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The Theory of Well-Quasi-Ordering: A Frequently Discovered Concept

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Results from the rich and well-developed theory of well-quasi-ordering have often been rediscovered and republished. The purpose of this paper is to describe this intriguing subject. To illustrate the theory, here are two definitions and an elementary result. A partially ordered set is called well-partially-ordered if every subset has at least one, but only a finite number, of minimal elements. For sequences s and t, we define $s \leqslant t$ if some subsequence of t majorizes s term by term. Then the space of all finite sequences over a well-partially-ordered set is itself well-partially-ordered.

INTRODUCTION

Results from the rich and well-developed theory of well-quasi-ordering have been rediscovered and published many times over the last 20 years (most recently by Haines [6] in this journal). The purpose of this paper is to describe this intriguing subject, so that future work will not needlessly repeat what is known, but can push forward into new areas.

To illustrate the theory, here are two basic definitions and an elementary result (which is stronger than Haines' theorem). A partially ordered set is called well-partially-ordered if every subset has at least one, but no more than a finite number, of minimal elements. For sequences s and t, we define $s \leqslant t$ if some subsequence of t majorizes s term by term. Then according to the Finite Sequences Theorem, the space $\mathscr{F}(X)$, of all finite sequences over a well-partially-ordered set X, is itself well-partially-ordered. (To get Haines' result, specialize X to be finite set, with the partial order defined by equality).

To indicate how highly developed the theory is, we briefly summarize one line of development here. (For details, see below. Note that, except for technical details, well-quasi-ordered is the same as well-partially-ordered.) In 1954 Richard Rado [29] showed that transfinite sequences, over a finite set, of length ω^2, are well-quasi-ordered (wqo). In 1959 Erdös and Rado [3] extended this result to somewhat longer sequences, and in 1965 Crispin Nash-Williams [25] proved it for sequences of any length. In 1968 E. C. Milner [20] proved that transfinite sequences of a certain length over a well-ordered set are wqo. Later in 1968 Nash-Williams [27], using a concept he invented in 1965 [24], namely, "better-quasi-ordered" (bqo), and has proved a result which subsumes all those described above: transfinite sequences (of any length) over a bqo set are bqo. In 1969 Richard Laver [16, 17] made a considerable further extension to generalized transfinite "sequences" in which the index set is not well-ordered but is instead a countable union of scattered order types. And this is not the only direction in which the theory has moved.

Background

A quasi-ordering (qo) means any reflexive, transitive relation. If, in addition, $x \leqslant y \leqslant x$ implies $x = y$, then we have a partial-ordering (po). At the casual level it is easier to work with po than qo, but in advanced work the reverse is true. Any qo generates a po on the equivalence classes ($x \equiv y$ if $x \leqslant y \leqslant x$). By means of this connection, all definitions and theorems for one domain can be translated into the other domain. When translating from po to qo it is necessary to add the word "non-equivalent" at various places, and to interpret $<$ as implying non-equivalence. The following definitions of wqo all serve to define well-partial-order (wpo) as well, if we assume po to start with.

A wqo is a qo in which (a) every strictly descending sequence is finite, and (b) every set of pairwise incomparable elements is finite. An equivalent definition of wqo, whose analogy with the concept of well-order justifies the name used, is a qo in which every non-empty set has at least one but no more than a finite number of (non-equivalent) minimal elements. Another equivalent definition is a qo in which every infinite sequence has an infinite ascending subsequence. Still other elegant definitions have been given.

Many properties of wqo may be easily proved, such as: (1) if X wqo, then any subset of X and any homomorphic image of X is wqo; (2) wqo is preserved under finite unions and finite Cartesian products; and (3) X well-ordered implies X wqo.

We shall use $\mathscr{S}(X)$, the class of all sequences, (both finite and transfinite), of elements from X. If X is qo, define a relation on $\mathscr{S}(X)$ as follows: $s \leqslant t$ if there is a subsequence of t which majorizes s term by term. This relation is easily proved to be a qo. (If the ordering on X is a po, the relation on $\mathscr{S}(X)$ need *not* be a po, however. It is considerations like this which make qo more convenient to work with than po.) Note that $\mathscr{S}(X)$ directly extends $\mathscr{F}(X)$ with the same qo.

We shall also use $\mathscr{P}(X)$, the class of all subsets of X, with this qo: $X_1 \leqslant X_2$ if for every x_1 in X_1 there is an x_2 in X_2 such that $x_1 \leqslant x_2$.

