Semantic labeling meets dependency pairs

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Zantema [Zan95] devised the method of semantic labeling in which a given term rewriting system $(\mathcal{F}, \mathcal{R})$ is transformed into a labeled TRS $(\mathcal{F}_{lab}, \mathcal{R}_{lab})$ by labeling the function symbols. The labeling is always based on a quasi-model of $\mathcal{R}$ and it turns out that $\mathcal{R}$ is terminating if and only if $\mathcal{R}_{lab}$ is terminating. Semantic labeling is an interesting technique for proving termination because termination of $\mathcal{R}_{lab}$ is often much easier to prove than termination of $\mathcal{R}$; for example, a recursive path ordering might be applicable to $\mathcal{R}_{lab}$ but not to $\mathcal{R}$. The main result in [Zan95] reads as follows.

Theorem 1.1 A TRS $(\mathcal{F}, \mathcal{R})$ is terminating if and only if there exists a non-empty quasi-model $(\mathcal{M}, \succsim)$ of $(\mathcal{F}, \mathcal{R})$ and a labeling (L, lab) for $\mathcal{M}$ such that the TRS $(\mathcal{F}_{lab}, \mathcal{R}_{lab} \cup dec(\mathcal{F}_{lab}))$ is terminating, where

$$dec(\mathcal{F}_{lab}) = \bigcup_{f \in \mathcal{F}^{(n)}, n \in \mathbb{N}} \{f_a(x_1, \ldots, x_n) \to f_b(x_1, \ldots, x_n) \mid a >_f b\}.$$

However, Zantema [Zan95] remarks that “the technique of semantic labelling is hard to automate”. In contrast to that, the dependency pair method of Arts and Giesl [AG00, AG98] can be automated. This fact constitutes the main advantage of the dependency pair method over semantic labeling. In the dependency pair approach a set IN of inequalities, each of the form $s \geq t$ or $s > t$, is generated and the existence of a quasi-ordering $\succsim$ satisfying these inequalities is sufficient for showing termination. More precisely, Arts and Giesl [AG98] showed the following theorem (note that here we use the stable-strict ordering corresponding to a quasi-ordering instead of its strict part; see [GO00]).

Theorem 1.2 A TRS $\mathcal{R}$ is terminating if and only if for each cycle $\mathcal{P}$ in the estimated dependency graph EDG($\mathcal{R}$) of $\mathcal{R}$, there is an argument filtering system (AFS) $\mathcal{A}$ and a quasi-reduction ordering $\succsim$ on $\mathcal{T}(\mathcal{F}', \mathcal{V})$ such that

- $l \downarrow_{\mathcal{A}} \succsim r \downarrow_{\mathcal{A}}$ for all rules $l \to r$ from $\mathcal{R}$,
- $s \downarrow_{\mathcal{A}} \succsim t \downarrow_{\mathcal{A}}$ for all dependency pairs $\langle s, t \rangle$ from $\mathcal{P}$,

- $s\downarrow_{\mathcal{A}} \succ t\downarrow_{\mathcal{A}}$ for at least one dependency pair $\langle s,t\rangle$ from $\mathcal{P}$,

where $\mathcal{F}'$ consists of all function symbols occurring in the inequalities.

The dependency pair method and semantic labeling can quite naturally be combined as we will show. We consider only finite TRSs over finite signatures and assume that every signature as well as every algebra is non-empty. Given a TRS $\mathcal{R}$, we apply the dependency pair method in order to show termination. Suppose for a cycle $\mathcal{P}$ in EDG($\mathcal{R}$) we have found an AFS $\mathcal{A}$ and a well-founded weakly monotone $\mathcal{F}$-algebra $(\mathcal{M},\succsim)$ such that

