Well-Founded Unions*

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Abstract. Given two or more well-founded (terminating) binary relations, when can one be sure that their union is likewise well-founded? We suggest new conditions for an arbitrary number of relations, generalising known conditions for two relations. We also provide counterexamples to several potential weakenings. All proofs have been machine checked.

1 Introduction

A binary relation R (which need not be an ordering) over some underlying set is *well-founded* (or *terminating*) if there is no infinite *descending* chain $x_0 R x_1 R \cdots R x_{n-1} R x_n R \cdots$.[3] Given well-founded binary relations $R_0, R_1, \ldots, R_n$ over some common (fixed) underlying set X, under what conditions is their union $R_0 \cup R_1 \cup \cdots \cup R_n$ also well-founded?

For two well-founded relations A and B, their union $A \cup B$ is well-founded (Corollary 6 below) if the following relatively powerful condition holds [11]: see also [12]. It is called *Jumping* in [7]:

$$BA \subseteq A(A \cup B)^* \cup B \; . \tag{$*$}$$

Juxtaposition is being used for composition ($xBAz$ iff there's a y such that xBy and yAz) and the asterisk for the reflexive-transitive closure (xB^*z iff there are $y_0, y_1, \ldots, y_n$, $n \geq 0$, such that $x = y_0 B y_1 B \cdots B y_n = z$).

Jumping ($*$) generalises simpler ways of showing well-foundedness of the union of two relations. Eliding the rightmost B possibility gives *quasi-commutation* [2], which is relevant to many rewriting situations (e.g. [15,5,2,6]):

$$BA \subseteq A(A \cup B)^* \; . \tag{1}$$

Likewise, the simple A option also suffices for the well-foundedness of the union:

$$BA \subseteq A \cup B \; . \tag{2}$$

To gain purchase on the manner of reasoning, let $R = A \cup B$ and imagine a minimal infinite descending chain in R: $x_0 R x_1 R \cdots R x_{n-1} R x_n R \cdots$. By "minimal" we mean that its elements are as small as possible vis-à-vis A, which –

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[3] We choose to view the forward direction as descent.

as it is well-founded – always enjoys minimal elements. Thus x_0 is the smallest element in the underlying set from which an infinite chain in R ensues. By the same token, x_1 is the smallest possible y such that $x_0 R y R \cdots$. And so on. By the well-foundedness of both A and B, any such chain must have (indeed, must have infinitely many) adjacent BA-steps: $\cdots x B x' A x'' R \cdots$. Now, if (2) holds, we could have taken a giant step xRx'', instead, before continuing down the infinite path from x''. But this would imply that the chain is not actually minimal because x'' is less than x' with respect to A, and should have been next after x.

Similarly, to show that (1) suffices, we choose a "preferred" infinite counterexample, in the sense that an A-step is always better than a B-step, given the choice. Again, an infinite chain containing a pair of steps $xBx'Ax''$ could not be right since there is a preferred alternative, $xAyR \cdots Rx''R \cdots$, dictated by (1).

Combining these two arguments gives the sufficiency of the combined jumping condition $(*)$. Among preferred counterexamples, always choose B-steps, xBx', having minimal x' with respect to A. Preference precludes taking an A-first detour instead of a BA pair $xBx'Ax''$, while minimality precludes a B-shortcut xBx''.

To garner further insight, we first tackle – in the next two sections – the easier case of just three relations. Then, in Sect. 4, we extend the tripartite results and describe the general pattern for an arbitrary number of relations. We also show in Sect. 5 that under the same conditions any chain in the union can be rearranged so that the individual relations appear contiguously. This is followed in Sect. 6 by an example of the use of Preferential Commutation for four relations involved in the dependency-pair method [1].

Letting $R_{i:n} = \bigcup_{j=i}^{n} R_j$ be the union of well-founded relations $R_i, R_{i+1} \mathinner{..} R_n$, and letting R_i^+ be the transitive closure of R_i, our efforts culminate in Sect. 7 with the following sufficient condition (Theorem 28) for the well-foundedness and rearrangeability of $R_{0:n}$: There is some k, $0 \le k \le n$, such that

$$R_{i+1:n} R_i \subseteq R_0 R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \text{for } i = 0 \mathinner{..} k-1 \qquad (**)$$

$$R_{i+1:n} R_i \subseteq R_i R_{i:n}^* \cup R_{i+1:n} \qquad \text{for } i = k \mathinner{..} n-1\,. \qquad (***)$$

In the quadripartite case ($n = 3$), with $k = 2$, this amounts to the following:

$$(B \cup C \cup D)A \subseteq A(A \cup B \cup C \cup D)^* \cup B \cup C \cup D \qquad (3a)$$

$$(C \cup D)B \subseteq A(A \cup B \cup C \cup D)^* \cup B^+ \cup C \cup D \qquad (3b)$$

$$DC \subseteq C(C \cup D)^* \cup D\,. \qquad (3c)$$

All proofs have been machine-checked using Isabelle/HOL; see Sect. 8.

We conclude with an open quadripartite problem and ideas for future work.

2 Tricolour Unions

We now study the three-relation case $n = 2$. We will refer to the relations A, B, and C as "colours". Ramsey's Theorem may be applied directly:

Theorem 1 (d'après Ramsey). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if it is transitive:*

$$(A \cup B \cup C)(A \cup B \cup C) \subseteq A \cup B \cup C\,. \tag{4}$$

Proof. The infinite version of Ramsey's Theorem applies when the union is transitive, so that every two (distinct) nodes within an infinite chain in the union of the colours have a (directed) edge that is coloured in one of the three colours. Then, there must lie an infinite monochromatic subchain within any infinite chain, contradicting the well-foundedness of each colour alone. □

The suggestion to use Ramsey's Theorem for such a purpose is due to Franz Baader in 1989 [16, items 38–41]; see [13, Sect. 3.1]. Its use in a termination prover was pioneered in the TermiLog system [10]. Other uses followed; see [3].

