The Ordinal Path Ordering

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Abstract

We reformulate Okada's version of Takeuti's ordinal diagrams as inference rules in the style of the abstract path ordering.

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1 Purpose

Ordinals are used in proof-theoretical investigations to chraracterize the logical complexity of systems of analysis and of specific mathematical theorems. Ordinal diagrams, the original version of which is due to Takeuti [12], are one of the most powerful syntactic notations of ordinals that have been devised. They are related to the Friedman's [11] and Kriz's [6] gap version of Kruskal's famous Tree Theorem [7, 8], in that the well-orderedness of diagrams follows from the Gap Tree Theorem. Partially ordered versions of ordinal diagrams are only possible in restricted cases; see [10, 5, 4].

We reformulate ordinal diagrams in the style Okada [9] by ignoring forests (i.e. unconnected trees), which can be compared as multisets of trees [2]. This re-articulation highlights the (as yet unexploited) similarity of the ordering of diagrams with the abstract path ordering [1], designed to prove termination of term rewriting systems.

2 Atomic Case

Given base sets Σ and Π, well-ordered by a *precedence* $\gtrdot$, with all of Π greater than all of Σ, we define a well-ordering $>_\infty$ over *unordered* trees T with leaves from Σ and (internal) nodes from Π, that is:

$$T ::= \Sigma \mid \Pi(T, \ldots, T)$$

Notation: s, t, s_k, t_ℓ are trees of T; α, β are nodes from Π; a, b are leaves from Σ; $u, v \in T \setminus \Sigma$, the non-leaf trees.

Stratified subtrees. By an α-subtree we mean an immediate subtree of some α node in the tree for which there are no smaller nodes en route from the root. Define the relation $\vartriangleright_\alpha$ as follows:

$$\frac{\beta \geq \alpha \quad s \trianglerighteq_\alpha u}{\beta(\ldots, s, \ldots) \vartriangleright_\alpha u}$$

where $>$ here is $\gtrdot$ (the ordering on nodes). As usual, we are using $\geq$ and $\trianglerighteq$ for the reflexive closures.

This relation is transitive.

The following three definitions are mutually recursive.

Minimal operator. *The smallest node in a tree (or trees) that is greater than α.*

$$\begin{aligned} \mu_\alpha(a) &= \infty \\ \mu_\alpha(\beta(s_1,\dots,s_n)) &= \begin{cases} \min\{\beta, \mu_\alpha(s_1,\dots,s_n)\} & \beta > \alpha \\ \mu_\alpha(s_1,\dots,s_n) & \alpha \geq \beta \end{cases} \\ \mu_\alpha(s_1,\dots,s_n) &= \min\{\mu_\alpha(s_1),\dots,\mu_\alpha(s_n)\} \end{aligned}$$

where $>$ is $\gtrdot$ and minima are taken with respect to $\gtrdot$, with ∞ greater than all node values.

Stratified ordering. For each α, the following is a well-ordering.

$$\frac{a > b}{a >_\alpha b}\ (\alpha 1) \qquad \frac{}{u >_\alpha b}\ (\alpha 2) \qquad \frac{u \rhd_\alpha \circ \geq_\alpha v}{u >_\alpha v}\ (\alpha 3) \qquad \frac{u >_{\mu_\alpha\{u,v\}} v \quad u >_\alpha / \rhd_\alpha v}{u >_\alpha v}\ (\alpha 4)$$

where $>$ is $\gtrdot$ and $u >_\alpha / \rhd_\alpha v$ means $u >_\alpha s$ for every $s \lhd_\alpha v$. Recall that a and b are leaves; u and v are not. Clearly, each stratum $>_\alpha$ has the "stratified" subtree property, namely: $x \rhd_\alpha y$ implies $x >_\alpha y$.

We note that $u >_\beta v$ iff $u >_\gamma v$ whenever $\gamma = \mu_\alpha(u,v) \gtrdot \beta \gtrdot \alpha$, there being no β-subtrees in u or v, so ($\alpha 3$) is not applicable and the second hypothesis of ($\alpha 4$) is vacuous.

