Quasi-Ordered Gap Embedding⋆

—Extended Abstract—

Nachum Dershowitz and Iddo Tzameret

School of Computer Science
Tel-Aviv University
P.O.B. 39040
Ramat Aviv, Tel-Aviv 69978
Israel
email: Nachumd@tau.ac.il Tzameret@tau.ac.il

Kruskal's Tree Theorem [3], stating that finite trees are well-quasi-ordered under homeomorphic embedding, and its extensions, have played an important rôle in both logic and computer science. In proof theory, it was shown to be independent of certain logical systems by exploiting its close relationship with ordinal notation systems (cf. [6]), while in computer science it provides a common tool for proving the termination of many rewrite-systems via the *recursive path* and related orderings [1]. For demonstrating termination of rewriting, it is beneficial to use a *partial* (or *quasi*-) ordering on labels, rather than a total one.

In [7], it was shown that many important order-theoretic properties of the well-partial-ordered precedence relations on function symbols carry over to the induced termination ordering. This is done by defining a general framework for precedence-based termination orderings via (so-called) *relativized ordinal notations*. Based on a few examples, it is further conjectured that every such application of a partial-order to an ordinal notation system carries the order-theoretic properties of the partial-order to the relativized notation system. An example of such a construction, using Takeuti's ordinal diagrams, is introduced in [5] under the name *quasi-ordinal-diagrams*. The definition of these diagrams is the only result we know of that deals with gap embedding of trees and *quasi*-ordered labels.

Kříž [2] proved a conjectured extension by Harvey Friedman of the Tree Theorem, which states that finite trees labelled by ordinals are well-quasi-ordered under gap embedding. Our work extends this further to finite trees with well-quasi-ordered labels, with the following restriction (which is shown necessary): *Every node is comparable with all its ancestors.*

Let $\mathcal{T}$ be a set of *ordered* (rooted, plane-planted) finite trees, with nodes well-quasi ordered by $\precsim$, and with the above restriction. Let $t^\bullet$ denote the root of tree t. There is a *(gap) subtree* relation $\trianglerighteq$ on trees (which includes all immediate subtrees) with the following properties:

$$s \trianglerighteq t \trianglerighteq u \wedge t^\bullet \succsim u^\bullet \Rightarrow s \trianglerighteq u \qquad (1)$$

$$s \trianglerighteq t \trianglerighteq u \wedge s^\bullet \precsim t^\bullet \Rightarrow s \trianglerighteq u \qquad (2)$$

$$s \trianglelefteq t \Rightarrow s^\bullet \precsim t^\bullet \vee t^\bullet \precsim s^\bullet \qquad (3)$$

⋆ Supported in part by the Israel Science Foundation (grant no. 254/01).

There is also an *(gap) embedding* relation $\hookrightarrow$ on trees with the properties:

$$s \hookrightarrow t \trianglelefteq u \wedge t^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u \tag{4}$$

$$s \hookrightarrow t \trianglelefteq u \wedge s^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u \tag{5}$$

A *sequence* s is a partial function $s : \mathbf{N} \to \mathcal{T}$. If $s(i)$ is not defined we write $s(i) = \bot$. It is very convenient to extend the subtree relation and node ordering to empty positions of a sequence, so that: $t \trianglerighteq \bot$ and $t^\bullet \precsim \bot^\bullet$.

Let Seq be the set of ω-sequences of trees from $\mathcal{T}$. Define:

$$\begin{aligned} Ds &:= \{i \in \mathbf{N} \,|\, s(i) \neq \bot\} \\ \text{Bad} &:= \{s \in \text{Seq} \,|\, \forall i < j \in Ds.\ s(i) \not\hookrightarrow s(j)\} \\ \text{Sub}\ h &:= \{s \in \text{Seq} \,|\, \forall i \in Ds.\ h(i) \triangleright s(i)\} \\ \text{Inc}\ k &:= \{s \overset{\infty}{\subseteq} k \,|\, \forall i < j \in Ds.\ s^\bullet(i) \precsim s^\bullet(j)\} \end{aligned}$$

where $\triangleright$ is the *proper* (gap) subtree relation. We'll say that s is *infinite* when Ds is.

Since $\succsim$ is a well-quasi-ordering, Inc k is nonempty, as long as k is infinite.

Theorem 1. Bad $= \emptyset$.

In other words, for every $s \in$ Seq there exist $i < j \in Ds$ such that $s(i) \hookrightarrow s(j)$. This extends the result of Kříž to quasi-ordered labels.

