H) = 1/2 \)


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\( H^+ \) and \( H^- \) as we wish, without changing the position of each other point of \( A \) with
respect to \( H \). One can easily check that this guarantees the existence of an \( H \)
satisfying (1).

We can now prove Theorem 1 by induction on \( n \). For \( n = 1 \) the result is trivial.
Assuming the result for all \( n, n' > n \), let \( A, A_1, A_2, \ldots, A_d \) be as in Theorem 1 and
let \( H \) be a hyperplane, guaranteed by Lemma 2, satisfying (1). Put \( B_i = H^+ \cap A_i \)
and \( C_i = H^- \cap A_i \), for \( 1 \leq i \leq d \), \( B = B_1 \cup \cdots \cup B_d \) and \( C = C_1 \cup \cdots \cup C_d \).
By applying the induction hypothesis to \( B, B_1, \ldots, B_d \) and \( C, C_1, \ldots, C_d \), we obtain two
sets \( S_1 \) and \( S_2 \) of \([n/2]\) pairwise disjoint simplices each, where each simplex of \( S_1 \)
contains precisely one vertex from each \( B_i \) and each simplex of \( S_2 \) contains precisely
one vertex from each \( C_i \). Clearly, all the simplices in \( S_1 \) lie in \( H^+ \) and all those in \( S_2 \)
lie in \( H^- \).

We thus obtained \( 2 \cdot \lceil n/2 \rceil \) pairwise nonintersecting simplices. These, together
with the simplex spanned by \( A_i \cap H \) if \( n \) is odd, complete the induction and the proof of Theorem 1.

PROOF OF THEOREM 2

We need the following result of Erdös.

Lemma 3 (4): Every \( d \)-uniform hypergraph with \( n \) vertices and at least \( n^{d-(1/2)} \)
edges contains a complete \( d \)-partite subhypergraph on \( d \) classes of \( d \) vertices each.

Now suppose that \( G \) is a geometric \( d \)-hypergraph with \( n \) vertices and at least \( n^{d-(1/2)} \)
edges. By Lemma 3 there is a set \( A \) of \( d \) vertices of \( G \), \( A = A_1 \cup \cdots \cup A_d \),
where \( |A_i| = l \) for each \( i \), and all the \( d \)-simplices consisting of one
vertex from each \( A_i \) are edges of \( G \). The assertion of Theorem 2 now follows from
Theorem 1.