Target ordering. The *ordinal path ordering* $>_\infty$ is a dependent lexicographic pair, consisting of the ordering $\gtrdot$ on nodes followed by the multiset extension of the ordering $>_\alpha$, selected by the shared node α, on immediate subtrees.

$$\frac{}{u >_\infty b}\ (\infty 1) \qquad \frac{\alpha > \beta}{\alpha(\dots,s_k,\dots) >_\infty \beta(\dots,t_\ell,\dots)}\ (\infty 2) \qquad \frac{\{\dots,s_k,\dots\} \gg_\alpha \{\dots,t_\ell,\dots\}}{\alpha(\dots,s_k,\dots) >_\infty \alpha(\dots,t_\ell,\dots)}\ (\infty 3)$$

where $>$ is $\gtrdot$ and $\gg_\alpha$ is the multiset extension of $>_\alpha$.

Another way to express this top-level ordering is to extend $\gtrdot$ so that non-leaves are compared by comparing their root nodes in the node ordering and non-leaves are always greater than leaves, and to define $\gg$ to compare non-leaf trees with equal root values to each other by comparing the multiset of immediate subtrees in the order indexed by the root-node value. (Trees with incomparable roots are incomparable.) Then $>_\infty$ is the union of these two (disjoint) orderings, and we can economize by using the following rules:

$$\frac{s \gtrdot t}{s >_\infty t}\ (\infty 1{,}2) \qquad \frac{s \gg t}{s >_\infty t}\ (\infty 3)$$

3 Tree Case

Given a base set Σ with minimal element 0, well-ordered by $\gtrdot$, we define a well-ordering $>_\infty$ over *unordered* (trees of) trees T with leaves from Σ and *trees* for internal nodes, that is:

$$T ::= \Sigma \mid T(T,\dots,T)$$

There is no longer a separate node vocabulary Π. Hence, the ordering on nodes is no longer $\gtrdot$, but instead is the lowest stratum $>_0$ of the same ordering as is being defined on trees. The definition is the same, except that $\gtrdot$ is replaced by $>_0$ throughout.

Notation: $\alpha, \beta, s, t, s_k, t_\ell$ are trees of T; a, b are leaves from Σ; $u, v \in T \setminus \Sigma$.

The following four definitions are mutually recursive.

Stratified subtrees. *A subtree of an α node with no smaller nodes en route.*

$$\frac{\beta \geq \alpha \quad s \trianglerighteq_\alpha u}{\beta(\dots,s,\dots) \rhd_\alpha u}$$

where $>$ here is $>_0$.

Minimal operator. As above: *The smallest node in a tree (or trees) that is greater than α.*

$$
\begin{aligned}
\mu_\alpha(a) &= \infty \\
\mu_\alpha(\beta(s_1,\ldots,s_n)) &= \begin{cases} \min\{\beta, \mu_\alpha(s_1,\ldots,s_n)\} & \beta > \alpha \\ \mu_\alpha(s_1,\ldots,s_n) & \alpha \geq \beta \end{cases} \\
\mu_\alpha(s_1,\ldots,s_n) &= \min\{\mu_\alpha(s_1),\ldots,\mu_\alpha(s_n)\}
\end{aligned}
$$

where $>$ is $>_0$ and minima are taken with respect to $>_0$.

Stratified ordering. For each α, the following is a well-ordering.

$$
\frac{a > b}{a >_\alpha b} \qquad \frac{}{u >_\alpha b} \qquad \frac{u \rhd_\alpha \circ \geq_\alpha v}{u >_\alpha v} \qquad \frac{u >_{\mu_\alpha\{u,v\}} v \quad u >_\alpha / \rhd_\alpha v}{u >_\alpha v}
$$

where $>$ is $>_0$.