Only three of the nine cases implicit in the left-hand side of (4) are actually needed for the limited outcome that we are seeking, an infinite monochromatic path, rather than a clique as in Ramsey's Theorem – as we observe next.

Theorem 2. *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$BA \cup CA \cup CB \subseteq A \cup B \cup C\,. \tag{5}$$

Proof. When the union is not well-founded, there are infinite chains $Y = \{x_i\}_i$ with each x_i being connected to its neighbor x_{i+1} by one of the relations A, B, or C. Extract a maximal (noncontiguous) subsequence $Z = \{x_{i_j}\}_j$ of Y that consists of "hops" $x_{i_j} A x_{i_{j+1}}$, via A, for each j. If it's finite and ends at some x_k, then at the *first* opportunity in the tail of Y beginning $x_{k+1}, x_{k+2}, \ldots$ extract another such sequence $\{x_{i'_j}\}_j$. Tack on to Z the intervening steps linking the terminus x_k of the maximal subsequence to the start $x_{i'_1}$ of the second, followed by the rest of it, $x_{i'_2}, x_{i'_3}$, etc. Repeat and repeat. If any such subsequence turns out to be infinite, we have a contradiction to well-foundedness of A. If they're all finite, then consider the pair of steps $x_{i'_1-1}(B \cup C)x_{i'_1} A x_{i'_2}$ in Z. Since we could not take an A-step from $x_{i'_1-1}$ or else we would have, condition (5) tell us that $x_{i'_1-1}(B \cup C)x_{i'_2}$. Swallowing up all such (non-initial) A-steps in Z in this way, we are left with an infinite chain in $B \cup C$ only, for which we also know that no A-hops are possible anywhere. Now extract maximal B-chains in the same fashion and then erase them, replacing $xCyBz$ with xCz (A- and B-steps having been precluded), leaving an infinite chain coloured purely C. □

Condition (5) above is better than what we get by just iterating the simple condition (2) as shown below, with the difference being the option $BA \subseteq C$:

$$BA \subseteq A \cup B$$
$$CA \cup CB \subseteq A \cup B \cup C\,.$$

To guarantee an infinite clique, not just well-foundedness, instead of (4), one can insist on the three transitivity cases ($AA \subseteq A$, $BB \subseteq B$, $CC \subseteq C$), too:

Corollary 3. *If A, B, and C are transitive relations satisfying (5) and there is an infinite path in $A \cup B \cup C$, then there is an infinite monochromatic clique.*

Proof. By Theorem 2, (at least) one of A, B, C is not well-founded. By transitivity, the elements of any infinite chain in that non-well-founded colour form an infinite clique in the underlying undirected graph. □

Let's refer to the elements in any infinite descending chain in the union $A \cup B \cup C$ as *immortal*. We can do considerably better than the previous theorem:

Theorem 4 (Tripartite). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{6a}$$

$$CB \subseteq A(A \cup B \cup C)^* \cup B^+ \cup C\ . \tag{6b}$$

Proof (sketch). First construct an infinite chain $Y = \{x_i\}_i$, in which an A-step is always preferred over B or C, as long as immortality is maintained. To do this, start with an immortal element x_0 in the underlying set. At each stage, if the chain so far ends in x_i, check if there is any y such that $x_i A y$ and from which proceeds some infinite chain in the union, in which case y is chosen to be x_{i+1}. Otherwise, x_{i+1} is any immortal element z such that $x_i B z$ or $x_i C z$.

If there are infinitely many B's and/or C's in Y, use them – by means of the first condition – to remove all subsequent A-steps, leaving only B- and C-steps, which go out of points from which A leads of necessity to mortality. From what remains, if there is any C-step at a point where one could take one or more B-steps to any place later in the chain, take the latter route instead. What remains now are C-steps at points where B^+ detours are also precluded. If there are infinitely many such C-steps, then applying the condition for CB will result in a pure C-chain, because neither $A(A \cup B \cup C)^*$ nor B^+ are options. □

Dropping C from the conditions of the previous theorem, one gets the Jumping criterion, which we explored in the introduction:

Definition 5 (Jumping criterion [11,12]). *Binary relation A* jumps over *binary relation B if*

$$BA \subseteq A(A \cup B)^* \cup B\ . \tag{$*$}$$

Corollary 6 (Jumping [11,12]). *The union $A \cup B$ of well-founded relations A and B is well-founded whenever A jumps over B.*

Applying this Jumping criterion twice, one gets somewhat different (incomparable) conditions for well-foundedness of the union of three relations.

Theorem 7 (Jumping I). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$BA \subseteq A(A \cup B)^* \cup B \tag{7a}$$

$$C(A \cup B) \subseteq (A \cup B)(A \cup B \cup C)^* \cup C\ . \tag{7b}$$

Proof. The first inequality is the Jumping criterion $(*)$. The second is the same with C for B and $A \cup B$ in place of A. $\square$

For two relations, Jumping provides a substantially weaker well-foundedness criterion than does Ramsey. For three, whereas Jumping allows more than one step in lieu of BA (in essence, AA^*B^*), it doesn't allow for C, as does Ramsey.

