Leanest Quasi-Orderings

Nachum Dershowitz [1]

School of Computer Science, Tel Aviv University,
Ramat Aviv, Tel Aviv 69978, Israel

E. Castedo Ellerman

40 Moreland Ave., Lexington, MA 02421, USA

Abstract

A convenient method for defining a quasi-ordering, such as those used for proving termination of rewriting, is to choose the minimum of a set of quasi-orderings satisfying some desired traits. Unfortunately, a minimum in terms of set inclusion can be non-existent even when an intuitive "minimum" exists. We suggest an alternative to set inclusion, called "leanness", show that leanness is a partial order on quasi-orderings, and provide sufficient conditions for the existence of a "leanest" member of a set of total well-founded quasi-orderings.

Key words: Quasi-ordering, well-quasi-ordering, lexicographic path ordering

In my poor, lean lank face
nobody has ever seen
that any cabbages were sprouting.

—Abraham Lincoln

1 Introduction

Well-founded partial orders (admitting no infinite strictly decreasing sequences) are the standard tool for proving algorithm termination. States of the program are assigned values in the underlying set, such that program

Email addresses: nachum.dershowitz@cs.tau.ac.il (Nachum Dershowitz), castedo@castedo.com (E. Castedo Ellerman).

[1] Supported in part by the Israel Science Foundation (grant no. 250/05).

steps always result in a decrease in the ordering, thereby establishing termination. Quasi-orderings (reflexive-transitive binary relations, also called "quasi-orders" or "pre-orders") are often more convenient for this purpose than partial or total orders: the ordering on states induced by a partial order of values is in fact a quasi-ordering. In this paper, the unqualified term "ordering" will always refer to a *quasi-ordering*.

A non-empty set of quasi-orderings can be defined by a set of conditions (such as weak-monotonicity and weak-subterm for *quasi-simplification orderings* [**?**]); then we can identify a particular, ideal ordering by choosing the minimum ordering in the set. Unfortunately, at times, a set of orderings will have no minimum in the usual set-theoretic sense of minimum. (One example where there is a meaningful such minimum may be found in [**?**].) Accordingly, this paper suggests a more general definition of "minimum" that leads more often to a unique ordering, which is intuitively the desired minimum ordering.

The notion of "leanness" defined here embodies a preference for thinness of quasi-orderings near their bottom. By "thinness" we mean that equivalence classes are smaller. Our definition is especially useful when defining orderings by incrementally adding constraints, since one wants to commit as late as possible. Investigations of alternate choices of partial orders for rewriting, specifically regarding multiset orderings, include [4,6,7]. A classification of some string orderings appears in [8].

We begin with a motivating example. Then, in Sections 3 and 4, we define initial segments for quasi-orderings and approach them from the standpoint of a binary relation on quasi-orderings. With these building blocks in place, Section 5 defines the leanness relation. This is followed by a section devoted to conditions guaranteeing the existence of a leanest quasi-ordering. Section 7 illustrates the ideas with an example of a leanest tree ordering. We show that a natural set of desiderata for a lexicographically biased simplification ordering on binary trees does not lead to a unique minimal tree ordering satisfying those conditions, but that the leanest ordering satisfying them is precisely the well-known *lexicographic path ordering*. We conclude with a brief discussion.

2 A String Example

As usual, a quasi-ordering A may be viewed as a set of ordered pairs, where each ordered pair is a comparison. We use $x \precsim_A y$ to denote $(x, y) \in A$, a comparison according to ordering A. As usual, $x \prec_A y$ will denote $x \precsim_A y$ but not $y \precsim_A x$.

Example 1 *Consider a simple example of a set of conditions defining a set of* *quasi-orderings. Let $\mathcal{Q}$ denote the set of all quasi-orderings A of strings over $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ that satisfy all three of the following conditions:*

(1) $\varepsilon \precsim_A \mathbf{a} \precsim_A \mathbf{b} \precsim_A \mathbf{c}$*;*
(2) if $v \precsim_A w$ *and* $x \precsim_A y$*, then* $vx \precsim_A wy$*;*
(3) if $v \prec_A w$*, then* $vx \precsim_A wy$*,*

for all strings $v, w \in \Sigma^*$ and symbols $x, y \in \Sigma$.

Intuitively it might seem that there should be a minimum ordering that satisfies these conditions. In it, the empty string ε would be the smallest element, followed by **a**, **b** and **c** in strictly increasing order. Following this pattern we can enumerate a total "length-first lexicographic" ordering in the following fashion:

$$\varepsilon \prec \mathbf{a} \prec \mathbf{b} \prec \mathbf{c} \prec \mathbf{aa} \prec \mathbf{ab} \prec \mathbf{ac} \prec \mathbf{ba} \prec \cdots .$$

Example 2 *Let F be the above quasi-ordering, which may be defined as follows:*

$$v \precsim_F w \;:=\; f(v) \leq f(w),$$

where f is the homomorphism:

$$\begin{aligned} f(\varepsilon) &= 1\,, \\ f(w\mathbf{a}) &= f(w)3\,, \\ f(w\mathbf{b}) &= f(w)3 + 1\,, \\ f(w\mathbf{c}) &= f(w)3 + 2\,, \end{aligned}$$

for any string w. □

A natural definition for the minimum (or, "least defined") ordering is the minimum in terms of the subset relation: the ordering that, as a set of comparisons, is a subset of all other orderings in $\mathcal{Q}$. Surprisingly, perhaps, F is not a minimum of $\mathcal{Q}$ in this sense. Furthermore, no such minimum in terms of subset exists.

To see this, consider another ordering that intuitively is greater than F, but is not a super-set of F.

Example 3 *We make an intuitively less minimal ordering G by forcing* **a** *and* **b** *to be equivalent. Like F, let ε be strictly less than* **a** *and* **c** *be strictly greater than* **b***. The next equivalence classes in G are*

$$\{\mathbf{aa}, \mathbf{ab}, \mathbf{ba}, \mathbf{bb}\}\,,$$

followed (strictly) by

$$\{\mathbf{ac}, \mathbf{bc}\} \; .$$

Like F, we define the entirety of G with a mapping:

$$v \precsim_G w \;\; := \;\; g(v) \leq g(w),$$

where g is the string-homomorphism:

$$\begin{aligned} g(\varepsilon) &= 1 \; , \\ g(w\mathbf{a}) &= g(w)2 \; , \\ g(w\mathbf{b}) &= g(w)2 \; , \\ g(w\mathbf{c}) &= g(w)2 + 1 \; , \end{aligned}$$

for any string w.

This ordering G also satisfies all the conditions for bona fide membership in $\mathcal{Q}$. □

With **a** and **b** equivalent in G but strictly increasing in F, a more striking difference between G and F is made possible. In G the string **ac** is strictly greater than **ba**, since $g(\mathbf{ac}) = 5$ and $g(\mathbf{ba}) = 4$. However, in F, **ac** is strictly less than **ba**, since $f(\mathbf{ac}) = 11$ and $f(\mathbf{ba}) = 12$.

The following diagram displays comparisons for F, G, and any relation S that is a subset of both F and G:

	F	G	S
$\mathbf{a} \precsim \mathbf{b}$	$\surd$	$\surd$	$\surd$ or $\times$
$\mathbf{b} \precsim \mathbf{a}$	$\times$	$\surd$	$\times$
$\mathbf{ac} \precsim \mathbf{ba}$	$\surd$	$\times$	$\times$

We see that $\mathbf{ac} \precsim_S \mathbf{ba}$ must not hold even when $\mathbf{a} \prec_S \mathbf{b}$ does. This means that any ordering that is a subset of both F and G cannot satisfy the third condition for membership in $\mathcal{Q}$. Thus, $\mathcal{Q}$ cannot have an ordering that is the minimum in terms of the subset relation, or "subset minimum".

Nevertheless, intuitively, F is "more minimal" than G, since it omits the inequality $\mathbf{b} \precsim \mathbf{a}$. So, instead of comparing quasi-orderings in terms of the subset relation, we propose an alternative relation: "leanness" of orderings. In this alternative relation of quasi-orderings, F is in fact the "leaner" of the two. Both F and G "start off" the same, with $\varepsilon \prec \mathbf{a}$, but then diverge with the comparison of $\mathbf{a}$ and $\mathbf{b}$. Whereas G has an equivalence class of $\mathbf{a} \simeq \mathbf{b}$, F has only $\mathbf{a}$. This is why F is to be preferred.

In the next section, we formalize a construct that captures how orders "start off". This construct will be a building block for a general definition of a "leanness" relation on quasi-orderings.

3 Initial Segments

The comparisons of **a**, **b**, **ac** and **ba** in Examples 2 and 3 proved problematic because viewing comparisons outside the context of the comparisons around them results in a "subset tie". By taking into account what happens lower down in an ordering, such ties can be avoided. The rationale is that the constraints that characterize the family of orderings in question are typically inductive, for which reason the ordering imposed on smaller elements ought to be more significant.

Instead of looking at comparisons by themselves, we want to work with a construct that takes into account the position of the comparisons. For that purpose, we extend the standard notion of initial segment for well-orders to also cover quasi-orderings.

Definition 4 (Initial segment) *For any quasi-ordering A and set S, the* initial segment of A below S *is the set of all ordered pairs in A restricted to the down-set of S. In symbols:*

$$A \upharpoonright S \;:=\; \{(x,y) \in A : y \precsim_A z, z \in S\}\ .$$

When S is a singleton $\{z\}$, we write simply $A \upharpoonright z$.