As a direct generalization of $\mathscr{S}$, $\mathscr{F}$, $\mathscr{P}$, and many other operators which have appeared in the literature, suppose $\mathscr{U} = (U)$ is a class of sets. (Sometimes each U may be assumed to be qo or have some other structure.) Suppose that certain functions f from U into a qo set are specified as *acceptable*. Then $\mathscr{U}(X)$ consists of all acceptable functions $f\colon U \to X$ for all U in $\mathscr{U}$. Among conditions which have been used to specify acceptability are: (a) f is 1-to-1, (b) f is finitary (that is, the range of f contains only a finite number of non-equivalent elements), (c) f is a homomorphism (that is, f preserves $\leqslant$), and (d) the null condition (that is, all functions are acceptable).

To define the order on $\mathscr{U}(X)$, suppose certain mappings between sets in $\mathscr{U}$ are specified as *proper*. Suppose that the composition of two proper mappings is always proper, and that the identity map on every U is proper. Then it is easy to see that the following definition yields a qo:

$$f_1 : U_1 \to X \leqslant f_2 : U_2 \to X$$

if there is a proper map $p : U_1 \to U_2$ such that $f_1(u) \leqslant f_2(p(u))$ for all u in U_1. Most commonly, proper maps are all isomorphic embeddings: sometimes they are all mappings, or all homomorphisms.

To realize $\mathscr{S}$ and $\mathscr{F}$ in this framework, we let $\mathscr{S}$ be a class of well-ordered sets containing at least one of every ordinality, and we let $\mathscr{F}$ be a similar class of finite well-ordered sets. We let all functions be acceptable, and we let isomorphic embeddings be proper. To realize $\mathscr{P}$, we let $\mathscr{P}$ be a class of (unstructured) sets having one of every cardinality. We let all 1-to-1 functions be acceptable, and we let all functions be proper.

History

A forerunner to the concept of well-quasi-ordering was invented by Georges Kurepa [14] in 1937. Writing in French, he defined a set to be "partiellement bien ordonné" (partially well ordered) if every linearly ordered subset is well-ordered. Though this poses no restriction on the size of a set of pairwise incomparable elements, as does well-partial-ordering, the size of such sets explicitly enters into his paper. Thus the relationship is close, even though Kurepa never uses quite the present concept.

Another forerunner to the concept of wqo occurs in a conjecture made by Andrew Vazsonyi, in the 1940's I believe, and spread by Paul Erdös. Vazsonyi conjectured that any infinite collection of finite trees must contain some pair of trees such that one is homeomorphically embeddable in the other. In view of the fact that a strictly descending sequence of trees must trivially be finite, this conjecture is easily equivalent to stating that the qo of finite trees by homeomorphic embedding is a wqo. Since Vazsonyi never published his ideas, it is not clear how close he approached to the concept of wqo. Vazsonyi's conjecture was subsequently proved by Kruskal [13] in 1954, and proved much more elegantly by Nash-Williams [21] in 1963. Independently, in 1960, a proof of the same theorem was announced by Tarkowski [30].

Still another forerunner to the concept of wqo occurs in a problem proposed by Paul Erdös [2] in 1949. He starts with an infinite ascending sequence of positive interges. He supposes that any infinite subsequence must contain two integers such that one divides the other. His problem was to prove that the set of all (multiple) products (that is, the multiplicative closure) of the given sequence has the same property as the original sequence. Using the qo based on divisibility, tthis comes to assuming that the given set of integers is wqo, and proving that the set of all products is also. (He indicates that this generalizes a well-known result by Dickson.)

The first clear use of wqo appeared in two simultaneous though independent articles in 1952. Graham Higman [7], who calls it "the finite basis property," gave the first significant development of the theory and obtained a proof of Erdös's problem. However, his paper places greater emphasis on exploiting the "finite basis property" to make a very nice generalization and simplification of work by Neumann [28] and others which used formal "power series" to embed group-algebras and similar structures in division rings. Incidentally, Higman refers to an unpublished manuscript of Erdös and Rado which was probably an early version of [29] or of [4].

Simultaneously with Higman, Erdös and Rado [4] published a solution to Erdös's problem. In a final "note" they introduce the phrase "partially well ordered" and state without proof a result which says essentially that finite *sets* from a wpo set are wpo. They state that Higman and B. H. Neumann, independently of each other and of themselves, proved essentially that finite *sequences* from a wpo set are wpo.

The next clear use of wqo is given in 1954 by Rado [29], who uses the phrase "partial well-ordering." His main result, quoted above, was that finitary sequences of length $\leqslant \omega^2$ over a wqo set are wqo.

Another independent invention of wqo was made by Kruskal [11] in 1954. His stimulus was Vazsonyi's conjecture, which he heard from Erdös.