Target ordering. *Dependent lexicographic pair, ordering the roots followed by the multiset extension of the selected ordering on immediate subtrees.*

$$
\frac{}{u >_\infty b} \qquad \frac{\alpha > \beta}{\alpha(\ldots,s_k,\ldots) >_\infty \beta(\ldots,t_\ell,\ldots)} \qquad \frac{\{\ldots,s_k,\ldots\} \gg_\alpha \{\ldots,t_\ell,\ldots\}}{\alpha(\ldots,s_k,\ldots) >_\infty \alpha(\ldots,t_\ell,\ldots)}
$$

where $>$ is the node ordering $>_0$ and $\gg_\alpha$ is the multiset extension of $>_\alpha$.

4 Examples

We focus in the coming examples on unary trees (strings) and the atomic ordering, though the tree case is the more interesting.

4.1 An Example

Consider the rewriting rule $ffx \to fgfx$, with $\Pi = \{f, g\}$ and Σ anything, and let $f \gtrdot g$. First, notice that $fx >_\infty gy$, for any y, and in particular $fx >_\infty gfx$. So the target ordering $>_\infty$ does not have the subtree property (which is what makes it useful in this—and many other—cases).

Since f is the largest node value in the precedence, we have $fx >_\infty gy$ for all trees x and y. Similarly, we have

$$
\cfrac{\cfrac{\cfrac{\cfrac{f \gtrdot g}{fx >_\infty gfy}\,(\infty 2) \qquad \cfrac{}{fx >_f / \rhd_f gfy}}{fx >_f gfy}\,(\alpha 4)}{\{fx\} \gg_f \{gfy\}}}{ffx >_\infty fgfy}\,(\infty 3)
$$

since there are no f-subtrees in gfy.

Since $>_\infty$ is total and well-ordered, it cannot be monotonic. Still we want $s >_\infty t$ whenever s rewrites to t; in other words, we want $vffw >_\infty vfgfw$, for all $v, w \in \Pi^*$.

We show that if $u >_\infty v$, for u and v having the same root node, then $\alpha u >_\infty \alpha v$, for any node α (f or g). There are four cases: $gx >_\infty gy$ implies $fgx >_\infty fgy$ and $ggx >_\infty ggy$ and $fx >_\infty fy$ implies $ffx >_\infty ffy$ and $gfx >_\infty gfy$.

It is easy to verify that $g\,x >_\infty g\,y$ implies $f\,g\,x >_\infty f\,g\,y$ for all strings x and y:

$$\dfrac{\dfrac{g\,x >_\infty g\,y \qquad \overline{g\,x >_f/\rhd_f g\,y}}{g\,x >_f g\,y}\;(\alpha 4)}{f\,g\,x >_\infty f\,g\,y}\;(\infty 3)$$

there being no f-subtrees in gy. (So this is also true for larger alphabets, as long as f is maximal.)

Furthermore, $fx >_\infty fy$ implies $ffx >_\infty ffy$:

$$\dfrac{\dfrac{fx >_\infty fy \qquad \begin{array}{c}\vdots\\ \overline{fx >_f/\rhd_f fy}\end{array}}{fx >_f fy}\;(\alpha 4)}{ffx >_\infty ffy}\;(\infty 3)$$

because, for any $z \trianglelefteq_f y$, we have

$$\dfrac{\dfrac{\overline{fx \rhd_f x}}{fx >_f x}\;(\alpha 3) \qquad \dfrac{\dfrac{fx >_\infty fy}{x >_f y} \qquad y \trianglerighteq_f z}{x >_f z}}{fx >_f z}$$

and $x >_\alpha y \trianglerighteq_\alpha z$ always implies $x >_\alpha z$ on account of the subtree property and transitivity. Virtually the same argument (with one additional step) shows that $fx >_\infty fy$ implies $gfx >_\infty gfy$.