Example 5 *Let N be the natural $\leq$ order of the natural numbers. Then*

$$N \upharpoonright 3 \;=\; \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}\ . \qquad \square$$

This quasi-ordered set (or "qoset") $(\{1,2,3\},\leq)$ can be presented as

$$\{1 \leq 2 \leq 3\}\ ,$$

with reflexivity and transitivity understood.

The above example involves a total order, where the definition of initial segment does not differ much from that of a well-order.

Example 6 *Let D be the order relation "x divides y" on the natural numbers.*

$$\begin{aligned} D \restriction 3 &= \{(1,1),(1,3),(3,3)\}\ , \\ D \restriction 4 &= \{(1,1),(1,2),(1,4),(2,2),(2,4),(4,4)\}\ . \qquad \square \end{aligned}$$

The desirable property of closure of initial segments under arbitrary union follows easily from the definition. For example,

$$(D \restriction 3) \cup (D \restriction 4) \;=\; D \restriction \{3,4\}\ .$$

Proposition 7 *For any quasi-ordering A and set of sets $\mathcal{S}$,*

$$\bigcup_{S \in \mathcal{S}} (A \restriction S) = A \restriction \Big(\bigcup_{S \in \mathcal{S}} S \Big)\ .$$

Other useful results follow easily from the definition.

Proposition 8 *For any quasi-ordering A and set S,*

$$A \restriction S \subseteq A\ .$$

Proposition 9 *For any quasi-ordering A and sets S and T,*

$$S \subseteq T \;\; \textit{implies} \;\; A \restriction S \subseteq A \restriction T\ .$$

Proposition 10 *For any quasi-orderings A and B and set S,*

$$A \subseteq B \;\; \textit{implies} \;\; A \restriction S \subseteq B \restriction S\ .$$

The closure of initial segments under intersection is not as direct a result as for union.

Note that an initial segment $A \restriction z$ is not usually a quasi-ordering of the whole of A. In the example above, since $(4, 4)$ is not in $N \restriction 3$, the relation is not reflexive (and thus not a quasi-ordering) on all the natural numbers. However, $N \restriction 3$ is a quasi-ordering on the set $\{1, 2, 3\}$. Such sets can be determined by looking at the "domain" of a binary relation (set of ordered pairs).

Definition 11 (Domain) *The* domain *of a binary relation is the set of all elements found as a first component. In symbols:*

$$\mathrm{dom}\, A \;\; := \;\; \{x : \exists y.\ x \leq_A y\} \; .$$

If a set of ordered pairs is a quasi-ordering, then it is a reflexive binary relation, which implies that any element that shows up as a first component must also show up as a second component, and vice-versa.

Proposition 12 *For any quasi-ordering A,*

$$\mathrm{dom}\, A \;\; = \;\; \{y : \exists x.\ x \leq_A y\} \; .$$

So any element that is comparable to anything is in the domain of the quasi-ordering.

Theorem 13 *For any quasi-ordering A and set S, $A \restriction S$ is a quasi-ordering on* $\mathrm{dom}\,(A \restriction S)$*.*

Proof. For any $x \leq_{A \restriction S} y$, both x and y must be in $\mathrm{dom}\,(A \restriction S)$; thus, $A \restriction S \subseteq (\mathrm{dom}\,(A \restriction S))^2$, making $A \restriction S$ a binary relation on $\mathrm{dom}\,(A \restriction S)$.

For every $x \in \mathrm{dom}\,(A \restriction S)$, there exists some $y \in \mathrm{dom}\,(A \restriction S)$ and $z \in S$ such that $x \leq_A y \leq_A z$. Since $x \leq_A x \leq_A z$ must hold, so does $x \leq_{A \restriction S} x$; thus reflexivity holds for $A \restriction S$.

Transitivity of $A \restriction S$ follows easily from transitivity of A. $\square$

Proposition 14 *For any quasi-orderings A and B,*

$$A \subseteq B \;\; \textit{implies} \;\; \mathrm{dom}\, A \subseteq \mathrm{dom}\, B \; .$$

Definition 15 (Initial segment)

(1) *A quasi-ordering B is an* initial segment *of a quasi-ordering A if there exists some set S such that $B = A \restriction S$.*
(2) *For any quasi-ordering A, $\mathcal{I}(A)$ denotes the set of all initial segments of A.*
(3) *For any set $\mathcal{Q}$ of quasi-orderings, $\mathcal{I}(\mathcal{Q})$ denotes the set of all initial segments of members of $\mathcal{Q}$.*

Example 16 *If D is the partial order "x divides y" on natural numbers, then $\mathcal{I}(D \restriction \{3, 4\})$ is*

$$\begin{aligned}
\{\ &\emptyset, \\
&\{(1,1)\}, \\
&\{(1,1),(1,2),(2,2)\}, \\
&\{(1,1),(1,3),(3,3)\}, \\
&\{(1,1),(1,2),(1,4),(2,2),(2,4),(4,4)\}, \\
&\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}\ \} \qquad \square
\end{aligned}$$

Any set of initial segments $\mathcal{I}(A)$ has the empty set $\emptyset$ as a member since $\emptyset = A \restriction \emptyset$. At the other extreme, A itself is also a member of $\mathcal{I}(A)$ as the next theorem will show.

Lemma 17 *For any quasi-orderings A and B,*

$$B \subseteq A \ \textit{implies}\ B \subseteq A \restriction \operatorname{dom} B\ .$$

Proof. Assume $B \subseteq A$. If $x \leq_B y$, then $y \in \operatorname{dom} B$ and $x \leq_A y \leq_A y$. Thus, $x \leq_{A \restriction \operatorname{dom} B} y$. $\square$

By Lemma 17, $A \subseteq A \restriction \operatorname{dom} A$. Actually, they are equal:

Theorem 18 *Any quasi-ordering is an initial segment of itself. Specifically, $A = A \restriction \operatorname{dom} A$.*

Proof. Clearly, $A \restriction \operatorname{dom} A \subseteq A$. With Lemma 17, the result follows. $\square$

Theorem 19 *For any quasi-orderings A and B,*

$$B \in \mathcal{I}(A) \text{ implies } B = A \restriction \operatorname{dom} B\ .$$

Proof. Assume $B = A \restriction S$, for some S. For any $x \leq_{A \restriction \operatorname{dom} B} y$, then $x \leq_A y \leq_A z$, for some $z \in \operatorname{dom} B$. Since $z \leq_B z$, there exists $w \in S$ such that $z \leq_A w$, and thus $y \leq_A w$, which implies $x \leq_{A \restriction S} y$ and thus $x \leq_B y$. Therefore, $A \restriction \operatorname{dom} B \subseteq B$. With Lemma 17, equality is proven. $\square$

Theorem 19 shows that if a quasi-ordering B is an initial segment of quasi-ordering A, there is a "standard" and predictable set S for which B is an initial segment below S, namely the domain of B. This makes it easy to work with initial segments as quasi-orderings and only resort to the $A \restriction S$ form when needed, deducing a suitable S by choosing the domain of the initial segment in question.

With initial segment domains developed, we return to intersections of initial segments.

Theorem 20 (Initial segment intersection) *For any quasi-ordering A and subset $\mathcal{J}$ of its initial segments $\mathcal{I}(A)$,*

$$\bigcap_{B \in \mathcal{J}} B = A \restriction \Big(\bigcap_{B \in \mathcal{J}} \operatorname{dom} B \Big)\ .$$

Proof. Let $D = \bigcap_{B \in \mathcal{J}} \operatorname{dom} B$.

Consider any $x \leq_{\cap B} y$. For all $B \in \mathcal{J} \subseteq \mathcal{I}(A)$, $x \leq_B y$ and so $x \leq_A y$. By Theorem 13, B is reflexive, so $y \leq_B y$, and thus $y \in \operatorname{dom} B$. Therefore, $y \in D$ and thus $x \leq_{A \restriction D} y$.

Consider any $x \leq_{A \restriction D} y$. For some $z \in D$ and for all $B \in \mathcal{J}$, $z \in \operatorname{dom} B$ and $x \leq_A y \leq_A z$. Thus, for all $B \in \mathcal{J}$, $x \leq_{A \restriction (\operatorname{dom} B)} y$, and, by Theorem 19, $x \leq_B y$. Therefore, $x \leq_{\cap B} y$. $\square$

4 Initial Segment Order

An initial segment is a quasi-ordering with a specific relation to another quasi-ordering. It will prove convenient to define a binary relation between quasi-orderings and their initial segments.

Definition 21 (Initial segment relation) *Let $A \sqsubseteq B$ be the relation $A \in \mathcal{I}(B)$. An initial segment of B is a* strict *initial segment, denoted $A \sqsubset B$, if it is not equal to B. If $A \sqsubseteq B$, then B is a* super-segment *of A; it is a* strict super-segment *if $A \neq B$.*

Theorem 22 *The initial segment relation $\sqsubseteq$ is a partial order on quasi-orderings.*

Proof. By Theorem 18, $\sqsubseteq$ is a reflexive relation on quasi-orderings.

Consider any quasi-orderings A and B with $A \sqsubseteq B$ and $B \sqsubseteq A$. From the definition of $\restriction$, we know that $A \subseteq B \subseteq A$, and thus $A = B$. Thus, $\sqsubseteq$ is antisymmetric.