Assuming the contrary, the proof builds a minimal counterexample, minimal in the sense that no infinite sequence of proper (gap) subtrees of its elements is also bad (which leads to a contradiction—as in the original proof by Nash-Williams [4]). The construction of the minimal bad sequence proceeds by ordinal induction as follows:

$H(0):$	$h :\in \text{Bad}$ **if** $\text{Bad} \cap \text{Sub}\ h = \emptyset$ **then return** h $h_0 :\in \text{Inc lex}(h)$
$H(\alpha+1):$	**if** $\text{Bad} \cap \text{Sub}\ h_\alpha = \emptyset$ **then return** h_α $k := \text{lex}(h_\alpha)$ $\forall i \in \mathbf{N}.\ f(i) := \begin{cases} k(i) & \textbf{if } h^\bullet_\alpha(i) \precsim k^\bullet(i) \\ \bot & \textbf{otherwise} \end{cases}$ $g :\in \text{Inc}\ f$ $\forall i \in \mathbf{N}.\ h_{\alpha+1}(i) := \begin{cases} h_\alpha(i) & \textbf{if } i < \min Dg \\ g(i) & \textbf{otherwise} \end{cases}$
$H(\lambda):$	$\forall i \in \mathbf{N}.\ \ell(i) := \lim_{\gamma \to \lambda} h_\gamma(i)$ **if** $\text{Bad} \cap \text{Sub}\ \ell = \emptyset$ **then return** ℓ $h_\lambda :\in \text{Inc lex}(\ell)$

where the construct $s :\in S$ chooses an arbitrary s from S ($s = \bot$ if $S = \emptyset$). The function lex : Bad $\to$ Bad chooses a bad sequence of subtrees with (lexicographically) minimal labels:

$$
\begin{array}{|l|l|}
\hline
\mathrm{lex}(h): & K := \mathrm{Bad} \cap \mathrm{Sub}\, h \\
 & \textbf{for } i := 1 \textbf{ to } \infty \textbf{ do} \\
 & \quad m :\in \min\{a^{\bullet}(i) \,|\, a \in K\} \\
 & \quad k(i) :\in \{a(i) \,|\, a \in K,\ a^{\bullet}(i) = m\} \\
 & \quad K := \{a \in K \,|\, a(i) = k(i)\} \\
 & k :\in K \\
 & \textbf{return } k \\
\hline
\end{array}
$$

By construction, we have (for all α and i):

$$Dh_{\alpha} \supseteq Dh_{\alpha+1} \tag{6}$$

$$h_{\alpha}(i) \trianglerighteq h_{\alpha+1}(i) \tag{7}$$

$$h^{\bullet}_{\alpha}(i) \precsim h^{\bullet}_{\alpha+1}(i) \tag{8}$$

For each sequence h_{α} (for every countable ordinal α and for all $i < j \in Dh_{\alpha}$):

$$h_{\alpha}(i) \not\rightarrow h_{\alpha}(j) \tag{9}$$

$$h^{\bullet}_{\alpha}(i) \precsim h^{\bullet}_{\alpha}(j) \tag{10}$$

The successor step of (9,10) is proved by induction; the only interesting case is $i < \min Dg \leq j$, when

$$h^{\bullet}_{\alpha+1}(i) = h^{\bullet}_{\alpha}(i) \precsim h^{\bullet}_{\alpha}(j) \precsim f^{\bullet}(j) = k^{\bullet}(j) = h^{\bullet}_{\alpha+1}(j)$$

from which (9) follows using (5). By considering the limit case, it can be seen that for all $\alpha < \beta$:

$$Dh_{\alpha} \supseteq Dh_{\beta} \tag{11}$$

Finally, it can be shown that:

1. The constructed sequences h_{α} are all infinite.
2. The constructed sequences h_{α} are each distinct.
3. The construction eventually terminates with a minimal bad sequence.

By induction, this result may be extended also to the case where every path in the tree can be partitioned into some bounded number of subpaths with comparable labels.

Moreover, the absence of such a bound yields a bad sequence with respect to gap-embedding: Let a, b, c be three incomparable elements of the node ordering. The following is an antichain with respect to gap embedding:

$$c - a - c \qquad c - b - a - c \qquad c - a - b - a - c \qquad c - b - a - b - a - c \ldots$$

Consequently, the extension of Theorem 1 to set of trees with bounded number of comparable subpaths shows that the above counterexample is *canonical*: Every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

References

1. N. Dershowitz. Orderings for term-rewriting systems. *Theor. Comput. Sci.*, 17:279–301, 1982.
2. I. Kříž. Well-quasiordering finite trees with gap-condition. proof of Harvey Friedman's conjecture. *Ann. of Math.*, 130:215–226, 1989.
3. J. B. Kruskal. Well-quasiordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. American Mathematical Society*, 95:210–223, 1960.
4. C. St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, 1963.
5. M. Okada and G. Takeuti. On the theory of quasi-ordinal diagrams. In: *Logic and Combinatorics* (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 295–308.
6. S.G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In L. A. Harrington, editor, *Harvey Friedman's research on the foundation of mathematics*, pages 87–117. Elsevier, 1985.
7. A. Weiermann. Proving termination for term rewriting systems. In *Computer Science Logic* (Berne, 1991), volume 626 of *Lecture Notes in Comput. Sci.*, pages 419–428. Springer, Berlin, 1992.