Switching rôles, start with Jumping for $B \cup C$ before combining with A, we get slightly different conditions yet:

Theorem 8 (Jumping II). *The union $A \cup B \cup C$ of well-founded relations A, B, and C is well-founded if*

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{8a}$$
$$CB \subseteq B(B \cup C)^* \cup C\ . \tag{8b}$$

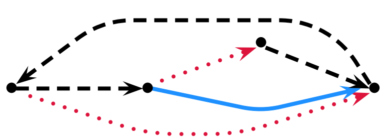
Both this version of Jumping and our tripartite condition allow

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup B \cup C \tag{6a,8a}$$
$$CB \subseteq B^+ \cup C\ . \tag{cf. 6b,8b}$$

They differ in that Jumping also allows the condition shown below on the left whereas tripartite has the one shown on the right instead:

Jumping allows	Tripartite allows
$CB \subseteq B(B \cup C)^*$.	$CB \subseteq A(A \cup B \cup C)^*$.

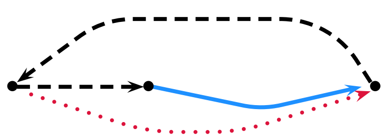
Example 9. (a) Sadly, we cannot have the best of both worlds. Let's colour edges A, B, and C with (solid) azure, (dashed) black, and (dotted) crimson ink, respectively. The graph below only has multicoloured loops despite satisfying the inclusions below.



$$(B \cup C)A \subseteq C$$
$$CB \subseteq A \cup B(B \cup C)^*\ .$$

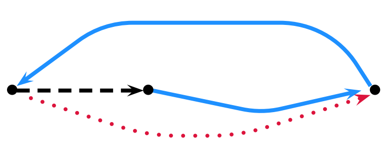
(b) Even the conditions shown below are insufficient since the double loop in the graph below harbours no monochromatic subchain:

$$(B \cup C)A \subseteq C$$
$$CB \subseteq B(A \cup B)^*$$



(c) By the same token, the putative hypothesis that the conditions shown below suffice is countered by the graph at its side:

$$BA \cup CB \subseteq C$$
$$CA \subseteq BA^*$$



3 Tripartite Proof

In preparation for the general case, we decompose the proof of the Tripartite Theorem 4 of the previous section into a sequence of notions and lemmata.

Definition 10 (Immortality [7]). *Let $R \subseteq X \times X$ be a binary relation over some underlying set X. The set $R^\infty \subseteq X$ of R-*immortal *elements are those elements $x_0 \in X$ that head infinite (descending) R-chains, $x_0 R x_1 R \cdots$.*

So, a relation R is well-founded if and only if every element of the underlying set is mortal ($R^\infty = \varnothing$).

Two trivial observations, first.

Proposition 11. *If $R \subseteq S^+$, for binary relations R and S, then perforce $R^\infty \subseteq S^\infty$, that is, every R-immortal is also S-immortal.*

It follows that

Proposition 12. *Binary relation R is well-founded if it is contained in a well-founded relation S, and, more generally, if $R \subseteq S^+$.*

As usual, the (forward) *image* $Q[Y]$ of a set Y under relation Q consists of those z such that yQz for some $y \in Y$, and the *inverse* (or *pre-*) *image* $Q^{-1}[Y]$ of Y under Q are those y such that yQz for some $z \in Y$.

If yRz for (R-)immortal z, then y is also immortal:

Proposition 13. *The inverse image of immortals is immortal: $R^{-1}[R^\infty] = R^\infty$.*

We will make repeated use of the Jumping criterion ($*$), $BA \subseteq A(A \cup B)^* \cup B$. By induction (on the number of A's), Jumping extends to the transitive closure:

Lemma 14. *If binary relation A jumps over relation B, then*

$$BA^* \subseteq A(A \cup B)^* \cup B \ . \tag{9}$$

A central tool will be the following concept:

Definition 15 (Constriction). *The* constriction *$B^\sharp$ of binary relation B over X (with respect to relation A) excludes from B all steps of the form zBw for which there is an $A \cup B$-immortal y such that zAy:*

$$B^\sharp = B \setminus \{(z, w) \mid z \in A^{-1}[(A \cup B)^\infty], w \in X\} \ .$$

The idea of constriction is inspired by its use by Plaisted [17] for subterms.

Lemma 16. *The union $A \cup B$ of binary relations A and B is well-founded whenever $A \cup B^\sharp$ is.*

Proof. Construct an infinite descending $A \cup B$-chain, using A when it leads to immortality, and using B only when needed (making it a constricted step). $\square$

Lemma 17. *If binary relation A jumps over relation B and both A and $B^\sharp$ are well-founded, then $A \cup B^\sharp$ is well-founded.*

Proof. Consider any infinite descending $A \cup B^\sharp$-chain. As A is well-founded, it must contain infinitely many $B^\sharp$-steps. As A jumps over B, Lemma 14 tells us that $B^\sharp A^* \subseteq A(A \cup B)^* \cup B^\sharp$. We have $B^\sharp$ on the right, because that position is constricting on the left. But in any infinite $A \cup B^\sharp$-chain, we cannot replace $B^\sharp A^*$ by $A(A \cup B)^*$ since that would mean that A leads to immortality, violating constriction. Hence, all (non-initial) A-steps may be removed from the chain, leaving an impossible infinite $B^\sharp$-chain. □

Combining the previous two lemmata, we can improve on Corollary 6.