Consider any quasi-orderings A, B and C with $A \sqsubseteq B \sqsubseteq C$. From Theorem 19, $A = B \restriction \operatorname{dom} A$ and $B = C \restriction \operatorname{dom} B$. Since $A \subseteq B \subseteq C$, we conclude that $A \subseteq C \restriction \operatorname{dom} A$, by Lemma 17. If $x \leq_{C \restriction \operatorname{dom} A} y$, then, for some $z \in \operatorname{dom} A \subseteq \operatorname{dom} B$, we have $x \leq_C y \leq_C z$; thus, $x \leq_{C \restriction \operatorname{dom} B} y$ and $x \leq_B y$. Furthermore, $y \leq_C z \leq_C z$; so $y \leq_B z$, which implies that $x \leq_{B \restriction \operatorname{dom} A} y$ and $x \leq_A y$. Thus, $A = C \restriction \operatorname{dom} A$ and thus $A \sqsubseteq C$. Therefore, $\sqsubseteq$ is transitive. □

Theorem 23 (Isomorphic posets) *For any quasi-ordering A, the three partially-ordered sets (posets),*

$$
\begin{aligned}
&(\mathcal{I}(A), \sqsubseteq) \ , \\
&(\mathcal{I}(A), \subseteq) \ , \\
&(\{\operatorname{dom} B : B \in \mathcal{I}(A)\}, \subseteq) \ ,
\end{aligned}
$$

are all isomorphic to each other.

In particular, $A \sqsubseteq B$ implies $A \subseteq B$, since both A and B are initial segments of B.

Proof. Consider any B and C in $\mathcal{I}(A)$. It follows easily from the definition of $\restriction$ that $B \sqsubseteq C$ implies $B \subseteq C$. And it follows easily from the definition of dom that $B \subseteq C$ implies $\operatorname{dom} B \subseteq \operatorname{dom} C$.

Suppose $B \subseteq C$. By Theorem 18, $B = B \restriction \operatorname{dom} B \subseteq C \restriction \operatorname{dom} B$. By Theorem 19, $C \restriction \operatorname{dom} B \subseteq A \restriction \operatorname{dom} B = B$. Thus, $B = C \restriction \operatorname{dom} B$. We conclude that $B \subseteq C$ implies $B \sqsubseteq C$. Finally, $\operatorname{dom} B \subseteq \operatorname{dom} C$ implies $A \restriction \operatorname{dom} B \subseteq A \restriction \operatorname{dom} C$, which implies $B \subseteq C$, by Theorem 19. $\square$

The equivalence of $\sqsubseteq$ and $\subseteq$ does not extend to quasi-orderings in general: quasi-ordering A being a subset of quasi-ordering B certainly does not necessarily make A its initial segment. Compare Proposition 14.

The close tie between partial orders $\subseteq$ and $\sqsubseteq$ suggests the usefulness of a (semi-) lattice structure, with least upper bounds and greatest lower bounds for initial segments.

Definition 24 (Least common super-segment) *For any set of quasi-orderings $\mathcal{Q}$, C is its* least common super-segment*, symbolized $C = \bigsqcup_{A \in \mathcal{Q}} A$, if C is a super-segment of all members of $\mathcal{Q}$ and C is an initial segment of every super-segment of members of $\mathcal{Q}$. The least common super-segment of just two quasi-orderings A and B is denoted $A \sqcup B$.*

A common super-segment does not necessarily exist, unless all the segments are initial segments of some ordering:

Theorem 25 (Least common super-segment existence) *Every set of initial segments $\mathcal{J} \subseteq \mathcal{I}(A)$, for any quasi-ordering A, has a least common super-segment*

$$\bigsqcup_{B \in \mathcal{J}} B \quad = \quad \bigcup_{B \in \mathcal{J}} B \ ,$$

with

$$\operatorname{dom}\Big(\bigsqcup_{B \in \mathcal{J}} B\Big) \quad = \quad \bigcup_{B \in \mathcal{J}} \operatorname{dom} B \ .$$

Proof. The set $\sqcup_{B \in \mathcal{J}} B$ is the least upper bound of $\mathcal{J}$ under poset $(\mathcal{I}(A), \sqsubseteq)$; $\cup_{B \in \mathcal{J}} B$, of $\mathcal{J}$ under $(\mathcal{I}(A), \subseteq)$; $\cup_{B \in \mathcal{J}} \operatorname{dom} B$, of $\{\operatorname{dom} B : B \in \mathcal{J}\}$ under $(\{\operatorname{dom} B : B \in \mathcal{I}(A)\}, \subseteq)$. By Theorem 23, the natural isomorphisms between the posets map the least upper bounds to each other. $\square$

Definition 26 (Greatest common initial segment) *For any set of quasi-orderings $\mathcal{Q}$, C is its* greatest common initial segment*, symbolized $C = \bigsqcap_{A \in \mathcal{Q}} A$, if C is an initial segment of all members of $\mathcal{Q}$ and C is a* *super-segment of every initial segment of members of $\mathcal{Q}$. The greatest common initial segment of just two quasi-orderings A and B is denoted $A \sqcap B$.*

Theorem 27 (Greatest common initial segment existence) *Every set of initial segments $\mathcal{J} \subseteq \mathcal{I}(A)$, for any quasi-ordering A, has a greatest common initial segment*

$$\prod_{B \in \mathcal{J}} B \;=\; \bigcap_{B \in \mathcal{J}} B \,,$$

with

$$\operatorname{dom}\Big(\prod_{B \in \mathcal{J}} B\Big) \;=\; \bigcap_{B \in \mathcal{J}} \operatorname{dom} B \,.$$

The proof is analogous to the previous.

Unlike the case for least common super-segments, a greatest common initial segment always exists. This asymmetry is related to the fact that $\emptyset$ is an initial segment of all quasi-ordering, but there is no single super-segment of all quasi-orderings.

Theorem 28 (Greatest common initial segment general existence) *For any set of quasi-orderings $\mathcal{Q}$, its greatest common initial segment is*

$$\prod_{B \in \mathcal{Q}} B \;=\; \bigcup\Big(\bigcap_{B \in \mathcal{Q}} \mathcal{I}(B)\Big) \,.$$

Proof. For any $C \in \mathcal{Q}$, $\mathcal{I}(C)$ is closed under union and $\cap_{B \in \mathcal{Q}} \mathcal{I}(B) \subseteq \mathcal{I}(C)$; thus $\sqcap_{B \in \mathcal{Q}} B \sqsubseteq C$. For any $C \in \bigcap_{B \in \mathcal{Q}} \mathcal{I}(B)$, $C \subseteq \prod_{B \in \mathcal{Q}} B$. Thus, by Theorem 23, $C \sqsubseteq \sqcap_{B \in \mathcal{Q}} B$. $\square$

Although posets $(\mathcal{I}(A), \sqsubseteq)$ and $(\operatorname{dom} A, \leq_A)$ are not isomorphic, many of the fundamental ordering traits of A are preserved in poset $(\mathcal{I}(A), \sqsubseteq)$. The next theorem shows that totality is preserved.

Lemma 29 *For any quasi-ordering A and x and y in the domain of A,*

$$A \restriction x \sqsubseteq A \restriction y \text{ if and only if } x \leq_A y \,.$$

Proof. Since $x \in \operatorname{dom} A$, we must have $x \leq_{A \restriction x} x$. From $A \restriction x \sqsubseteq A \restriction y$, we deduce that $x \leq_{A \restriction y} x$, which implies that $x \leq_A y$.

For the other direction, note that $A \restriction x \subseteq A \restriction y$ if $x \leq_A y$, and apply Theorem 23. □

Theorem 30 (Totality preservation) *A quasi-ordering A is total if and only if the poset $(\mathcal{I}(A), \sqsubseteq)$ is total.*

Proof. Suppose A is not total and x, y are incomparable in A. By Lemma 29, $A \restriction x$ and $A \restriction y$ must be incomparable in poset $(\mathcal{I}(A), \sqsubseteq)$.

Suppose poset $(\mathcal{I}(A), \sqsubseteq)$ is not total and $B, C \in \mathcal{I}(A)$ are incomparable under $\sqsubseteq$. By Theorem 23, neither $\operatorname{dom} B$ nor $\operatorname{dom} C$ can be a subset of the other. Let $x \in \operatorname{dom} B \setminus \operatorname{dom} C$ and $y \in \operatorname{dom} C \setminus \operatorname{dom} B$. Thus $A \restriction x \in \mathcal{I}(B) \setminus \mathcal{I}(C)$ and $A \restriction y \in \mathcal{I}(C) \setminus \mathcal{I}(B)$ showing that $A \restriction x$ and $A \restriction y$ are incomparable. So by By Lemma 29, x and y must be incomparable in A and thus A must not be total. □

5 Leanness

We return to the string example of Section 2 to prepare a definition of "leanness". In the case of Examples 2 and 3, the first non-trivial initial segments differ: F has an initial segment $\varepsilon \prec_F \mathbf{a}$, whereas G has an initial segment $\varepsilon \prec_G \mathbf{a} \simeq_G \mathbf{b}$. That F's initial segment is a subset of G's initial segment is the first indication that F is leaner than G.

In the general case of arbitrary quasi-orderings A and B, there may be no single "next" initial segment that marks the divergence between orderings A and B. The key property, however, is that of initial segments that are found in one ordering but not the other.

We now have the building blocks necessary to define a general "leaner" relation for quasi-orderings. In the simple case of Examples 2 and 3, there was one initial segment from F that was a subset of one initial segment from G. In the general case, *all* initial segments of G will be considered, as long as they are not initial segments of F. Similarly, more than just one initial segment from F can be a subset of initial segments from G, just as long as the initial segment from F is not an initial segment of G.