At a critical point in his development of a theory of wqo, a second chance meeting with Erdös led to his learning of Higman's paper. This could be reinterpreted to yield a limited form of Vazsonyi's conjecture, namely, the case in which the finite trees are limited to having degree $\leqslant$ some fixed bound. Higman's ideas provided the final clue needed to solve Vazsonyi's conjecture in [13].

Still another independent invention of wqo was made in 1960 by Ernest Michael [19], who used the phrase "fairly well-ordered." He proves some elementary properties, and refers to a minor topological application of his Proposition 1, part (4) by J. Ceder in Lemma 8.5 on page 121 of [1].

The last independent invention of wqo occurs in 1969 in the paper by Leonard Haines [6], who uses no phrase but simply refers to the fact that a set of pairwise incomparable objects must be finite. The objects for which the proves this are finite sequences over a finite set.

Finally we mention some other published work which is related to wqo. In 1967, E. S. Wolk [31] defined the "partial ordinal" of a wpo set P to be the class of all wpo sets which are isomorphic to P. He developed a theory of partial ordinals, using the relation induced on them by the following relation on wpo sets: $P \leqslant Q$ if P is isomorphic to a lower set in Q. (A lower set is defined to be a set such that q_2 in the set and $q_1 \leqslant q_2$ implies q_1 in the set.)

In 1967, writing in French, Pierre Jullien [9] generalized the concept of an ordinal slightly to what he called a "surordinal." He announced several results, among which is the statement that the class of "surordinaux" are wqo. The relation he uses differs from that of Wolk. (He uses "préorder" for qo, and introduces "prébelordre" for wqo.) He motivated these results by quoting Roland Fraïssé [5] as making several conjectures about linear order types which are "disperseés" (English, "scattered"). Two of these conjectures together, as he points out, come to the assertion that scattered order types are wqo. However the conjectures by Fraïssé must be signficantly altered to make them match the quotations by Jullien. In 1968, Jullien [10] announced some results on the po set of words from a finite alphabet.

Further papers by authors already mentioned, particularly Nash-Williams, appear in the references.

Present State

The theory of wqo has a few culminating results of primary importance, though they do not of course subsume the whole subject. To state them, however, necessarily involves a somewhat complicated but very helpful and ingenious concept due to Nash-Williams, namely, better-quasi-ordered (bqo). Bqo is stonger than wqo but weaker than well-ordered, and all "naturally occurring" wqo sets which are known are bqo.

Laver [17] has given a good explanation of bqo, on which we draw. Recall that $\mathscr{P}(X)$ consists of all subsets of X, and consider X as part of $\mathscr{P}(X)$ in the obvious way. We define $\mathscr{P}^{\alpha}(X)$ for every ordinal α in the natural way, namely, $\mathscr{P}^{\alpha+1}(X) = \mathscr{P}(\mathscr{P}^{\alpha}(X))$, and $\mathscr{P}^{\lambda}(X) =$ the union over $\beta < \lambda$ of $\mathscr{P}^{\beta}(X)$, for λ a limit ordinal. Then as Laver points out, the Nash-Williams definition of X bqo is equivalent to assuming that $\mathscr{P}^{\Omega}(X)$ is wqo, where Ω is the smallest uncountable ordinal.

If $\mathscr{P}^{\Omega}(X)$ is not wqo, then there is an (ordinary) infinite sequence over $\mathscr{P}^{\Omega}(X)$ which is nowhere ascending (that is, every term $\not\leqslant$ every later term). Each term of the sequence is either an element of X, or a set of elements, or a set of sets, or so on transfinitely. For each term which is not at the very lowest level, it is possible to pick a sequence of its elements with certain properties; from the resulting structure, it is possible to repeat this process; and so on through ω similar steps, until only elements at the very lowest level are present. The resulting structure is what the following definition rules out.

To define bqo requires certain preliminaries. If A is any set having a binary relation R (not necessarily transitive), and if $f: A \to X$ is a function into a qo set X, then f is called *good* if there exist two elements a_1 and a_2 in A such that $a_1 R a_2$ and $f(a_1) \leqslant f(a_2)$. To see how this relates to our previous concepts, suppose for the moment that A consists of the positive integers, with R being the usual relation $<$. Then a function $f: A \to X$ is an ordinary infinite sequence. It has been proved in several papers that X is wqo if and only if every infinite sequence over X (that is, every $f: A \to X$) is good.