Corollary 18. *If binary relation A jumps over relation B and both A and $B^\sharp$ are well-founded, then $A \cup B$ is well-founded.*

When there are more than three relations, as in the next section, we will need to revise the following lemma with a more flexible notion of constriction. For now, let $C^\flat$ be like $C^\sharp$ except that B-steps may be needed for immortality. Thus $C^\flat$ excludes all C-steps zCw with an $A \cup B \cup C$-immortal y such that zAy.

Lemma 19. *The union $B \cup C^\flat$ is well-founded if well-founded binary relations B and $C^\flat$ satisfy*

$$CB \subseteq A(A \cup B \cup C)^* \cup B^+ \cup C \,. \tag{6b}$$

Proof. Suppose that B and $C^\flat$ are well-founded, but $B \cup C^\flat$ is not. So there exist $B \cup C^\flat$-immortal elements. Choose z to be a B-minimal such element, and also to be $C^\flat$-minimal among all possible B-minimal choices.

As z is B-minimal, the first step of an infinite $B \cup C^\flat$-chain must be $zC^\flat y$, for some y. Since B is well-founded, let y be B-minimal among possible choices for such a y. By the aforementioned $C^\flat$-minimality of z, although y is $B \cup C^\flat$-immortal, it is not B-minimal among $B \cup C^\flat$-immortals. So we have yBx, where x is $B \cup C^\flat$-immortal.

Relying on (6b), we could replace $zCyBx$ in the putative infinite chain by any one of the following:

- $zAy'(A \cup B \cup C)^*x$, for some y' – but x heads an infinite descending $B \cup C$-chain, contradicting the constriction of $zC^\flat y$; or
- zB^+x, which would contradict our choice of z to be B-minimal; or
- zCx, and so $zC^\flat x$, which would contradict our choice of y to be B-minimal, since yBx and x could have been chosen in place of y.

Since each alternative leads to a contradiction, $B \cup C^\flat$ must be well-founded. □

Everything is in place now for a modular proof of the Tripartite Theorem.

Proof (Proof of Theorem 4). Since A jumps over $B \cup C$ (6a), by Corollary 18, it is enough to show that $(B \cup C)^\sharp$ is well-founded. Given (6b), by Lemma 19, we have that $B \cup C^\flat$ is well-founded. Clearly $(B \cup C)^\sharp \subseteq B \cup C^\flat$, because constricted B is in B and C is constricted to the same degree in both $(B \cup C)^\sharp$ and $B \cup C^\flat$, namely that A does not lead to immortality in the full union. By Proposition 12, the required well-foundedness of $(B \cup C)^\sharp$ follows. □

4 Preferential Commutation

The two three-relation conditions, Jumping I and Jumping II, can each be straightforwardly extended by induction to arbitrarily many relations.

Corollary 20 (Jumping I). *The union $R_{0:n}$ of well-founded relations $R_0, R_1, \ldots, R_n$ is well-founded if*

$$R_{i+1}R_{0:i} \subseteq R_{0:i}R_{0:i+1}^* \cup R_{i+1} \qquad \textit{for all } i = 0 \mathinner{..} n-1 \ .$$

Proof. Since $B = R_{i+1}$ is well-founded, assume $A = R_{0:i}$ is well-founded by induction. Jumping (Corollary 18) then implies that so is $A \cup B = R_{0:i+1}$. □

Corollary 21 (Jumping II). *The union $R_{0:n}$ of well-founded relations $R_0, R_1, \ldots, R_n$ is well-founded if*

$$R_{i+1:n}R_i \subseteq R_iR_{i:n}^* \cup R_{i+1:n} \qquad \textit{for all } i = 0 \mathinner{..} n-1 \ . \tag{10}$$

Proof. Let $A = R_i$ and $B = R_{i+1:n}$ in Corollary 18, and reason by induction. □

We next extend Theorem 4 to an arbitrary number of relations and show the sufficiency of what we call *Preferential Commutation*.

Theorem 22 (Preferential Commutation). *The union $R_{0:n}$ of well-founded relations $R_0, R_1, \ldots, R_n$ is well-founded if it satisfies the following* Preferential Commutation Condition*:*

$$R_{i+1:n}R_i \subseteq R_0R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \textit{for all } i = 0 \mathinner{..} n-1 \ . \tag{11}$$

Preferential Commutation (11) specializes to the two conditions (6a,6b) of Theorem 4 in the tripartite case. In the quadripartite case ($n = 3$), it asserts that $A \cup B \cup C \cup D$ is well-founded if

$$(B \cup C \cup D)A \subseteq A(A \cup B \cup C \cup D)^* \cup B \cup C \cup D \tag{11a}$$
$$(C \cup D)B \subseteq A(A \cup B \cup C \cup D)^* \cup B^+ \cup C \cup D \tag{11b}$$
$$DC \subseteq A(A \cup B \cup C \cup D)^* \cup C^+ \cup D \ . \tag{11c}$$

Notice the inclusion of the options B^+ and C^+ in (11b) and (11c), respectively, when compared with the Jumping criteria. The A^+ has been omitted from (11a) on account of its inclusion in $A(A \cup \cdots)^*$.

We apply Preferential Commutation to four relations in Sect. 6.

Foremost to the argument for the above theorem is a general "detour" condition given below: replacing R_0 in Preferential Commutation with arbitrary P and $R_{0:n}$ with any S. Consider conditions (11a–11c) on A, B, C, D in the case of four relations. The point is that we require the union of B, C, D to be well-founded so as to apply Jumping in conjunction with A, but were we to simply use the same method of jumping to establish that, we would not be allowed to introduce any A-steps in the inclusions for compositions of pairs from B, C, D.