Definition 31 (Leanness) *Quasi-ordering A is* leaner *than quasi-ordering B, symbolized $A \trianglelefteq B$, iff for every initial segment B_0 of B and not of A there is an initial segment A_0 of A and not of B that is a subset of B_0:*

$$\forall B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A).\ \exists A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B).\ (A_0 \subseteq B_0)\ .$$

Leanness is a partial order, as we will see below. For Examples 2 and 3, we do have $F \trianglelefteq G$. It is also the case that F is the leanest ordering in the set $\mathcal{Q}$ of Example 1.

Remark 32 *The definition of leanness resembles a Smyth powerdomain construction [10] on initial segments (but removes common elements from comparison) and the multiset extension [2] of proper superset (but applies to infinite sets).*

Leanness can still can compare any two quasi-orderings with equal domains that are comparable by subset.

Theorem 33 *For any two quasi-orderings A and B with* $\operatorname{dom} A = \operatorname{dom} B$*,*

$$A \subseteq B \textit{ implies } A \trianglelefteq B \ .$$

Proof. For any $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$ choose $A_0 = A \upharpoonright \operatorname{dom} B_0$. Since $\operatorname{dom} A = \operatorname{dom} B$ and $A \subseteq B$ it must hold that $\operatorname{dom} A_0 = \operatorname{dom} B_0$. Since $B_0 \notin \mathcal{I}(A)$, $B_0 \neq A \upharpoonright \operatorname{dom} B_0 = A_0$. Since $A_0 \neq B_0 = B \upharpoonright \operatorname{dom} B_0 = B \upharpoonright \operatorname{dom} A_0$, A_0 can not be in $\mathcal{I}(B)$, thus $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$. Lastly, we have $A_0 \subseteq B \upharpoonright \operatorname{dom} B_0 = B_0$. $\square$

The initial segment relation, $\sqsubseteq$, and the leanness relation, $\trianglelefteq$, play complementary roles. For any two distinct quasi-orderings with the same domains, $\sqsubseteq$ will always leave the two orderings incomparable, whereas $\trianglelefteq$ may make them comparable. When quasi-orderings are partial orders (that is, they are anti-symmetric) the leanness relation does not compare any two distinct orderings with equal domains.

Possibly counter-intuitive at first is the following result.

Theorem 34 (Initial segment leanness duality) *For any two quasi-orderings A and B,*

$$A \sqsubseteq B \textit{ implies } B \trianglelefteq A \ .$$

In words, super-segments are leaner, but, then again, leanness is designed for comparing orderings with the same domain.

Proof. For any two quasi-orderings A and B with $A \sqsubseteq B$, we have $\mathcal{I}(A) \setminus \mathcal{I}(B) = \emptyset$, so the definition of $B \trianglelefteq A$ is vacuously true. $\square$

The reverse direction is not generally true for quasi-orderings, but is true for (antisymmetric) well-orders.

Theorem 35 *For any two well-orders A and B,*

$$A \trianglelefteq B \text{ implies } B \sqsubseteq A .$$

Proof. Suppose $B \not\sqsubseteq A$. Then $B \neq A \sqcap B$ and thus $\operatorname{dom} B \setminus \operatorname{dom}(A \sqcap B) \neq \emptyset$. Since B is well-ordered (and antisymmetric), $\operatorname{dom} B \setminus \operatorname{dom}(A \sqcap B)$ must have a minimum element x. Let $D = \{x\} \cup \operatorname{dom}(A \sqcap B)$ and $B_0 = B \upharpoonright D$. We must have $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$ and $\operatorname{dom} B_0 = D$.

For any $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$, it can not be the case that $\operatorname{dom} A_0 \subseteq \operatorname{dom} B_0$. Thus no $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ can be a subset of B_0. Thus $A \not\trianglelefteq B$. $\square$

In general, leanness is always a partial order (on quasi-orderings). The proof proceeds as follows:

Lemma 36 *For any quasi-orderings A, A_0 and B,*

$$A_0 \sqsubseteq A \text{ and } A_0 \subseteq B \subseteq A \text{ imply } A_0 \sqsubseteq B .$$

Proof. If $A_0 \sqsubseteq A$ and $A_0 \subseteq B \subseteq A$, then, by Lemma 17, Proposition 10 and Theorem 19,

$$A_0 \subseteq B \upharpoonright \operatorname{dom} A_0 \subseteq A \upharpoonright \operatorname{dom} A_0 = A_0 .$$

Thus $A_0 \sqsubseteq B$. $\square$

Theorem 37 *Leanness is a partial order on quasi-orderings.*

Proof. For any quasi-ordering A, $A \trianglelefteq A$ is trivially true since $\mathcal{I}(A) \setminus \mathcal{I}(A) = \emptyset$. Thus leanness is reflexive.

Consider any quasi-orderings A and B, with $A \trianglelefteq B \trianglelefteq A$. Suppose there exist $A_1 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$, $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$ with $B_0 \subseteq A_1$ and $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ with $A_0 \subseteq B_0$. By Lemma 36, $A_0 \in \mathcal{I}(B)$, a contradiction. Thus, $\mathcal{I}(A) \setminus \mathcal{I}(B) = \emptyset$;

likewise $\mathcal{I}(B) \setminus \mathcal{I}(A) = \emptyset$. Thus, $\mathcal{I}(A) = \mathcal{I}(B)$, and hence $A = B$, giving anti-symmetry.

For transitivity, suppose $A \trianglelefteq B \trianglelefteq C$. Consider any $C_0 \in \mathcal{I}(C) \setminus \mathcal{I}(A)$. We show there is an $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(C)$ such that $A_0 \subseteq C_0$.

Case 1: $C_0 \sqsubseteq B$. Since $A \trianglelefteq B$, there is an $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ such that $A_0 \subseteq C_0$. Were $A_0 \in \mathcal{I}(C)$, then, by Theorem 23, $A_0 \sqsubseteq C_0$ and then $A_0 \sqsubseteq B$. But $A_0 \not\sqsubseteq B$, so we must have $A_0 \notin \mathcal{I}(C)$. Thus, $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(C)$ with $A_0 \subseteq C_0$.

Case 2: $C_0 \not\sqsubseteq B$. Since $B \trianglelefteq C$, there must exist $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(C)$ such that $B_0 \subseteq C_0$.

Case 2a: $B_0 \in \mathcal{I}(A)$. Let $A_0 = B_0 \in \mathcal{I}(A) \setminus \mathcal{I}(C)$ with $A_0 = B_0 \subseteq C_0$.

Case 2b: $B_0 \notin \mathcal{I}(A)$. Since $A \trianglelefteq B$, there must exist $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ such that $A_0 \subseteq B_0$. By Lemma 36 and $A_0 \subseteq B_0 \subseteq C$, if $A_0 \sqsubseteq C$, then $A_0 \sqsubseteq B_0 \sqsubseteq B$; thus $A_0 \not\sqsubseteq C$, since $A_0 \not\sqsubseteq B$. Thus, we have $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(C)$ with $A_0 \subseteq B_0 \subseteq C_0$.

Thus $A \trianglelefteq C$. $\square$

The next theorem will present an alternative definition of leanness that, at times, is more convenient to use than the original.

Lemma 38 *For any quasi-ordering A and set S,*

$$(\operatorname{dom} A) \cap S \subseteq \operatorname{dom}(A \upharpoonright S) .$$

Proof. Consider any $x \in (\operatorname{dom} A) \cap S$. With $x \leq_A x \leq_A x$ and $x \in S$, it must hold that $x \leq_{A \upharpoonright S} x$ and thus $x \in \operatorname{dom}(A \upharpoonright S)$. $\square$

Theorem 39 (Alternate leanness definition) *For any quasi-orderings A and B, $A \trianglelefteq B$ iff for every strict super-segment B_0 of $A \sqcap B$ in B there exists a strict super-segment A_0 of $A \sqcap B$ in A such that $A_0 \subseteq B_0$. In symbols: $A \trianglelefteq B$ iff*

$$\forall B_0.\ (A \sqcap B \sqsubset B_0 \sqsubseteq B \Rightarrow \exists A_0.\ (A \sqcap B \sqsubset A_0 \sqsubseteq A \wedge A_0 \subseteq B_0)) .$$

Proof. Assume $A \trianglelefteq B$, and consider any B_0 such that $A \sqcap B \sqsubset B_0 \sqsubseteq B$. Since $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$, there exists $A_1 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ such that $A_1 \subseteq B_0$. Let $A_0 = A_1 \sqcup (A \sqcap B)$. By the fact that $A \sqcap B \sqsubseteq B_0$ and Theorem 25, we have

$$A_0 \;=\; A_1 \sqcup (A \sqcap B) \;=\; A_1 \cup (A \sqcap B) \;\subseteq\; B_0 .$$

For the other direction, assume the condition of the theorem, and consider any $B_1 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$. Let $B_0 = B_1 \sqcup (A \sqcap B)$. Since $A \sqcap B \sqsubset B_0 \sqsubseteq B$, we are assuming there exists A_0 with $A \sqcap B \sqsubset A_0 \sqsubseteq A$ and $A_0 \subseteq B_0$. By Theorems 23 and 25,

$$\mathrm{dom}\,(A \sqcap B) \;\subsetneq\; \mathrm{dom}\,A_0 \;\subseteq\; \mathrm{dom}\,B_0 \;\subseteq\; \mathrm{dom}\,B_1 \cup \mathrm{dom}\,(A \sqcap B)\ ,$$

from which (by set theory) it follows that $\mathrm{dom}\,A_0 \cap \mathrm{dom}\,B_1 \not\subseteq \mathrm{dom}\,(A \sqcap B)$. By Lemma 38,

$$\mathrm{dom}\,A_0 \cap \mathrm{dom}\,B_1 \;\subseteq\; \mathrm{dom}\,(A_0 \restriction \mathrm{dom}\,B_1)\ .$$