Let A^* consist of all finite, strictly ascending sequences of positive integers. (Certain subsets of A^* are going to play the role of A above.) Define a binary relation $\lhd$ on A^* as follows: $a \lhd b$ if and only if there is a (strictly ascending) sequence of integers $i_1, \ldots, i_n$, and an $m < n$, such that a is $i_1, \ldots, i_m$ and b is $i_2, \ldots, i_n$. Note that, for the subset of A^* consisting of one-element sequences, $\lhd$ reduces to the ordinary relation $<$.

A subset A of A^* is called a *block* if every ordinary infinite, strictly ascending sequence of positive integers has an initial segment in A. X is called *better-quasi-ordered* (bqo) if, for every block A, all functions f: $A \to X$ are good. Since the one-element sequences form a block, it is clear that bqo implies wqo. Bqo is in fact a substantially stronger property than wqo, as we see from the following.

Let $\mathscr{I}(X)$ consist of all ordinary infinite sequences over X. Obviously $\mathscr{I}^n(X)$ wqo implies $\mathscr{I}^{n-1}(X)$ wqo. It follows from Nash-Williams' work that X bqo implies $\mathscr{I}^n(X)$ wqo (in fact bqo) for all finite n. However, for every finite n, Kruskal [11] constructs a space J_n such that $\mathscr{I}^n(J_n)$ is *not* wqo but $\mathscr{I}^{n-1}(J_n)$ *is* wqo. Thus the property that $\mathscr{I}^n(X)$ is wqo actually has different strength for every n, and this infinite chain of distinct properties is intermediate between X being wqo and X being bqo.

We restate here more precisely a culminating result from Laver [17] which has already been referred to in the introduction. Let M be any linearly ordered set which is a countable union of "scattered" subsets M_i (that is, no M_i contains a subset isomorphic to the rationals). Let $\mathscr{M}$ be the class of all such M, let all functions be acceptable, and let a map $f: M_1 \to M_2$ be proper if it is an isomorphic embedding. Then Laver's theorem asserts that, if X is bqo, $\mathscr{M}(X)$ is bqo also.

This powerful result settles in a very strong way a conjecture by Fraïssé [5], often described as equivalent to stating that $\mathscr{M}$ is wqo, which attracted considerable attention among set theorists in the 1950's. Furthermore it subsumes an already powerful result by Nash-Williams [27] that X bqo implies $\mathscr{S}(X)$ bqo, since well-ordered sets are scattered.

Laver's theorem does not, however, subsume a closely related result from Nash-Williams [25] that $\mathscr{S}_f(X)$, the space of finitary sequences over X, is wqo if X is wqo.

Another culminating result by Nash-Williams is that the space $\mathscr{T}$ of all trees (finite and infinite) is bqo, where the ordering is induced by homeomorphic embedding. Actually, Laver has generalized Nash-Williams' proof somewhat to get the result that X bqo implies $\mathscr{T}(X)$ bqo (where the proper maps are homeomorphic embeddings, of course). These results subsume Kruskal's result, which is that the space of finite trees is wqo, and proves more than his conjecture, which was that the space of all trees is wqo.

The theorems involving bqo demonstrate a familiar but very important principle of combinatorics, namely, that it is sometimes easier to prove a stronger property by recursion than a weaker one. It is now clear, after the fact, that wqo was the wrong property to use in the necessary recursion, and that the stronger property of bqo is much easier to handle. The devising of bqo as the appropriate property to use in these theorems was a very significant part of Nash-Williams' contribution. Of course, the fashioning of wqo was an earlier illustration of the same principle.

The Future

Last, it seems appropriate to mention the most important unsolved problem in this field. It was also conjectured by Vazsonyi that graphs of degree 3 are wqo. (This problem seems quite hard enough in the finite case, so let us restrict this conjecture to finite graphs.) Despite several attacks, very little is known in this direction. Kruskal attempted to work with a restricted class of graphs of degree 3 called "ladders," namely, those which can be laid out as two parallel paths, with edges connecting their vertices in some permutation. By distinguishing the two paths and the two ends of such graphs, they come into a 1-to-1 correspondence with permutations. However, Laver has shown by a simple (unpublished) counterexample that the permutations are not wqo. Unfortunately, this example does not work for the graph-theoretic situation. Nash-Williams on the other hand has generalized the concept of homeomorphic embedding to what he calls "immersions." Essentially, this permits the paths into which distinct edges are mapped to cross each other at "unimportant" vertices. He conjectures that, under the qo defined by immersion, the class of all finite graphs is wqo. He points out that, for both graphs of degree 3 and for trees, an immersion must be an embedding, so that his conjecture would subsume the wqo of both graphs of degree 3 and of trees.

As an indication of what will *not* happen in the future, we mention that a mimeographed paper by Jenkyns and Nash-Williams [8] contains counterexamples to several attractive conjectures.

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