On the Representation of Ordinals up to Γ_0

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Let T be the set of (unordered rooted) trees whose nodes may themselves be trees, i.e. the <u>trivial</u> tree

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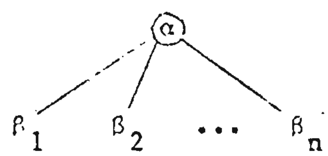
is an element of T and if $\alpha, \beta_1, \beta_2, \ldots, \beta_n \in T$ $(n>0)$, then the <u>compound</u> tree



is also in T. We shall use () to denote the trivial tree and

$$(\alpha\ \beta_1\ \beta_2\ \cdots\ \beta_n)$$

to denote the above compound tree. For example, ((()(()()()))) is an element of T, as are all balanced parenthetic expressions.

We define the following functions on T:

a) The depth d of a tree is defined by

$$d(t) = \begin{cases} 0 & \text{if } t=() \\ \max\{d(\alpha)+1, d(\beta_1), \ldots, d(\beta_n)\} & \text{if } t=(\alpha\ \beta_1\ \beta_2\ \cdots\ \beta_n) \end{cases}$$

b) The function Op returns the root of a (compound) tree:

$$Op((\alpha\ \beta_1\ \beta_2\ \cdots\ \beta_n)) = \alpha$$

c) The function Ops returns the multiset of subtrees of a (compound) tree:

$$Ops((\alpha\ \beta_1\ \beta_2\ \cdots\ \beta_n)) = \{\beta_1, \ldots, \beta_n\}$$

The following total ordering is defined recursively on T:

For any $t,t' \varepsilon T-\{()\}$

$$t > ()$$

and

$$t > t' \quad \text{iff} \quad \begin{cases} Ops(t) >> Ops(t') & \text{when } op(t)=op(t') \\ \{t\} >> Ops(t') & \text{when } op(t)>op(t') \\ Ops(t) \gg \{t'\} & \text{when } op(t)<op(t') \end{cases}$$

where >> is the extension to multisets of > wherein $S \gg S'$ if for all x' in S' but not in S there is a greater x in S that is not in S' and $S>>S'$ if $S \gg S'$ and $S \neq S'$. For example, $((())()) > (()(())(()))$, since $(()) > ()$ and $((())()) > ((()))$. [This is an extension of the Recursive Path Ordering, see Plaisted and Dershowitz.]

There exists the following order-preserving one-to-one mapping ψ from T onto Γ_0, where the ordinal Γ_0 is (?) Veblen's first E-number:

$$\psi(t) = \begin{cases} 0 & \text{if } t=() \\ \phi^{\psi(\alpha)}\left(\sum_{i=1}^{n} \omega^{\psi(\beta_i)}\right)+\delta(t) & \text{if } t=(\alpha\ \beta_1\ \beta_2 \ldots \beta_n) \end{cases}$$

where $\phi(\beta)=\epsilon_\beta$ (the β-th epsilon number), $\phi^\alpha(\beta)$ is the β-th fixpoint ξ of $\phi^\mu(\xi)=\xi$ common to ϕ^μ for all $\mu<\alpha$, $\sum$ is the natural (commutative) sum of ordinals, and

$$\delta(t) = \begin{cases} 1 & \text{if } t=(()()..()) \\ 1 & \text{if } t=(()()()...()\beta_j()...()) \text{ and } op(\beta_j)\neq() \\ 0 & \text{otherwise} \end{cases}$$

The purpose of δ is to ensure that $\psi\big((()\beta)\big) > \psi(\beta)$ even if $\psi(\beta)$ is an epsilon number. That this mapping is order-preserving follows from the fact [Feferman] that $\phi^{\alpha}(\beta)>\phi^{\alpha'}(\beta')$ if and only if $\alpha=\alpha'$ and $\beta>\beta'$, or else $\alpha>\alpha'$ and $\phi^{\alpha}(\beta)>\beta'$, or else $\alpha<\alpha'$ and $\beta>\phi^{\alpha'}(\beta')$. It follows that the order-type of $(T,>)$ is Γ_0.

The well-foundedness of $(T,>)$ may also be proved by induction on depth using Kruskal's Tree Theorem: Assume that there existed an infinite descending sequence $t_1>t_2>t_3>\ldots$ of trees. By the induction hypothesis the set of all nodes appearing in the trees of the sequence is well-founded. Thus, by the Tree Theorem, some t_i is homeomorphically embeddable in some subsequent t_j. But it can be shown from the definition of $>$ that that would imply $t_i<t_j$ which is a contradiction.

As an example of the use of this well-founded set in a termination proof, consider the term-rewriting system consisting of the single rule

$$\mathrm{if}(\mathrm{if}(\alpha,\beta,\gamma),\delta,\varepsilon) \to \mathrm{if}(\alpha,\mathrm{if}(\beta,\delta,\varepsilon),\mathrm{if}(\gamma,\delta,\varepsilon))$$

The conditional expression "$\mathrm{if}(\alpha,\beta,\gamma)$" stands for "if α then β else γ" and this system "normalizes" conditional expressions by repeatedly removing embedded if's from the condition α. To see that this system terminates, i.e. given any input expression any sequence of rewrites of subexpressions must be finite, note that $((\alpha\ \beta\ \gamma)\delta\ \varepsilon)>(\alpha(\beta\ \delta\ \varepsilon)(\gamma\ \delta\ \varepsilon))$, and therefore applying the rule always reduces the corresponding tree in the ordering $>$.

References

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