Let $A_1 = A_0 \restriction \mathrm{dom}\,B_1$; thus, $\mathrm{dom}\,A_1 \not\subseteq \mathrm{dom}\,(A \sqcap B)$, which, by Theorem 23, implies $A_1 \not\sqsubseteq A \sqcap B$, $A_1 \not\sqsubseteq B$ and, finally, $A_1 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$. By Proposition 10 and Theorem 19, $A_1 = A_0 \restriction \mathrm{dom}\,B_1 \subseteq B_0 \restriction \mathrm{dom}\,B_1 = B_1$. $\square$

Corollary 40 *For any quasi-orderings A and B with $A \not\sqsubseteq B$, if $A \subseteq B_0$ for every B_0 with $A \sqcap B \sqsubset B_0 \sqsubseteq B$, then $A \trianglelefteq B$.*

Proof. For every B_0 with $A \sqcap B \sqsubset B_0 \sqsubseteq B$, we have $A \sqcap B \sqsubset A \sqsubseteq A$ and $A \subseteq B_0$. $\square$

Lemma 41 *For any quasi-orderings A and B, if A is total and leaner than B, then every initial segment of A and not of B is leaner than B.*

Proof. Assume $A \trianglelefteq B$ and total. Consider any $C \in \mathcal{I}(A) \setminus \mathcal{I}(B)$. For any $B_0 \in \mathcal{I}(B) \setminus \mathcal{I}(A)$, there exists $A_0 \in \mathcal{I}(A) \setminus \mathcal{I}(B)$ such that $A_0 \subseteq B_0$. By Theorem 30, either $A_0 \sqsubseteq C$ or $C \sqsubseteq A_0$ Thus, $A_0 \sqcap C$ equals A_0 or equals C, whichever is the smaller segment. Thus, there exists $A_0 \sqcap C \in \mathcal{I}(C) \setminus \mathcal{I}(B)$ with $A_0 \sqcap C \subseteq B_0$. Thus $C \trianglelefteq B$. $\square$

6 Leanest

In this section, we will identify two properties that guarantee the existence of a leanest member in a given set of quasi-orderings. The structure these properties depend on is not that of the elements ordered, or from the way the set of orderings is defined, but rather from the set of all initial segments of a collection of quasi-orderings. We will denote by $\mathcal{I}(\mathcal{O})$ the set of all initial segments of members of $\mathcal{O}$.

The set $\mathcal{Q}$ from Example 1 provided a good instance of a set with a leanest ordering, but no subset-minimum.

Example 42 *Some sets of quasi-orderings, such as* $\{\mathbf{a} \prec \mathbf{b},\ \mathbf{b} \prec \mathbf{a}\}$*, have no leanest member. Its initial segments are:* $\{\emptyset, \mathbf{a}, \mathbf{b}, \mathbf{a} \prec \mathbf{b}, \mathbf{b} \prec \mathbf{a}\}$*. The empty set* $\emptyset$ *is the only common initial segment, and there is no super-segment of* $\emptyset$ *that is any "better" than any other.* □

Example 43 *In contrast, the initial segment of* $\mathbf{a}$ *(by itself) is the "best" super-segment of* $\emptyset$ *in the following set of quasi-orderings:*

$$\mathcal{I}\left(\{\mathbf{a} \prec \mathbf{b}, \mathbf{a} \simeq \mathbf{b}\}\right) \;=\; \{\emptyset, \mathbf{a}, \mathbf{a} \prec \mathbf{b}, \mathbf{a} \simeq \mathbf{b}\}\,. \qquad \square$$

Such "tie breaking" super-segments are formalized as "successor segments".

Definition 44 (Successor segment) *A strict super-segment of a quasi-ordering* A *in a set of quasi-orderings* $\mathcal{O}$ *is a* successor segment *of* A *(in* $\mathcal{O}$*) if it is a subset of all strict super-segments of* A *in* $\mathcal{O}$*.*

This gives us the first property for the existence of a leanest member of any set of quasi-orderings $\mathcal{O}$: any initial segment with super-segments, that is, any non-maximal member of poset $(\mathcal{I}\left(\mathcal{O}\right), \sqsubseteq)$, must have a successor segment.

Example 45 *Let* $\mathcal{S}$ *be the set of quasi-orderings*

$$\begin{array}{ll} \{\, 1 \simeq 2 \simeq 3 \simeq 4 \simeq \ldots, & \\ \;\; 1 \prec 2 \simeq 3 \simeq 4 \simeq \ldots, & \\ \;\; 1 \prec 2 \prec 3 \simeq 4 \simeq \ldots, & \\ \;\; 1 \prec 2 \prec 3 \prec 4 \simeq \ldots, & \\ \qquad\quad \ldots & \}\,. \end{array}$$

Since for every member of $\mathcal{S}$ *there is another strictly leaner member, it can be concluded that* $\mathcal{S}$ *has no leanest member. Were* $\mathcal{S}$ *to include the natural ordering of the natural numbers, however, then* $\mathcal{S}$ *would have a leanest member.* □

Note that $\mathcal{I}\left(\mathcal{S}\right)$, which equals

$$\mathcal{S} \cup \{\ \emptyset,\ 1,\ 1 \prec 2,\ 1 \prec 2 \prec 3,\ 1 \prec 2 \prec 3 \prec 4,\ \ldots\ \}\ ,$$

contains no upper bound in terms of $\sqsubseteq$. This suggests the second property for the existence of a leanest member: any ascending sequence of initial segments must have an upper bound.

Together the two properties form the following existence theorem.

Theorem 46 (Leanest existence) *A set of total well-founded quasi-orderings $\mathcal{O}$ contains a unique leanest member if poset $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ contains*

(1) a successor segment for every non-maximal member; and
(2) an upper bound for every ascending sequence.

The restriction that members of set $\mathcal{O}$ be total and well-founded is redundant, since condition (1) implies that all members of $\mathcal{O}$ must be total and well-founded, as shown by the following two propositions:

Proposition 47 *A set of quasi-orderings $\mathcal{O}$ contains only total well-founded members if poset $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ contains a successor segment for every non-maximal member.*

Proof. Assume $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ contains a successor segment for every non-maximal member.

Suppose $\mathcal{O}$ contains a non-total ordering C with incomparable x and y. By Lemma 29, neither $C \upharpoonright x$ nor $C \upharpoonright y$ is an initial segment of the other. Thus $(C \upharpoonright x) \sqcap (C \upharpoonright y)$ is distinct from both $C \upharpoonright x$ and $C \upharpoonright y$ and thus non-maximal in poset $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$.

Consider any $A \in \mathcal{I}(\mathcal{O})$ that is a strict super-segment of $(C \upharpoonright x) \sqcap (C \upharpoonright y)$. By Theorem 23, $\operatorname{dom} A \supsetneq \operatorname{dom}((C \upharpoonright x) \sqcap (C \upharpoonright y))$. By Theorem 27,

$$\operatorname{dom}(C \upharpoonright x) \cap \operatorname{dom}(C \upharpoonright y) \;=\; \operatorname{dom}((C \upharpoonright x) \sqcap (C \upharpoonright y)) \,.$$

By Theorem 23, neither $\operatorname{dom}(C \upharpoonright x)$ nor $\operatorname{dom}(C \upharpoonright y)$ is a subset of the other. Thus, $\operatorname{dom} A$ cannot be a subset of both $\operatorname{dom}(C \upharpoonright x)$ and $\operatorname{dom}(C \upharpoonright y)$. From the definition of dom, A can not be a subset of both $C \upharpoonright x$ and of $C \upharpoonright y$; thus, A can not be a successor segment of $(C \upharpoonright x) \sqcap (C \upharpoonright y)$. So, $\mathcal{O}$ contains only total orderings.

Consider any total ordering $M \in \mathcal{O}$ and non-empty set $S \subseteq \operatorname{dom} M$. Let $T = \{x : \forall y \in S. x <_M y\}$. Since $M \upharpoonright T$ is non-maximal, there must be a successor segment A of $M \upharpoonright T$ in $\mathcal{I}(\mathcal{O})$. By Theorem 23, $\operatorname{dom}(M \upharpoonright T) \subsetneq \operatorname{dom} A$. Since $T = \operatorname{dom} M \upharpoonright T$, there must be some $z \in \operatorname{dom} A \setminus T$.

For any $y \in S$, $M \upharpoonright y \sqsupset M \upharpoonright T$ and thus $A \subseteq M \upharpoonright y$, which implies $z \leq_M y$. Since $z \notin T$, z must be in S and thus is a minimum of S under M. □

Next, we introduce a construct that is used to find the leanest member of a set of quasi-orderings.