We first generalise the notion of constriction (Definition 15).

Definition 23 (Constriction). *For arbitrary binary relation S, the S-*constriction $B^{Q\sharp S}$ *of binary relation B over X,* with respect to Q, *excludes from B all steps of the form zBw where there exists some S-immortal element in the Q-image of z:*

$$B^{Q\sharp S} = B \setminus \left(Q^{-1}[S^{\infty}] \times X\right) \ .$$

Think of this as B minus cases where Q could have granted S-immortality.

The basic constriction $B^{\sharp}$ of Definition 15 in the previous section is $B^{A\sharp A\cup B}$, while $C^{\flat}$ of Lemma 19 is $C^{A\sharp A\cup B\cup C}$.

We note that $B \subseteq C$, $Q \subseteq PR^*$ and $S \subseteq R^+$ imply $B^{P\sharp R} \subseteq C^{Q\sharp S}$.

Definition 24 (Detour). *Binary relations A, B, P, S satisfy the* detour *condition $\Delta^{P\sharp S}_{B;A}$ if*

$$BA \subseteq PS^* \cup A^+ \cup B \ . \tag{12}$$

Our central lemma is next; it generalises Lemma 19 of the previous section. Though it does have a proof very similar to the latter, we give here an alternative, distinct argument, one we find quite interesting. Contrary to earlier proofs, here we modify relations to include *only immortal* points – if any!

Lemma 25. *For all binary relations A, B, P, S, such that $A \cup B \subseteq S^+$ and both A and $B^{P\sharp S}$ are well-founded, if the detour condition $\Delta^{P\sharp S}_{B;A}$ holds, then the union $A \cup B^{P\sharp S}$ is well-founded, as is the more constricted union $(A \cup B)^{P\sharp S}$.*

Proof. Let $\underline{A}$ and $\underline{B}$ be relations A and B, respectively, restricted to those pairs (x, y) for which y is an $A \cup B$-immortal element (of X). Assuming A and $B^{P\sharp S}$ are well-founded, so are $\underline{A}$ and $\underline{B}^{P\sharp S}$. Consider any pair of adjacent steps

$$x \ \underline{B}^{P\sharp S} \ y \ \underline{A} \ z \ .$$

By constriction, the detour $xPy'S^*z$ allowed by (12) in place of $x\underline{BA}z$ is not a viable option, since z is immortal in $A \cup B \subseteq S^+$, and hence y' is immortal in S. Thus, xBy would not actually be constricting with respect to P. So, we always have the following special case of the Jumping criterion $(*)$:

$$\underline{B}^{P\sharp S}\underline{A} \subseteq \underline{A}^+ \cup \underline{B}^{P\sharp S} \ .$$

Note that the B-step on the right is constricting because it is on the left. By Lemma 6, $\underline{A} \cup \underline{B}^{P\sharp S}$ is well-founded, and so is $A \cup B^{P\sharp S}$, as claimed, since it surely terminates for the excluded mortal elements of $A \cup B$.

Finally, $(A \cup B)^{P\sharp S} = A^{P\sharp S} \cup B^{P\sharp S} \subseteq A \cup B^{P\sharp S}$, so, a fortiori, $(A \cup B)^{P\sharp S}$ is well-founded (per Proposition 12). ◻

For binary relations $R_0, \ldots, R_n$, let $R = R_{0:n}$, and let $\Delta_j = \Delta^{R_0\sharp R}_{R_{j+1:n};R_j}$ be the detour $R_{j+1:n}R_j \subseteq R_0R^* \cup R_j^+ \cup R_{j+1:n}$. Preferential Commutation (11) is:

$$\Delta_0 \wedge \Delta_1 \wedge \cdots \wedge \Delta_{n-1} \ . \tag{11}$$

Lemma 26. *The constricted unions $R_{j:n}^{R_0\sharp R}$, $j = 0 .. n$, of preferentially-commuting well-founded binary relations $R_0, \ldots, R_n$ are all well-founded.*

Proof. By induction, starting with $j = n$ (when the conclusion holds by assumption) and working our way to $j = 0$. For the inductive step, given Δ_j and the well-foundedness of $R_{j+1:n}^{R_0\sharp R}$, and substituting $A = R_j$, $B = R_{j+1:n}$, $P = R_0$, and $S = R$ in the previous lemma, we obtain that $(R_j \cup R_{j+1:n})^{R_0\sharp R} = R_{j:n}^{R_0\sharp R}$ is likewise well-founded. The side condition $A \cup B \subseteq S^+$ of the lemma is satisfied by all the detours, as $R_j \cup R_{j+1:n} \subseteq R^+$ for all j. □

We are ready for our main result, namely that the union R of well-founded $R_0, R_1, \ldots, R_n$ is well-founded when the detour conditions (11) hold for them.

Proof (of Theorem 22). Let $A = R_0$ and $B = R_{1:n}$. Lemma 26 tells us in particular ($j = 1$) that $R_{1:n}^{R_0\sharp R} = B^\sharp$ is well-founded. Since Δ_0 means precisely that A jumps over B, Corollary 18 gives the well-foundedness of $A \cup B = R$. □

5 Preferential Rearrangement

As with Jumping with two relations [7, Thm. 54], Preferential Commutation also means that any chain in the union can be rearranged from "a to z", so to speak.

