Part I — Solutions

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Question 1 Which of the following systems are confluent? Which are terminating?

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & (\neg x) \vee (\neg y) \\ \neg(x \vee y) & \rightarrow & (\neg x) \wedge (\neg y) \end{array} \quad (1)$$

Answer: Terminating by the interpretation $[\neg x] = 2[x]$, $[x \wedge y] = [x \vee y] = [x] + [y] + 1$, $[constant] = 1$.

Confluent since its two critical pairs, $(x \wedge y) = \neg((\neg x) \vee (\neg y))$ and $(x \vee y) = \neg((\neg x) \wedge (\neg y))$, are.

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & (\neg\neg\neg x) \vee (\neg\neg\neg y) \\ \neg(x \vee y) & \rightarrow & (\neg\neg\neg x) \wedge (\neg\neg\neg y) \end{array} \quad (2)$$

Answer: Terminating by looking at the multiset containing, for each subterm $\neg s$ in a term, the number of symbols other than $\neg$ in s.

Confluent since its two critical pairs, $(x \wedge y) = \neg((\neg\neg\neg x) \vee (\neg\neg\neg y))$ and $(x \vee y) = \neg((\neg\neg\neg x) \wedge (\neg\neg\neg y))$, are.

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & (\neg x) \vee (\neg y) \\ \neg(x \vee y) & \rightarrow & (\neg x) \wedge (\neg y) \\ x \wedge (y \vee z) & \rightarrow & (x \wedge y) \vee (x \wedge z) \\ (y \vee z) \wedge x & \rightarrow & (y \wedge x) \vee (z \wedge x) \end{array} \quad (3)$$

Answer: Terminating by the recursive path ordering with $\neg > \vee > \wedge$.

Not confluent since $(a \vee b) \wedge (c \vee c)$ has two normal forms.

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & (\neg\neg\neg x) \vee (\neg\neg\neg y) \\ \neg(x \vee y) & \rightarrow & (\neg\neg\neg x) \wedge (\neg\neg\neg y) \\ x \wedge (y \vee z) & \rightarrow & (x \wedge y) \vee (x \wedge z) \\ (y \vee z) \wedge x & \rightarrow & (y \wedge x) \vee (z \wedge x) \end{array} \quad (4)$$

Answer: Not terminating for $\neg\neg(a \vee (b \wedge c))$.

Not confluent as above.

$$\begin{array}{rcl} x \vee x & \rightarrow & x \\ x \wedge x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & ((\neg x) \vee (\neg y)) \vee ((\neg x) \vee (\neg y)) \end{array} \quad (5)$$

Answer: Terminating by the recursive path ordering with $\neg > \vee$.

Confluent by Critical Pair Lemma.

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ x \vee x & \rightarrow & x \\ x \wedge x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & ((\neg x) \vee (\neg y)) \vee ((\neg x) \vee (\neg y)) \\ \neg(x \vee y) & \rightarrow & ((\neg x) \wedge (\neg y)) \wedge ((\neg x) \wedge (\neg y)) \end{array} \quad (6)$$

Answer: Terminating by the recursive path ordering with $\neg > \vee, \wedge$.

Confluent by Critical Pair Lemma.

$$\begin{array}{rcl} \neg\neg x & \rightarrow & x \\ x \vee x & \rightarrow & x \\ x \wedge x & \rightarrow & x \\ \neg(x \wedge y) & \rightarrow & ((\neg\neg\neg x) \vee (\neg\neg\neg y)) \vee ((\neg\neg\neg x) \vee (\neg\neg\neg y)) \\ \neg(x \vee y) & \rightarrow & ((\neg\neg\neg x) \wedge (\neg\neg\neg y)) \wedge ((\neg\neg\neg x) \wedge (\neg\neg\neg y)) \end{array} \quad (7)$$

Answer: Not terminating for $\neg\neg(a \wedge b)$.

Confluent since each rule in this system is an equality of the previous system and each step of the previous can be simulated by several steps of this one. That is,

$$\leftrightarrow_7^* \subseteq \leftrightarrow_6^* \subseteq \rightarrow_6^* \leftarrow_6^* \subseteq \rightarrow_7^* \leftarrow_7^*$$

Question 2 Let R and S be two rewrite systems with the following properties:

1. R is *confluent*.
2. S is *confluent*.
3. The reflexive-symmetric-transitive closures of their rewrite relations are identical ($\leftrightarrow_R^* = \leftrightarrow_S^*$).
4. The union of their rewrite relations ($\rightarrow_R \cup \rightarrow_S$) is terminating.
5. The right sides of every rule in R and S is in normal form.

Prove that their derivability relations are the same:

$$\rightarrow_R^* \quad = \quad \rightarrow_S^*$$

Answer: We show $R \subseteq \rightarrow_S^*$ whence it follows that $\rightarrow_R^* \subseteq \rightarrow_S^*$ and by symmetry the two relations are the same.

Let $l \rightarrow r \in R$.

By (3) $l \leftrightarrow_S^* r$.

By (2) $l \rightarrow_S^* u \leftarrow_S^* r$ for some u.

By (3) $u \leftrightarrow_R^* r$.

By (1) $u \rightarrow_R^* v \leftarrow_R^* r$ for some v.

By (5) $r = v$ and hence $u \rightarrow_R^* r$.

By (4) $u = r$ and hence $l \rightarrow_S^* r$.