Definition 48 (Dual-chain) *For any ordinal-indexed sequence $\mathcal{Q} = \{Q_\mu\}_\mu$*

of quasi-orderings, let the dual-chain *of* $\mathcal{Q}$ *be the sequence*

$$Q^{\Delta}_{\mu} \;:=\; \prod_{\beta \geq \mu} Q_{\beta}$$

of common initial segments. Let the dual-chain limit *be their union:*

$$\lim^{\Delta}_{\mu} Q_{\mu} \;:=\; \bigcup_{\alpha} Q^{\Delta}_{\alpha}\ .$$

Lemma 49 *The dual-chain of a sequence of quasi-orderings from* $\mathcal{O}$ *is a descending sequence in poset* $(\mathcal{I}(\mathcal{O}), \trianglelefteq)$ *and an ascending sequence in* $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$*.*

Proof. By Theorem 28, each member of a dual-chain exists as a greatest common initial segment, so by construction, a dual-chain must be ascending under $\sqsubseteq$. By Theorem 34, a dual-chain ascending under $\sqsubseteq$ is descending under $\trianglelefteq$. $\square$

Lemma 50 *Let* $\mathcal{C}$ *be a descending sequence* $C_0 \trianglerighteq C_1 \trianglerighteq C_2 \trianglerighteq \ldots$ *of total quasi-orderings. If* $\lim^{\Delta}_{\mu} C_{\mu} \notin \mathcal{I}(\mathcal{C})$*, then* $\lim^{\Delta}_{\mu} C_{\mu}$ *is leaner than all members of* $\mathcal{C}$*.*

Proof. Assume $\lim^{\Delta}_{\mu} C_{\mu} \notin \mathcal{I}(\mathcal{C})$ and consider any C_{α}. Since $\lim^{\Delta}_{\mu} C_{\mu} \not\sqsubseteq C_{\alpha}$ and $\mathcal{I}(C_{\alpha})$ is closed under union, $C^{\Delta}_{\beta} \not\sqsubseteq C_{\alpha}$ for some $\beta > \alpha$. By $C_{\beta} \trianglelefteq C_{\alpha}$ and Lemma 41, $C^{\Delta}_{\beta} \trianglelefteq C_{\alpha}$. With $C^{\Delta}_{\beta} \sqsubseteq \lim^{\Delta}_{\mu} C_{\mu}$ and Theorem 34, we have $\lim^{\Delta}_{\mu} C_{\mu} \trianglelefteq C^{\Delta}_{\beta}$. Thus, $\lim^{\Delta}_{\mu} C_{\mu} \trianglelefteq C_{\alpha}$. $\square$

Lemma 51 *For any descending sequence* $C_0 \trianglerighteq C_1 \trianglerighteq C_2 \trianglerighteq \ldots$ *over a set of quasi-orderings* $\mathcal{O}$*, if* $\lim^{\Delta}_{\mu} C_{\mu} \sqsubseteq C_{\alpha}$*, for some* α*, then* $\lim^{\Delta}_{\mu} C_{\mu} \sqsubseteq C_{\beta}$ *for all* $\beta > \alpha$*.*

Proof. Assume $\lim^{\Delta}_{\mu} C_{\mu} \sqsubseteq C_{\alpha}$. Suppose there is some $\beta > \alpha$ such that $\lim^{\Delta}_{\mu} C_{\mu} \not\sqsubseteq C_{\beta}$. Since $\mathcal{I}(C_{\beta})$ is closed under union, there must be some $\gamma > \beta$ such that $C^{\Delta}_{\gamma} \not\sqsubseteq C_{\beta}$. Since $C^{\Delta}_{\gamma} \sqsubseteq C_{\alpha}$ and $C_{\beta} \trianglelefteq C_{\alpha}$, there must exist some $B_0 \in \mathcal{I}(C_{\beta}) \setminus \mathcal{I}(C_{\alpha})$ such that $B_0 \subseteq C^{\Delta}_{\gamma}$. We have $B_0 \not\sqsubseteq C_{\gamma}$, because Theorem 23 and $B_0 \sqsubseteq C_{\gamma}$ imply $B_0 \sqsubseteq C^{\Delta}_{\gamma}$, which implies $B_0 \sqsubseteq C_{\alpha}$. Thus, $B_0 \in \mathcal{I}(C_{\beta}) \backslash \mathcal{I}(C_{\gamma})$. Since $C_{\gamma} \trianglelefteq C_{\beta}$, there must exist some $A_0 \in \mathcal{I}(C_{\gamma}) \backslash \mathcal{I}(C_{\beta})$ such that $A_0 \subseteq B_0 \subseteq C^{\Delta}_{\gamma}$. By Theorem 23, $A_0 \sqsubseteq C^{\Delta}_{\gamma}$, and by Lemma 36, $A_0 \sqsubseteq B_0$. However, $A_0 \notin \mathcal{I}(C_{\beta})$, a contradiction. Thus, $\lim^{\Delta}_{\mu} C_{\mu} \sqsubseteq C_{\beta}$ for all $\beta > \alpha$. $\square$

Lemma 52 *Let* $\mathcal{C}$ *be any descending sequence* $C_0 \trianglerighteq C_1 \trianglerighteq C_2 \trianglerighteq \ldots$ *over a set of quasi-orderings* $\mathcal{O}$*. Then* $\mathcal{I}(\mathcal{C})$ *cannot include a successor segment from* $\mathcal{I}(\mathcal{O})$ *of the dual-chain limit of* $\mathcal{C}$*.*

Proof. Let $S \in \mathcal{I}(\mathcal{O})$ be a successor segment of $\lim^{\Delta}_{\mu} C_\mu$, and suppose there did exist α such that $S \sqsubseteq C_\alpha$. Since $C^{\Delta}_{\alpha} \sqsubseteq \lim^{\Delta}_{\mu} C_\mu \sqsubset S$, there must be some $\beta > \alpha$ such that $S \not\sqsubseteq C_\beta$. Since $C_\beta \trianglelefteq C_\alpha$, there must exist some $B_0 \in \mathcal{I}(C_\beta) \setminus \mathcal{I}(C_\alpha)$ such that $B_0 \subseteq S$. By Lemma 51, $\lim^{\Delta}_{\mu} C_\mu \sqsubseteq C_\beta$, and by Theorem 25, $B_0 \sqcup \lim^{\Delta}_{\mu} C_\mu$ exists and, with $\lim^{\Delta}_{\mu} C_\mu \subseteq S$,

$$B_0 \sqcup \lim^{\Delta}_{\mu} C_\mu \;=\; B_0 \cup \lim^{\Delta}_{\mu} C_\mu \;\subseteq\; S\ .$$

However, since $B_0 \sqcup \lim^{\Delta}_{\mu} C_\mu \sqsupset \lim^{\Delta}_{\mu} C_\mu$, we have $S \subseteq B_0 \sqcup \lim^{\Delta}_{\mu} C_\mu$, implying equality and thus $S \sqsubseteq C_\beta$, a contradiction. □

Lemma 53 *For any set of quasi-orderings $\mathcal{O}$ and member $A \in \mathcal{O}$, a successor segment of A in $\mathcal{O}$ is leaner than or an initial segment of all super-segments in $\mathcal{I}(\mathcal{O})$ of A.*

Proof. Consider a super-segment B of A in $\mathcal{O}$ and a successor segment C of A in $\mathcal{O}$. If $B \sqsubseteq C$, then by Theorem 34, $C \trianglelefteq B$. Otherwise, consider any initial segment B_0 in $\mathcal{I}(B) \setminus \mathcal{I}(C)$. By Theorem 23, $B_0 \sqcup A$ exists and $B_0 \sqcup A \notin \mathcal{I}(C)$, and by the definition of successor segment, $C \subseteq B_0 \sqcup A$. By the definition of dom and Theorems 23 and 25, we have $\operatorname{dom} A \subsetneq \operatorname{dom} C \subseteq \operatorname{dom} B_0 \cup \operatorname{dom} A$. Thus, $\operatorname{dom} A \subsetneq \operatorname{dom} B_0$. By Theorem 23, $A \sqsubset B_0$, and thus $C \subseteq B_0$. If $C \not\sqsubseteq B$, then $C \trianglelefteq B$. □

Lemma 54 *For any set of quasi-orderings $\mathcal{O}$, if every non-maximal member of $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ has a successor segment in $\mathcal{I}(\mathcal{O})$, then $(\mathcal{O}, \trianglelefteq)$ is down-directed (that is, for all $A, B \in \mathcal{O}$ there is a $C \in \mathcal{O}$ such that $C \trianglelefteq A, B$).*

Proof. For $A, B \in \mathcal{O}$, if either is an initial segment of the other, then, by Theorem 34, the claim is trivially true.

So, assume neither A nor B is an initial segment of the other. Thus $A \sqcap B \sqsubset A, B$. If every non-maximal member of $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ has a successor segment in $\mathcal{I}(\mathcal{O})$, then $A \sqcap B$ has a successor segment S in $\mathcal{I}(\mathcal{O})$, where $S \subseteq A_0$ and $S \subseteq B_0$, whenever $(A \sqcap B) \sqsubset A_0 \sqsubseteq A$ and $(A \sqcap B) \sqsubset B_0 \sqsubseteq B$.

If $S \sqsubseteq A$, then S is not an initial segment of B and $A \sqcap B = S \sqcap B$ and thus $A \trianglelefteq S \trianglelefteq B$ by Theorem 34 and Corollary 40. Similarly, if $S \sqsubseteq B$, then $B \trianglelefteq A$.

If S is not an initial segment of either A or B, then $A \sqcap B = S \sqcap A = S \sqcap B$, and, by Corollary 40, $S \trianglelefteq A, B$. Since $S \sqsubseteq \mathcal{O}$, there exists some $C \in \mathcal{O}$ with $S \sqsubseteq C$. By Theorem 34, $C \trianglelefteq A, B$. □

We now have the necessary ingredients for a proof of leanest-ordering existence (Theorem 46).

Proof of Leanest Existence Theorem. Let $\mathcal{C}$ be any descending sequence $C_0 \trianglerighteq C_1 \trianglerighteq C_2 \trianglerighteq \ldots$ over $\mathcal{O}$. Its dual-chain is an ascending sequence in poset $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$ by Lemma 49. By the second condition, the dual-chain must have an upper bound U in $\mathcal{I}(\mathcal{O})$. Since $\mathcal{I}(U)$ is closed under union, the dual-chain limit $\lim^{\Delta}_{\mu} C_\mu$ must be in $\mathcal{I}(U) \subseteq \mathcal{I}(\mathcal{O})$. First, we seek $L \in \mathcal{I}(\mathcal{O})$ leaner than every member of sequence $\mathcal{C}$.