- $l\downarrow_{\mathcal{A}} \succsim_{\mathcal{M}} r\downarrow_{\mathcal{A}}$ for all rules $l\to r$ from $\mathcal{R}$,
- $s\downarrow_{\mathcal{A}} \succsim_{\mathcal{M}} t\downarrow_{\mathcal{A}}$ for all dependency pairs $\langle s,t\rangle$ from $\mathcal{P}$,

but there is no dependency pair $\langle s,t\rangle$ in $\mathcal{P}$ with $s\downarrow_{\mathcal{A}} \succ_{\mathcal{M}} t\downarrow_{\mathcal{A}}$ (if there were a dependency pair $\langle s,t\rangle\in\mathcal{P}$ with $s\downarrow_{\mathcal{A}} \succ_{\mathcal{M}} t\downarrow_{\mathcal{A}}$, then $\succsim_{\mathcal{M}}$ would be a quasi-reduction ordering satisfying the inequalities and we are done). Since we work with the stable-strict relation corresponding to $\succsim_{\mathcal{M}}$, this means that for every dependency pair $\langle s,t\rangle\in\mathcal{P}$ there is a ground substitution $\sigma\colon\mathcal{V}\to\mathcal{T}(\mathcal{F})$ with $[t\downarrow_{\mathcal{A}}\sigma]_{\mathcal{M}} \succsim [s\downarrow_{\mathcal{A}}\sigma]_{\mathcal{M}}$. Therefore, we have to take care of all assignments $\mu\colon\mathcal{V}\to M$ with $[\mu]_{\mathcal{M}}(s\downarrow_{\mathcal{A}}) \approx [\mu]_{\mathcal{M}}(t\downarrow_{\mathcal{A}})$, where $\approx$ denotes the equivalence relation on M induced by $\succsim$. In this situation we try to use semantic labeling w.r.t. the labeling induced by the quasi-model $(\mathcal{M},\succsim)$. So in contrast to Theorem 1.1, $(\mathcal{M},\succsim)$ is assumed to be well-founded. Moreover, since the quasi-model is fixed, the method is amenable to automation.

Definition 1.3 A weakly monotone $\mathcal{F}$-algebra $(\mathcal{M},\succsim)$ is a *quasi-model* of a set of inequalities IN if $s\succsim_{\mathcal{M}} t$ for every $s>t$ and every $s\geq t$ in IN.

Given a quasi-model $(\mathcal{M},\succsim)$, we label the function symbols as follows. For every $f\in\mathcal{F}^{(n)}$, choose either $L_f = M/{\approx}$ or $L_f=\emptyset$. That is, function symbols are labeled by equivalence classes from $M/{\approx}$. More precisely, in case $L_f\neq\emptyset$, define $lab_f(m_1,\ldots,m_n)=\mathbf{m}$, where $\mathbf{m}$ denotes the equivalence class of $f_{\mathcal{M}}(m_1,\ldots,m_n)$. Thus, for every assignment $\mu\colon\mathcal{V}\to M$, the mapping $lab_\mu:\mathcal{T}(\mathcal{F},\mathcal{V})\to\mathcal{T}(\mathcal{F}_{lab},\mathcal{V})$ is defined as

$$lab_\mu(t)=\begin{cases} t & \text{if } t\in\mathcal{V},\\ f(lab_\mu(t_1),\ldots,lab_\mu(t_n)) & \text{if } t=f(t_1,\ldots,t_n) \text{ and } L_f=\emptyset,\\ f_{\mathbf{m}}(lab_\mu(t_1),\ldots,lab_\mu(t_n)) & \text{if } t=f(t_1,\ldots,t_n) \text{ and } L_f\neq\emptyset,\end{cases}$$

where $\mathbf{m}$ denotes the equivalence class of $[\mu]_{\mathcal{M}}(t)$.

Our main result is the following theorem.

