Relating termination and infinitary normalization*

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Lazy languages admit giving *infinite values* as the meaning of expressions. Infinite values are limits of converging infinite sequences of *partially defined* values which are more and more defined. This can be formalized by using *strongly convergent infinitary rewrite sequences* [KKSV95]. These are Cauchy convergent rewrite sequences in which redexes are (ultimately) contracted deeper and deeper. We only consider sequences of length at most ω issued from finite terms. The existence of a strongly convergent rewrite sequence leading from a term to another is undecidable. However, we have the following.

Theorem 1. *Let $\mathcal{R}$ be a confluent, left-linear, right ground TRS over a finite signature, $t \in \mathcal{T}(\Sigma)$, and δ be a constructor (possibly infinite) term. It is decidable whether t reduces to δ.*

A TRS is *top-terminating* if no infinitary reduction sequence performs infinitely many rewrites at Λ [DKP91]. A TRS $\mathcal{R}$ is *strongly convergent* if all infinitary reductions in $\mathcal{R}$ are strongly convergent [GOV99]. Giesl et al. showed that *dependency pairs* [AG00] can be used for proving strong convergency of TRSs (without collapsing rules) [GOV99]. Kennaway et al. remarked that *top-termination* also ensures strong convergence of TRSs; this was implicit in the proof of Proposition 5.1 in [DKP91]. In fact, both notions coincide.

Theorem 2. *A TRS is top-terminating if and only if is strongly convergent.*

A TRS is *infinitary normalizing* if every (finite) term t admits a strongly convergent sequence starting from t and ending into a normal form (i.e., a term without redexes). Infinitary normalizing TRSs can be thought of as playing the role of normalizing TRSs in the infinitary setting, i.e., ensuring that every term has a meaning. The following fact easily follows from[1] [DKP91].

Theorem 3. *Every left-linear, top-terminating TRS is infinitary normalizing.*

Infinitary normalizing TRSs do not need to be top-terminating: Consider the TRS $\mathcal{R}$:

$$\mathtt{f(a)} \to \mathtt{f(f(a))} \qquad\qquad \mathtt{f(a)} \to \mathtt{a}$$

It is not difficult to see that $\mathcal{R}$ is infinitary normalizing; however, the infinite rewrite sequence

$$\underline{\mathtt{f(a)}} \to \mathtt{f(}\underline{\mathtt{f(a)}}\mathtt{)} \to \underline{\mathtt{f(a)}} \to \cdots$$

performs infinitely many reductions at the top position.

We show that it is possible to prove top-termination of a left-linear TRS $\mathcal{R}$ by proving *termination* of the *canonical* context-sensitive rewrite relation associated to $\mathcal{R}$. In context-sensitive rewriting [Luc98], we (only) rewrite subterms at *replacing* arguments; a *replacement*

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[1] But note that the notion of infinitary normal form used in [DKP91] (a term t such that $t = t'$ whenever $t \to t'$) differs from the usual one.

map $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$ discriminates the argument positions $\mu(f) \subseteq \{1, \ldots, ar(f)\}$, on which we *can* perform replacements. This is extended to positions of terms inductively. If the context-sensitive rewrite relation associated to a TRS $\mathcal{R}$ and a replacement map μ is terminating, we say that $\mathcal{R}$ is μ-terminating. A replacement map μ is *more restrictive* than μ', written $\mu \sqsubseteq \mu'$ if $\forall f \in \Sigma, \mu(f) \subseteq \mu'(f)$; $\mu_\top$ given by $\mu_\top(f) = \{1, \ldots, ar(f)\}$ for all $f \in \Sigma$ is the less restrictive map, imposing no restriction. Given a TRS $\mathcal{R}$, its *canonical* replacement map $\mu^{can}_{\mathcal{R}}$ is the most restrictive replacement map that permits replacements at every non-variable position of the *lhs*'s of $\mathcal{R}$. We have the following.

Theorem 4. *If a left-linear TRS $\mathcal{R}$ is $\mu^{can}_{\mathcal{R}}$-terminating, then $\mathcal{R}$ is top-terminating.*

Strongly convergent TRSs do not need to be $\mu^{can}_{\mathcal{R}}$-terminating: Consider the TRS $\mathcal{R}$:

$\mathtt{f(a)} \to \mathtt{f(f(a))}$

It is not difficult to see that $\mathcal{R}$ is strongly convergent. However, it is not $\mu^{can}_{\mathcal{R}}$-terminating, since $\mu^{can}_{\mathcal{R}} = \mu_\top$ and $\mathcal{R}$ is not terminating.

Since μ'-terminating TRSs are also μ-terminating whenever $\mu \sqsubseteq \mu'$ (see [Luc96]), Theorem 4 can trivially be extended to replacement maps μ such that $\mu^{can}_{\mathcal{R}} \sqsubseteq \mu$. However, Theorem 4 does not necessarily hold if $\mu^{can}_{\mathcal{R}} \not\sqsubseteq \mu$: consider our first example; if $\mu(\mathtt{f}) = \varnothing$, then $\mathcal{R}$ is μ-terminating but it is not top-terminating. In this way, μ-termination criteria [GM99,Luc96,SX98,Zan97] can also be used for proving infinitary normalization. The advantage w.r.t. [GOV99] is that we can use available software and techniques for proving termination by just pre-processing the original TRS by using the transformations of [GM99,Luc96,SX98,Zan97].

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