Case 1: $L = \lim^{\Delta}_{\mu} C_\mu \notin \mathcal{I}(\mathcal{C})$. By Lemma 50, L is leaner than all members of $\mathcal{C}$.

Case 2: $L = \lim^{\Delta}_{\mu} C_\mu$ is in $\mathcal{I}(\mathcal{C})$ and is maximal in $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$.
For some α, $L \sqsubseteq C_\alpha$. By Lemma 51, for all $\beta > \alpha$, $L \sqsubseteq C_\beta$. Since L is maximal, $L = C_\beta$ for all $\beta > \alpha$. Since $C_\alpha \trianglelefteq C_\gamma$ for $\gamma < \alpha$, L is leaner than all members of $\mathcal{C}$.

Case 3: $\lim^{\Delta}_{\mu} C_\mu$ is in $\mathcal{I}(\mathcal{C})$ and is non-maximal in $(\mathcal{I}(\mathcal{O}), \sqsubseteq)$.
Let L be the successor segment in $\mathcal{I}(\mathcal{O})$ of $\lim^{\Delta}_{\mu} C_\mu$. For some α, $\lim^{\Delta}_{\mu} C_\mu \sqsubseteq C_\alpha$. By Lemma 51, C_β is also a super-segment of $\lim^{\Delta}_{\mu} C_\mu$ for all $\beta > \alpha$. By Lemma 52, $L \not\sqsubseteq C_\beta$. Thus, by Lemma 53, L must be leaner than all C_β and, hence, of all members of C.

In all three cases, we have $L \in \mathcal{I}(\mathcal{O})$ leaner than every member of $\mathcal{C}$. There must exist some $M \in \mathcal{O}$ such that $L \sqsubseteq M$. By Theorem 34, $M \trianglelefteq L$, thus M is a lower bound to $\mathcal{C}$ in poset $(\mathcal{O}, \trianglelefteq)$.

Let A_i be an enumeration of $\mathcal{O}$ and define B_i such that

$$B_0 = A_0$$
$$B_{i+1} = \begin{cases} A_{i+1} & \text{if } A_{i+1} \trianglelefteq B_i \\ B_i & \text{otherwise} \end{cases} .$$

Since $\{B_i\}_i$ is a descending sequence in poset $(\mathcal{O}, \trianglelefteq)$, there exists a lower bound in $\mathcal{O}$, namely some A_k. For any $A_j \trianglelefteq A_k$, we must have $B_j = A_j$ and thus $A_j = A_k$. Thus A_k must be minimal in poset $(\mathcal{O}, \trianglelefteq)$.

By Lemma 54, $\mathcal{O}$ is directed, so $\mathcal{O}$ has a unique leanest member. □

7 Application to Binary Trees

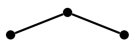
Earlier, we described a very simple leanest string ordering. With the Existence Theorem, leanest orderings of greater complexity can be found. In the example to follow, binary trees serve as elements rather than strings. Binary trees exemplify more complexity than strings, but less complexity than terms. In fact, binary trees are a special case of terms, with only one constant and only one function.

The most basic and trivial tree is the empty tree denoted $\Box$. This tree has no nodes or branches. From the empty tree $\Box$, more interesting trees can be built using the operation of $(x \curlywedge y)$, which places tree x to the left of a root node and tree y to the right. For instance,

$$(\Box \curlywedge \Box) \;=$$

$$(\Box \curlywedge (\Box \curlywedge \Box)) \;=$$


$$((\Box \curlywedge \Box) \curlywedge \Box) \;=$$


$$((\Box \curlywedge \Box) \curlywedge (\Box \curlywedge \Box)) \;=$$


We set out to design, from intuitive principles, a rewrite-ordering for the simplified example of binary trees. One approach is to construct an explicit definition of a rewrite-ordering relation and then prove the defined relation satisfies various desired properties such as transitivity. The alternative approach presented below, is to define a set of quasi-orderings which automatically satisfy all desired conditions with the exception of antisymmetry.

Definition 55 *Let $\mathcal{T}$ be the set of all quasi-orderings A of finite binary trees that satisfy the following three tree-ordering conditions:*

- ***Growth:*** $x, y \precsim_A (x \curlywedge y)$;
- ***Monotonicity:*** *if* $y \precsim_A z$, *then* $(x \curlywedge y) \precsim_A (x \curlywedge z)$ *and* $(y \curlywedge x) \precsim_A (z \curlywedge x)$;
- ***Lexicography:*** *if* $x_1 \prec_A y_1$ *and* $x_0 \prec_A (y_1 \curlywedge y_0)$, *then* $(x_1 \curlywedge x_0) \precsim_A (y_1 \curlywedge y_0)$.

The Growth and Monotonicity conditions are chosen because they give us a quasi-simplification order, which suffices for well-founded termination proofs [1]. Growth and Monotonicity alone give no preference for rightness or leftness. The $x_1 \prec_A y_1$ condition in Lexicography gives more significance to the left and retains the $x_0 \prec_A (y_1 \curlywedge y_0)$ condition so as not to work against Growth.

Intuitively, the minimum quasi-ordering satisfying the above conditions should be antisymmetric, lacking superfluous equivalence classes. As will be seen later in this section, there is no subset-minimum ordering. However, using the Existence Theorem, we can establish a leanest ordering. We first establish that the second condition of the Existence Theorem holds for $\mathcal{T}$.

Lemma 56 *Every chain in $(\mathcal{I}(\mathcal{T}), \sqsubseteq)$ has an upper bound.*

Proof. Consider any chain $\mathcal{C}$ in poset $(\mathcal{I}(\mathcal{T}), \sqsubseteq)$. Let $L = \cup_{A \in \mathcal{C}} A$ and D be the (possibly empty) set of all trees that are not ordered by L (the complement of $\operatorname{dom} L$). Define:

$$x \precsim_K y \ := \begin{cases} x \precsim_L y, & \text{or} \\ x \in \operatorname{dom} L \text{ and } y \in D, & \text{or} \\ x, y \in D \ . \end{cases}$$

The ordering K places D as an equivalence class ordered strictly above $\operatorname{dom} L$.

Consider any two binary trees x and y. If $(x \curlywedge y)$ is not in $\operatorname{dom} L$, then it is in D and thus $(x \curlywedge y) \succsim_K x, y$. Otherwise, $(x \curlywedge y)$ is in $\operatorname{dom} L$ and there is some $A \in \mathcal{C}$ such that $(x \curlywedge y) \in \operatorname{dom} A$. Since A satisfies the Growth condition, it results that $(x \curlywedge y) \succsim_K x, y$. Therefore K satisfies the Growth condition.

Consider any three binary trees x, y and z with $y \precsim_K z$. If $(x \curlywedge z)$ is not in $\operatorname{dom} L$, then it is in D and thus $(x \curlywedge y) \precsim_K (x \curlywedge z)$. Otherwise, $(x \curlywedge z)$ is in $\operatorname{dom} L$ and there is some $A \in \mathcal{C}$ such that $(x \curlywedge z) \in \operatorname{dom} A$. Since A must be an initial segment of an ordering that satisfies Growth, $z \in \operatorname{dom} A$. Since y can not be in D, for some $B \in \mathcal{C}$ we have $y \precsim_B z$. Since B is either an initial segment or a super-segment of A, it follows from both cases that $y \precsim_A z$. By Monotonicity, $(x \curlywedge y) \precsim_A (x \curlywedge z)$ and thus $(x \curlywedge y) \precsim_K (x \curlywedge z)$. By symmetry, $(y \curlywedge x) \precsim_K (z \curlywedge x)$. Therefore K satisfies the Monotonicity condition.

Consider any four binary trees x_0, y_0, x_1 and y_1 with $x_0 \prec_K x_1$ and $y_0 \prec_K (x_1 \curlywedge y_1)$. If $(x_1 \curlywedge y_1)$ is not in $\operatorname{dom} L$, then it is in D, and thus $(x_0 \curlywedge y_0) \precsim_K (x_1 \curlywedge y_1)$. If not, $(x_1 \curlywedge y_1)$ is in $\operatorname{dom} L$ and there is some $A \in \mathcal{C}$ such that $(x_1 \curlywedge y_1) \in \operatorname{dom} A$. Since A is an initial segment of an ordering that satisfies Growth, $x_1 \in \operatorname{dom} A$. Since x_0 and y_0 can not be in D, for some $B \in \mathcal{C}$ we have $x_0 \prec_B x_1$ and $y_0 \prec_B (x_1 \curlywedge y_1)$, and thus also $x_0 \prec_A x_1$ and $y_0 \prec_A (x_1 \curlywedge y_1)$, since B is either an initial segment or super-segment of A. By Lexicography, $(x_0 \curlywedge y_0) \precsim_B (x_1 \curlywedge y_1)$ and thus $(x_0 \curlywedge y_0) \precsim_K (x_1 \curlywedge y_1)$. Therefore K satisfies the Lexicography condition.

Since $K \in \mathcal{T}$, L is in $\mathcal{I}(\mathcal{T})$ and is an upper bound of $\mathcal{C}$. □

Next we establish that condition (1) of the Existence Theorem holds for $\mathcal{T}$.