Theorem 27 (Preferential Rearrangement). *If well-founded relations $R_0, R_1, \ldots, R_n$ satisfy*

$$R_{i+1:n}R_i \subseteq R_0R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \textit{for all } i = 0 .. n-1 \; . \tag{11}$$

then finite chains can always be rearranged:

$$R_{0:n}^* \subseteq R_0^* R_1^* \cdots R_n^* \; . \tag{13}$$

Proof. By our main theorem (Theorem 22), the union $R = R_{0:n}$ is well-founded, so we can argue by induction on it. Were this theorem false, there would be counterexamples xR^+t containing an inversion $xR^*R_jR_iR^*t$ with $j > i$, and such that no alternative properly ordered chain $xR_0^*R_1^* \cdots R_n^*t$ would be possible. Now, let $xRyR^*t$ be a *minimal* counterexample, in the sense that each element in the chain is minimal vis-à-vis R: there is no smaller head than x for any counterexample; y is minimal among chains with irreparable inversions that begin with x; and so on.

The counterexample must possess an inversion, hence must comprise at least two steps xR_jyR^+t. By minimality, x being larger than y, the lesser chain yR^+t must be rearrangeable, so there is a chain $xR_jyR_izR_i^*R_{i+1}^* \cdots R_n^*t$ for some particular i and j ($y \neq t$ for sure). If $j \leq i$, all is fine and dandy, meaning that the example was not in fact a counterexample. Otherwise, $j > i$, and by (11) one of the following should hold true: (i) xR_0vR^*z; (ii) xR_i^+z; or (iii) $xR_{i+1:n}z$. But (i) is impossible, because vR^*zR^*t itself would of necessity be a better counterexample, as were it resolvable, so too would be the original example, starting with xR_0v. Also, (ii) is impossible, because $xR_i^+zR_{i:n}^*t$ is a perfectly good rearrangement. Lastly, (iii) is impossible, since goodness (no inversions) of $xR_{i+1:n}zR_i^* \cdots R_n^*t$ would provide a viable rearrangement, while badness of $xR_{i+1:n}zR_i^* \cdots R_n^*t$ would make it a smaller counterexample than xR_jyR^*t, z being less than y in R. □

Well-foundedness is necessary [7, Note 43].

It follows that Preferential Commuting (11) of well founded-relations is equivalent to an ordered version of the condition:

$$R_{i+1:n}R_i \subseteq R_0^+R_1^* \cdots R_n^* \cup R_i^+ \cup R_{i+1:n} \qquad \text{for all } i = 0 \mathinner{..} n-1\,. \tag{11$'$}$$

6 Example: Preferential Dependencies

Term-rewriting systems (see [9]) compute by applying equations to terms left-to-right, replacing arbitrary subterms when they match the left-hand side of an equation. If $\ell = r$ is such an oriented equation, it is used to rewrite a term s by replacing a subterm of s that is an instance $\ell\sigma$ of ℓ with the corresponding right-hand side $r\sigma$, resulting in some new term t. We write $s \to t$.

Termination of the rewriting relation $\to$ of a given system is normally established by showing that each such rewrite results in a decrease in some well-founded term ordering, or, in other words, that $\to\, \subseteq\, >$ for some well-founded relation $>$. A popular method [1] extends the given system of equations with additional replacement rules, called *dependency pairs*, in a way that can make the overall proof easier. Strict decrease in $>$ is only required for top-level applications of rules or their extensions. But only a "quasi-decrease" is needed for applications of (original) rules at *proper* subterms, for some quasi-ordering $\gtrsim$ that is *compatible* with $>$ in the sense that $>\gtrsim\, \subseteq >$ and $\gtrsim> \,\subseteq >$. (See the version in [14].) We very briefly sketch the use of Preferential Commuting to justify variants of this approach.

Given a rewriting system, we deal with four relations: instances $D = \{(\ell\sigma, r\sigma) \mid \ell = r \text{ is a rule}, \sigma \text{ is a substitution}\}$ of original and extended rules; the immediate subterm relation $\rhd$ $[f(\ldots s \ldots) \rhd s]$; the intersection $>$ of $\to^+$ (for the original rules) with some well-founded partial order (think of it as "decreasing rewriting"); and inner-rewriting $\Rightarrow$, which is $\to$ applied to a *proper* (not necessarily immediate) subterm $[f(\ldots s \ldots) \Rightarrow f(\ldots t \ldots)$ if $s \to t]$. It can be seen without difficulty that if there exists any infinite rewrite chain with $\to$, then there also is an infinite $(D \cup \rhd \cup \Rightarrow)$-chain, wherein D occurs infinitely often.

Preferred Commutation for the union E of A: $\Rightarrow$, B: $>$, C: $\rhd$, and D is achieved by ensuring the following detours:

$$> \Rightarrow \,\subseteq\, \Rightarrow^+ >^* \cup > \qquad \rhd \Rightarrow \,\subseteq\, \Rightarrow \rhd \qquad D \Rightarrow \,\subseteq\, \Rightarrow E^* \cup > \tag{ba/ca/da}$$

$$\rhd > \,\subseteq\, \Rightarrow^+ \rhd \qquad D > \,\subseteq\, > \tag{cb/db}$$

$$D \rhd \,\subseteq\, \Rightarrow^* \rhd^+ \cup D\,. \tag{dc}$$

Conditions (ba) and (db) are usually guaranteed by showing that applying a rule can, if anything, only cause a decrease with respect to $>$ (because $D \subseteq \gtrsim$ and $\Rightarrow \subseteq \gtrsim$, for example). Condition (ba) holds automatically when $>$ is on account of an inner step. Conditions (ca) and (cb) hold by the nature of rewriting (recalling that $> \subseteq \rightarrow^+$). For (da) we require a strict decrease in $>$ for each extended rule ($D \subseteq >$), except perhaps for some instances that allow inner rewriting) and then rely on (ba) for an overall decrease. Condition (dc) is what guides the addition of rules: if there is a directed equation $\ell = r \in D$ and $r \rhd s$, but s is not also a subterm of ℓ ($\ell \not\rhd^+ s$), or of an inner reduct of ℓ (after some $\Rightarrow$ steps), then include $\ell = s$ as an extended rule in D. For each extended rule, a strict decrease in the ordering $>$ ensures (da) and (db). Extended rules may engender additional extended rules, per (dc).