Lastly, one can show that $g\,x >_\infty g\,y$ implies $g\,g\,x >_\infty g\,g\,y$:

$$\dfrac{\dfrac{\dfrac{g\,x >_\infty g\,y}{\left[\; g\,x >_f g\,y \;\right]}\;(\alpha 4) \qquad \dfrac{g\,x >_g x >_g y}{g\,x >_g/\trianglerighteq_g y}}{g\,x >_g g\,y}\;(\alpha 4)}{g\,g\,x >_\infty g\,g\,y}\;(\infty 3)$$

there being no f-subtrees in $g\,y$, and $x >_g y$ being the only way that one can have $g\,x >_\infty g\,y$. The bracketed step is omitted if f does not occur in x or y.

4.2 A Counterexample

For the purposes of a counterexample in [3] (showing the necessity of a subterm condition for the critical-pair lemma in the case of normal conditional rewriting), the following inequalities were needed: $a > b$, $fa > ga$, $hfa > c > kfa$, $c > kgb$, $fx > hfx$ (!), $fx > kgb$, $hx > kx$. For that, we can interpret terms as follows:

$$\begin{array}{rcll}
[\![a]\!] &=& 1 & \\
[\![b]\!] &=& 0 & \\
[\![c]\!] &=& 0(1(1),1) & \\
[\![h(x)]\!] &=& 0([\![x]\!],2) & \text{i.e. } [\![h]\!] = \lambda x.0(x,2) \\
[\![f(x)]\!] &=& 1([\![x]\!]) & \\
[\![k(x)]\!] &=& 0([\![x]\!]) & \\
[\![g(x)]\!] &=& 0([\![x]\!]) &
\end{array}$$

5 Conclusion

The use of ordinal diagrams, as made simple by the above inference rules, holds out some hope for helping in difficult (non-simplifying) termination proofs.

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References

1 Nachum Dershowitz. Jumping and escaping: Modular termination and the abstract path ordering. *Theoretical Computer Science*, 464:35–47, 2012. Available at `http://nachum.org/papers/Toyama.pdf`.

2 Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Communications of the ACM (CACM)*, 22(8):465–476, August 1979.

3 Nachum Dershowitz and Mitsuhiro Okada. Proof-theoretic techniques and the theory of rewriting. In *Proceedings of the Third IEEE Symposium on Logic in Computer Science (LICS)*, pages 104–111, Edinburgh, Scotland, July 1988. Available at `http://nachum.org/papers/ProofTheoretic.pdf`.

4 Nachum Dershowitz and Iddo Tzameret. Gap embedding for well-quasi-orderings. *Electr. Notes Theor. Comput. Sci.*, 84:80–90, 2003. Available at `http://www.sciencedirect.com/science/article/pii/S1571066104808466`.

5 Lev Gordeev. Generalizations of the Kruskal-Friedman theorems. *The Journal of Symbolic Logic*, 55(1):157–181, 1990.

6 Igor Kříž. Well-quasiordering finite trees with gap-condition. Proof of Harvey Friedman's conjecture. *Ann. of Math.*, 130:215–226, 1989.

7 Joseph B. Kruskal. Well-quasiordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. American Mathematical Society*, 95:210–223, May 1960.

8 Crispin St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, October 1963.

9 Mitsuhiro Okada. A simple relationship between Buchholz's new system of ordinal notations and Takeuti's system of ordinal diagrams. *The Journal of Symbolic Logic*, 52(3):577–581, 1987.

10 Mitsuhiro Okada and Gaisi Takeuti. On the theory of quasi-ordinal diagrams. In *Logic and Combinatorics (Arcata, CA, 1985)*, volume 65 of *Contemp. Math.*, pages 295–308. Amer. Math. Soc., Providence, RI, 1987.

11 Stephen G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In L. A. Harrington, editor, *Harvey Friedman's Research on the Foundation of Mathematics*, pages 87–117. Elsevier, 1985.

12 Gaisi Takeuti. Ordinal diagrams. II. *J. Math. Soc. Japan*, 12:385–391, 1960.