Lemma 57 *For every non-maximal A in poset $(\mathcal{I}(\mathcal{T}), \sqsubseteq)$, the set of strict super-segments of A has a subset-minimum.*

Proof. Consider the non-empty set S of strict super-segments in $\mathcal{I}(\mathcal{T})$ of non-maximal A. Because $\mathcal{T}$ consists of total *well-founded* quasi-orderings (see [3]), every super-segment in S must have an equivalence class ordered as less than all other elements outside of A. Let T be the set of all these equivalence classes and let B be the intersection of all these equivalence classes.

No two members of T can be disjoint, since otherwise one could construct an ordering of trees that satisfies the conditions of membership in $\mathcal{T}$, but the ordering would not be total—a contradiction.

Furthermore, there can be no subset descending sequence of members of T, for otherwise one could construct an ordering of trees that satisfies the conditions of membership in $\mathcal{T}$. But again the ordering would not be well-founded, which is a contradiction.

Were B empty, then either two members of T would be disjoint or there would be a subset descending sequence of members of T. Since neither can be the case, B must be non-empty.

Let C be the ordering of A, with B placed as an equivalence class strictly above A. Let D be the ordering of C, with an equivalence class strictly greater than C consisting of all binary trees not in C. If $D \notin \mathcal{T}$, then one of the members of S cannot be the initial segment of a member of $\mathcal{T}$; one of the conditions for membership in $\mathcal{T}$ must be violated.

Thus, C is the subset-minimum of all strict super-segment in $\mathcal{I}(\mathcal{T})$ of A. $\square$

It follows from Theorem 46 that $\mathcal{T}$ has a leanest ordering.

Theorem 58 *The lexicographic path ordering for binary trees is the leanest ordering in $\mathcal{T}$.*

Binary trees are terms with only one constant, namely $\square$, and only one (binary) function ($\curlywedge$). Under this special case, the lexicographic path ordering (LPO) [5,1] acts as follows:

$x \prec_{lpo} y$ iff

(1) $x = \square$ and $y \neq \square$, or
(2) $x = (x_1 \curlywedge x_0)$, $y = (y_1 \curlywedge y_0)$, and
  (a) $x \precsim_{lpo} y_1$,
  (b) $x \precsim_{lpo} y_0$,
  (c) $x_1 \prec_{lpo} y_1$ and $x_0 \prec_{lpo} y$, or
  (d) $x_1 = y_1$ and $x_0 \prec_{lpo} y_0$.

Proof. LPO must satisfy the Growth condition due to cases (1,2a,2b) in the definition of $\prec_{lpo}$. The Monotonicity condition holds with $\prec_{lpo}$ due to (2c,2d). Similarly, Lexicography holds due to (2c).

By Lemma 57 and Proposition 47, all orderings in $\mathcal{T}$ are total and well-founded (well-orders). Thus, by Theorem 35, the only ordering that could be leaner than LPO must be a super-segment. Since LPO orders all binary trees, there can be no leaner ordering, thus LPO is the leanest ordering in $\mathcal{T}$. $\square$

Here we see the leanness relation isolate a strict ordering of interest out of all the quasi-orderings that satisfy the tree-ordering conditions. Next, we will see why this isolation cannot be done by simply taking the minimum ordering in the usual set-theoretic sense of minimum.

As for $\mathcal{Q}$ from Example 1, a minimum ordering of $\mathcal{T}$ can not be found by taking the intersection of all member orderings. The member ordering of $\mathcal{T}$ analogous to ordering F in Example 2 is the following quasi-ordering from [3], which makes use of transfinite ordinal arithmetic.

Example 59 ([3]) *Let U be the quasi-ordering on binary trees defined as follows:*

$$x \precsim_U y \;:=\; u(x) \leq u(y),$$

where u is the homomorphism:

$$\begin{aligned} u(\square) &= 0\,, \\ u(x \curlywedge y) &= \omega^{u(x)} + u(y) \end{aligned}$$

for any binary trees x and y. $\square$

It follows easily by ordinal arithmetic that $U \in \mathcal{T}$.

The following quasi-ordering on binary trees is analogous to the ordering G from Example 3:

Example 60 *Let V be the quasi-ordering on binary trees defined as*

$$x \precsim_V y \;:=\; v(x) \leq v(y),$$

where v is the homomorphism:

$$\begin{aligned} v(\square) &= 1\,, \\ v(x \curlywedge y) &= \begin{cases} v(y) & \text{if } 2v(x) < v(y)\,, \\ v(y)+1 & \text{if } 2v(x) = v(y)\,, \\ 2v(x) & \text{if } 2v(x) > v(y) \end{cases} \end{aligned}$$

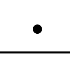
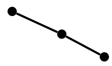
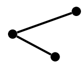
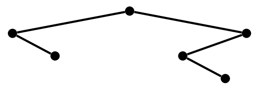
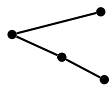
Binary Tree	$u(\cdot)$	$v(\cdot)$
	$\omega^0 + 0 = 1$	$2 \times 1 = 2$
	$\omega^0 + 1 = 2$	$2 + 1 = 3$
	$\omega^0 + 2 = 3$	3
	ω^2	$2 \times 3 = 6$
	$\omega^2 + \omega^2$	$6 + 1 = 7$
	ω^3	$2 \times 3 = 6$

Fig. 1. Tree orderings U and V.

---

for any binary trees x and y.

This V is a member of $\mathcal{T}$. It is easy to see that it satisfies Growth. Monotonicity can be seen by observing that $v(x \curlywedge y)$ either increases or remains the same as either $v(x)$ or $v(y)$ is incremented. For Lexicography, consider any x_0, y_0, x_1 and y_1 such that $x_0 \prec_V x_1$ and $y_0 \precsim_V (x_1 \curlywedge y_1)$. If $2v(x_0) < v(y_0)$, then $v(x_0 \curlywedge y_0) = v(y_0) \leq v(x_1 \curlywedge y_1)$. Otherwise, $2v(x_0) \geq v(y_0)$, in which case $v(x_0 \curlywedge y_0) \leq 2v(x_0) + 1 < 2v(x_1) \leq v(x_1 \curlywedge y_1)$. $\square$

Similar to F and G from $\mathcal{Q}$ in Examples 2 and 3, there is no ordering in $\mathcal{T}$ that is a subset of both U and V. To see this, consider how U and V order the trees in Fig. 1. Both U and V have a strict comparison between the penultimate and last trees, but in opposite directions. Because of this, any ordering W that is a subset of both U and V must leave the last and penultimate trees incomparable. By Proposition 47 and Lemma 57, any member of $\mathcal{T}$ must be total, thus W cannot be a member of $\mathcal{T}$.

8 Conclusion

In contrast to "explicit" definitions of simplification orders like the lexicographic path order, this paper illustrates an alternative approach to defining a simplification order: A set of quasi-orderings provides an "implicit" definition through the existence of a leanest member order. Simple and intuitive conditions can define a set of quasi-orderings, as in the binary-tree example. The existence of quasi-orderings in such a set is a trivial result, however; it is the "minimum" ordering that is of interest. As we have seen, this definition technique comes with a possible snag: there may be no subset-minimum. In particular, conditions that involve a strict comparison can preclude the existence of a subset-minimum.

To compensate for this problem, we described an alternative to a subset-minimum ordering, namely, the "leanest ordering", building on fundamental notions for quasi-orderings. By establishing two properties on a set of quasi-orderings, a leanest ordering is guaranteed to exist. These properties are defined independent of what kind of elements are ordered and what conditions define a set of quasi-orderings. In this paper, the lexicographic path order for binary trees is defined "implicitly", by showing the existence of a leanest ordering within a set of quasi-orderings satisfying intuitive conditions. Of more interest, however, would be more advanced simplification orders, which can be similarly implicitly defined via conditions, but whose explicit definitions would be excessively complicated.

Acknowledgement

We thank Ed Reingold for his interest and a referee for his/her comments.

References

[1] Nachum Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3(1&2):69–115, February/April 1987.

[2] Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Communications of the ACM*, 22(8):465–476, August 1979.

[3] Nachum Dershowitz and Edward M. Reingold. Ordinal arithmetic with list expressions. In A. Nerode and M. Taitslin, editors, *Proceedings of the Symposium on Logical Foundations of Computer Science (Tver, Russia)*, volume 620 of *Lecture Notes in Computer Science*, pages 117–126, Berlin, July 1992. Springer-Verlag.

[4] Jean-Pierre Jouannaud and Pierre Lescanne. On multiset orderings. *Information Processing Letters*, 15(2):57–63, September 1982.

[5] Sam Kamin and Jean-Jacques Lévy. Two generalizations of the recursive path ordering, February 1980. Unpublished note, Department

of Computer Science, University of Illinois, Urbana, IL. Available at http://www.ens-lyon.fr/~plescann/PUBLICATIONS/KAMIN_LEVY/.

[6] Ursula Martin. Extension functions for multiset orderings. *Information Processing Letters*, 26:181–186, December 1987.

[7] Ursula Martin. A geometrical approach to multiset orderings. *Information Processing Letters*, 67:37–54, May 1989.

[8] Ursula Martin and Elizabeth Scott. The order types of termination orderings on monadic terms, strings and multisets. *J. Symbolic Logic*, 62(2):624–635, 1997.

[9] Alexander Serebrenik and Danny De Schreye. On termination of meta-programs. In *Proceedings of the Artificial Intelligence on Logic for Programming (LPAR '01)*, pages 517–530, London, UK, 2001. Springer-Verlag. To appear in *Theory and Practice of Logic Programming*; available at http://arxiv.org/abs/cs.PL/0110035.

[10] Michael B. Smyth. Powerdomains. *J. of Computer and System Sciences*, 16(1):23–36, February 1978.