Colorings with a small number of monochromatic progressions
(Comment, Draft)

1 The result

Theorem 1.1 There exists an absolute constant $c > 0$, so that for every integer $k$ and every integer $n$, there is a coloring of $[n] = \{1, 2, \ldots, n\}$ by $k$ colors so that the number of monochromatic 3-term arithmetic progressions (=3-APs, for short) is at most $\frac{n^2}{k\log k}$.

Note that it suffices to prove the above theorem for large $k$, say $k > 2^{100}$, as by choosing a sufficiently small $c$, the assertion for small values of $k$ is trivial. Thus we will assume that $k$ is large. All logarithms in this note are in the natural base $e$, and we omit all floor and ceiling signs whenever these are not crucial.

2 The proof

By a well known construction of Behrend [1], there exists an absolute positive constant $a$ so that for every integer $K$ there is a subset $S \subset [K]$ of size at least $\frac{K}{e^{a\sqrt{\log K}}}$ which contains no 3-AP. Put $K = e^{c\log^2 k} = k^{c\log k}$, where $c = \frac{1}{4a^2}$, and assume $k$ is sufficiently large so that the inequality

$$\frac{\sqrt{K}}{4} > c\log^2 k = \frac{1}{4a^2} \log^2 k$$

holds. Let $S$ be a subset of $[K]$ of size at least $\frac{K}{e^{a\sqrt{\log K}}} = \frac{K}{\sqrt{k}}$ which contains no 3-AP.

Lemma 2.1 There is a coloring $g$ of $[K]$ with $k/2$ colors $\{1, 2, \ldots, k/2\}$ and no monochromatic 3-AP.

Proof: For each $i$, $1 \leq i \leq k/2$, let $x_i$ be a random integer chosen uniformly in the open interval $(-K, K)$, where all choices are independent. Clearly, the set $x_i + S$ contains no 3-AP. Fix an integer $j \in [K]$; the probability that $j \in x_i + S$ is precisely $\frac{|S|}{2\sqrt{k}} \geq \frac{1}{2\sqrt{k}}$, as there are $|S|$ choices of $x_i$ for which $j$ lies in $x_i + S$. It follows that the probability that $j$ is not covered by any of the $k/2$ sets $x_i + S$ is at most

$$\left(1 - \frac{1}{2\sqrt{k}}\right)^{k/2} \leq e^{-\sqrt{k}/4} < e^{-\frac{1}{4a^2} \log^2 k} = \frac{1}{K},$$

1
where here we used (1) and the definition of $K$. Thus, the expected number of members of $[K]$ which are not covered by the union of all sets $x_i + S$ is less than 1, and in particular there is a choice of $k/2$ integers $x_i$ so that $[K]$ is a subset of their union. To complete the proof of the lemma, fix one such choice, and define, for each $j \in [K]$, the color $g(j)$ of $j$ to be the smallest $i$ such that $j \in x_i + S$. \[ \Box \]

Returning to the proof of Theorem 1.1, observe, first, that if $n$ is smaller than $2K$ then by Lemma 2.1 there is a coloring of $[n]$ by at most $k$ colors with no monochromatic 3-AP at all, implying the assertion of the theorem in this case. We can thus assume $n$ is larger. Split the integers in $[n]$ into $2K$ intervals $I_1, I_2, \ldots, I_{2K}$ of consecutive integers, each of length $\frac{n}{2K}$. Define a coloring $f : [n] \mapsto \{1, 2, \ldots, k\}$, using the coloring $g$ provided by Lemma 2.1 as follows. For each $j \in [n]$, if $j \in I_{2i}$ for some $i$, $1 \leq i \leq K$, then $f(j) = g(i)$, whereas if $j \in I_{2i-1}$ for some $1 \leq i \leq k$, then $f(j) = g(i) + k/2$.

We claim that the only monochromatic 3-APs in this coloring consist of 3 elements from the same interval $I_i$. Indeed, suppose this is not the case, and consider a monochromatic 3-AP in which the smallest and largest elements are in two different intervals $I_i$ and $I_j$, with $i < j$. Obviously $i$ and $j$ have the same parity, since otherwise the colors of the corresponding elements are distinct (as odd numbered intervals and even numbered intervals are colored by disjoint sets of colors). Assume both $i$ and $j$ are even (the odd case is analogous). Then $i = 2i'$, $j = 2j'$ and the middle element of the 3-AP lies in the interval $I_{i'} + j'$. If $i' + j'$ is odd, then the color of the middle element differs from that of the two other ones, and this is impossible. Thus we may assume that $i' + j'$ is even and then the colors of the three elements of the 3-AP are $g(i'), g((i' + j')/2)$ and $g(j')$, and these cannot all be equal as there is no monochromatic 3-AP in the coloring $g$.

Therefore the three members of each monochromatic 3-AP in the coloring $f$ lie in the same interval $I_i$, and the number of such 3-APs is less than

$$2K \frac{n^2}{16K^2} < n^2 \frac{n^2}{K} = \frac{n^2}{k \log k},$$

completing the proof of the theorem. \[ \Box \]

References