Since $>$ and $\rhd$ are well-founded, and $\Rightarrow$ may be assumed well-founded by an inductive argument, all that remains to be shown is that D on its own is well-founded, which it is if $D \subseteq >$.

7 Preferential Jumping

Preferential Commutation (11) generalises the conjunction of conditions (6a,6b) of the Tripartite Theorem 4. Its beauty lies in that it allows initial "preferred" R_0-steps and multiple R_i-steps. It does not, however, generalise condition condition (8b) of Jumping II (Theorem 8).

We can, however, extend Theorem 22 to allow a mix of Preferential Commutation and Jumping, with Jumping taking over from Commuting at some point.

Theorem 28 (Preferential Jumping). *The union $R_{0:n}$ of well-founded relations $R_0, R_1, \ldots, R_n$ is well-founded if, for some k, $0 \le k \le n$,*

$$R_{i+1:n} R_i \subseteq R_0 R_{0:n}^* \cup R_i^+ \cup R_{i+1:n} \qquad \textit{for } i = 0 \mathinner{..} k-1 \tag{$**$}$$

$$R_{i+1:n} R_i \subseteq R_i R_{i:n}^* \cup R_{i+1:n} \qquad \textit{for } i = k \mathinner{..} n-1\ . \tag{$***$}$$

When $k = n$ this leaves only ($**$), which is pure Preferential Commutation (Theorem 22); when $k = 0$ this leaves ($***$), which is Jumping II (Corollary 21).

Proof. By ($***$) and Jumping II, $R_{k:n}$ is well-founded. Taking that into account, by ($**$) and Preferential Commuting, $R_{1:n}$ is. □

By iterating the similar result for Jumping for two relations [7, Thm. 54], we get a rearrangement theorem with conditions as for Corollary 21, that is, ($***$) for $k = 0$. Interestingly, this seems to require an inductive proof, unlike Theorem 27. Then, combining that with Theorem 27, we get the following result.

Theorem 29. *Well-founded relations $R_0, R_1, \ldots, R_n$ satisfying the Preferential Jumping conditions ($**$,$***$) for some k have re-arrangeable finite chains:*

$$R_{0:n}^* \subseteq R_0^* R_1^* \cdots R_n^*\ . \tag{13}$$

8 Formalising the Proof

All the results of the preceding sections have been verified using Isabelle/HOL 2005.[4] When formalising this work in Isabelle, we faced a problem in defining "well-foundedness" and "relational composition" since these are defined in exactly opposite ways in the term-rewriting and interactive theorem-proving communities. Fortunately, the two notions are almost always used together, meaning that the two effects cancel each other out, as we explain next.

In Isabelle, the well-foundedness and composition of relations are as follows: *Relation R is* well-founded *if there is no infinite descending chain* where $x <_R y$ means $(x, y) \in R$, and descent goes to the left:

$$\cdots <_R x_n <_R x_{n-1} <_R \cdots <_R x_1 <_R x_0 .$$

The Isabelle definition below is the positive form: a relation R is well-founded iff the principle of well-founded induction over R holds for all properties P:

```
wf ?R == ALL P.
   (ALL x. (ALL y. (y, x) : ?R --> P y) --> P x)
  --> (ALL x. P x)
```

We display Isabelle code explicitly so that readers can make a visual connection with our repository. In this definition, the question mark symbol `?` indicates implicit universal quantification and so `?R` is a free variable (parameter) that is instantiated. The explicit quantifiers are `ALL` and `EX`.

Next, we give its equivalent, which says that a relation R is well-founded if every non-empty set Y has an R-minimal member:

```
wf ?R = (ALL Y x. x : Y --> (EX z:Y. ALL y. (y, z) : ?R --> y ~: Y))
```

Then the Isabelle expression that states precisely that `wf` R iff there are no infinite descending chains is as follows, where `Suc` signifies successor in the naturals:

```
wf ?R = (~ (EX f. ALL i. (f (Suc i), f i) : ?R))
```

The symbol `~` encodes classical negation and infix `:` encodes $\in$, so `~:` encodes $\notin$.

In Isabelle, the *composition* of relations R and S (denoted `O`) is defined by

```
?R O ?S == {(x, z). EX y. (x, y) : ?S & (y, z) : ?R}
```

$$R \circ S = \{(x, z) \mid \exists y. (x, y) \in S \;\&\; (y, z) \in R\} .$$

[4] After 2005, it became too onerous to keep pace with changes in Isabelle. This does not detract from our verification in any way since Isabelle 2005 is a trusted system. Instructions on running the proofs are at `http://users.cecs.anu.edu.au/~jeremy/isabelle/2005/gen/tripartite-README`.