Theorem 1.4 Let IN be a set of inequalities of the form $s>t$ or $s\geq t$, where $s,t\in\mathcal{T}(\mathcal{F},\mathcal{V})$. There exists a quasi-reduction ordering on $\mathcal{T}(\mathcal{F},\mathcal{V})$ satisfying the inequalities in IN if and only if there is a well-founded quasi-model $(\mathcal{M},\succsim)$ of IN, and a quasi-reduction ordering $\succsim_{lab}$ on $\mathcal{T}(\mathcal{F}_{lab},\mathcal{V})$ satisfying the inequalities in IN_{lab}, where

$$\begin{aligned}\text{IN}_{lab} &= \{lab_\mu(s)>lab_\mu(t)\mid s>t\in\text{IN},\mu\colon\mathcal{V}\to M,[\mu]_{\mathcal{M}}(s)\approx[\mu]_{\mathcal{M}}(t)\}\\ &\cup\ \{lab_\mu(s)\geq lab_\mu(t)\mid s\geq t\in\text{IN},\mu\colon\mathcal{V}\to M,[\mu]_{\mathcal{M}}(s)\approx[\mu]_{\mathcal{M}}(t)\}\end{aligned}$$

Corollary 1.5 A TRS $\mathcal{R}$ is terminating if and only if for each cycle $\mathcal{P}$ in the estimated dependency graph of $\mathcal{R}$ there is an AFS $\mathcal{A}$ and a well-founded weakly monotone $\mathcal{F}'$-algebra $(\mathcal{M}, \succsim)$ satisfying

- $l \downarrow_{\mathcal{A}} \succsim_{\mathcal{M}} r \downarrow_{\mathcal{A}}$ for all rules $l \to r$ from $\mathcal{R}$,
- $s \downarrow_{\mathcal{A}} \succsim_{\mathcal{M}} t \downarrow_{\mathcal{A}}$ for all dependency pairs $\langle s, t \rangle$ from $\mathcal{P}$,

where $\mathcal{F}'$ consists of all function symbols occurring in the inequalities, and a quasi-reduction ordering $\succsim_{lab}$ on $\mathcal{T}(\mathcal{F}'_{lab}, \mathcal{V})$ satisfying

- $lab_\mu(l \downarrow_{\mathcal{A}}) \succsim_{lab} lab_\mu(r \downarrow_{\mathcal{A}})$ for every rule $l \to r \in \mathcal{R}$ and every assignment $\mu \colon \mathcal{V} \to M$ with $[\mu]_{\mathcal{M}}(l \downarrow_{\mathcal{A}}) \approx [\mu]_{\mathcal{M}}(r \downarrow_{\mathcal{A}})$,
- $lab_\mu(s \downarrow_{\mathcal{A}}) \succsim_{lab} lab_\mu(t \downarrow_{\mathcal{A}})$ for every dependency pair $\langle s, t \rangle$ from $\mathcal{P}$ and every assignment $\mu \colon \mathcal{V} \to M$ with $[\mu]_{\mathcal{M}}(s \downarrow_{\mathcal{A}}) \approx [\mu]_{\mathcal{M}}(t \downarrow_{\mathcal{A}})$,
- $lab_\mu(s \downarrow_{\mathcal{A}}) \succ_{lab} lab_\mu(t \downarrow_{\mathcal{A}})$ for at least one dependency pair $\langle s, t \rangle$ from $\mathcal{P}$ and every assignment $\mu \colon \mathcal{V} \to M$ with $[\mu]_{\mathcal{M}}(s \downarrow_{\mathcal{A}}) \approx [\mu]_{\mathcal{M}}(t \downarrow_{\mathcal{A}})$.

Proof: Direct consequence of Theorems 1.2 and 1.4.

By means of the preceding corollary, we are able to show termination of the TRS

$$\begin{array}{lcl} f(\lambda(x), y) & \to & \lambda(f(x, g(1, f(y, 2)))) \\ f(g(x, y), z) & \to & g(f(x, z), f(y, z)) \\ f(\lambda(x), y) & \to & f(x, \lambda(c(y))) \\ f(x, c(y)) & \to & f(c(x), y) \end{array}$$

which contains the first two rules of the system σ_0 and two new rules. The system σ_0 describes the process of substitution in combinatory categorical logic; cf. [Zan95].

References

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