Our notation RS from Sect. 2 and the Isabelle notation $R \circ S$ for "relational composition" are inverses, obeying $RS = (R^{-1} \circ S^{-1})^{-1}$:

$$RS = S \circ R = \{(a, c) \mid \exists b. (a, b) \in R \ \& \ (b, c) \in S\}$$
$$(RS)^{-1} = R^{-1} \circ S^{-1} = \{(c, a) \mid \exists b. (c, b) \in S^{-1} \ \& \ (b, a) \in R^{-1}\} \ .$$

Since the Isabelle definitions of composition `O` and `wf` of well-founded are *both* mirror images of those from this paper, our Isabelle theorems and the theorems in this paper correspond exactly: if only one were different, we would have to reverse the order of relation composition to make the two notions coincide. For example, the Jumping theorem for two relations [11] appears in our repository as below, where binary `Un` encodes union ($\cup$) and `^*` encodes transitive closure:

```
[| ?S O ?R <= (?R O (?R Un ?S)^* ) Un ?S; wf ?R; wf ?S |]
 ==> wf (?R Un ?S)
```

Using the positive definition of well-foundedness leads to Isabelle proofs rather different from our original pen-and-paper proofs. Consider, on the one hand, the arguments given in Sect. 1 and in the proof of Theorem 4, involving infinite sequences in which A is preferred, and in which members are A-minimal, with – on the other hand – the argument in the proof of Lemma 19, which chooses a $C^\flat$-minimal B-minimal immortal element. This latter proof reflects much better the flavour of the arguments used in the Isabelle proofs.

We have formulated two distinct Isabelle proofs of the crucial Lemma 25, one along the lines of that of Lemma 19 and another that follows the proof in Sect. 4, with relations restricted to immortal elements.

9 Conclusion and Prospects

Previous work provided sufficient conditions for the union of two well-founded orderings to be well-founded. We discovered a corresponding result for the union of three well-founded orderings and discussed how our sufficient conditions differ from those (viz. Jumping) that result from repeatedly applying the result for two orderings.

We then repackaged the proof of this result for three orderings to extend it to the union of any number of well-founded orderings – in a condition called Preferential Commutation. We showed that whenever there is a finite chain in the union, then there is also one between its two endpoints that takes steps from the relations one after the other, in order. We also gave an example of its use in proofs of termination of rewriting. Finally, we combined Jumping with Preferential Commutation. We expect these results to have significant and varied applications, concomitant with the versatility of binary Jumping.

Usually, when formalising a result, the pen-and-paper proofs have been completed, but in our case, the situation was the opposite. We actually found some proofs using Isabelle and have reworded them for presentation here. The proofs in Isabelle all use "positive" notions (`wf`) rather than "negative" notions ("no infinite chains"). In this case, formalising Theorem 4 involved splitting the proof up into lemmas, which in fact led us to Lemmas 19 and 25, and thence to formulating and proving Theorem 22. As always, formalising a proof confirms that no details have been overlooked or other errors made.

The answer to the question whether the following conditions suffice in the quadripartite case has so far eluded us:

$$(B \cup C \cup D)A \subseteq A(A \cup B \cup C \cup D)^* \cup B \cup C \cup D \qquad \text{(3a,11a)}$$

$$(C \cup D)B \subseteq A(A \cup B \cup C \cup D)^* \cup B^+ \cup C \cup D \qquad \text{(3b,11b)}$$

$$DC \subseteq B(B \cup C \cup D)^* \cup C^+ \cup D\ . \qquad \text{(cf. 3c,11c)}$$

All we can say is the following about any counterexample (where these conditions hold, the individual relations are well-founded, but their union is not):

- $A \cup B$ is not well-founded; for, were it, then Theorem 4 (for relations $A \cup B$, C and D) would give us well-foundedness of the union.
- $(C \cup D)^{A\sharp A \cup B \cup (C \cup D)}$ is not well-founded; for, if it were, then $(B \cup (C \cup D))^{A\sharp A \cup B \cup (C \cup D)} = (B \cup C \cup D)^\sharp$ would also be well-founded by Lemma 25, whence the union would also be by Corollary 18.

Unfortunately, these considerations have not yielded a counterexample.

Further matters worth exploring include:

- What effect would transitivity of the individual relations have on the conditions for well-foundedness? It is known to allow weakening of the Jumping criterion [7]. This suggests a weakening of the first Tripartite condition (6a):

$$(B \cup C)A \subseteq A(A \cup B \cup C)^* \cup (B \cup C)^+\ . \qquad \text{(6a}^+\text{)}$$

- Can we obtain a better understanding of the detour condition Δ that might allow the results reported here to be extended even further? For example, can we exploit the fact that the proof of the crucial Lemma 25 holds with $A(A \cup B^{P\sharp S})^*$ on the right of Eq. (12), not just A^+? This has the effect of weakening the second Tripartite condition (6b) to

$$CB \subseteq A(A \cup B \cup C)^* \cup B(B \cup C^\flat)^* \cup C\ , \qquad \text{(6b}^\flat\text{)}$$

  which is why, in Example 9(a), there had to be an immortalising A-step out of what would otherwise have been a perfectly nice $BC^\flat BB$ detour in place of the offending CB cycle, and not the unacceptable $BCBB$ cycling detour.
- Can one "extract" any code (semi-) automatically? For example, if we express results in the contrapositive, then given an infinite descending chain in one relation, can we derive an infinite descending chain in another, as in the manual proof of Lemma 16?
- Focussing on the infinite descending chains, do these results have applications in terms of liveness?
- One of the motivations for this work is the search for novel termination orderings, particularly for term rewriting. The conditions herein may be applicable to a path ordering based on Takeuti's ordinal diagrams [18], for which ramified jumping conditions play a rôle.

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