On Lazy Commutation

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For Nissim, distinguished scholar and longtime friend.

Abstract. We investigate combinatorial commutation properties for reordering a sequence of two kinds of steps, and for separating well-foundedness of unions of relations. To that end, we develop the notion of a constricting sequence. These results can be applied, for example, to generic path orderings used in termination proofs.

"Loop" as a train destination means that the train enters the Loop elevated structure in downtown Chicago, does a complete circle, and then returns the way it came. Red Line and Blue Line trains serve the Loop area of Chicago. . . .

– Paul Inast (wikipedia.org)

1 Introduction

Imagine a city that has two coöperating bus companies, a Red line and a Blue line. An unusual condition of Doornbos, Backhouse, and van der Woude [13] guarantees that if a circular excursion is possible, then there is one traveling solely on one of the lines. This condition, denominated here "lazy commutation", states that if you can get somewhere by first taking a Red bus and then transferring to a Blue one, then – unless the Red line can take you directly to your destination – the Blue line will be able take you someplace whence you can also get to your destination by some combination of segments.

We demonstrate that lazy commutation implies that there is no need for the Red line to ever issue transfers to Blue routes, provided there are no circular trips that are purely Red or purely Blue. Bear in mind that the fact that one bus line operates from point X to point Y does not mean that it – or its competitor – necessarily operates in the opposite direction, from Y to X.

One of the earlier works on using commutation properties in termination proofs, especially of complicated forms of rewriting, was by Porat and Francez [29]. That same avenue is explored here.

Lazy commutation turns out to be a very meaningful concept. It has three important consequences regarding sequences of Red and Blue steps:

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1. It implies "finite separation", in the above sense, namely that Blue steps followed by Red steps will get one to any destination – unless there is an infinite monochromatic (single-step or multiple-step) excursion.
2. It implies "infinite separation", in that if there is an infinite combined Red-Blue excursion, then there must be an infinite monochromatic excursion as well, as was shown in [13].
3. If there is an endless combined excursion, but no endless purely Blue ones, then there is also an endless Red excursion from which a transfer at any point to the Blue line precludes wandering the lines forever.

After fixing notations and some terminology in the next section, we collect some new and old observations on separability in Sect. 3. Then, in Sect. 4, we consider conditions that ensure that a positive number of steps separates into a likewise nonempty sequence. This is followed by a definition, in Sect. 5, of "constricting" sequences, expanding on a notion of Plaisted's [28].

A weaker notion than laziness, dubbed "selection" and defined in Sect. 6, suffices to prove the main separation results in Sects. 7 and 8. Section 9 discusses the "lifting" of termination of one color to that of the other in the presence of separation.

We close with some thoughts on the implications that lazy commutation has for proofs of well-foundedness of orderings, such as the path orderings commonly used in proofs of termination (strong normalization) of rewriting.

2 Notions and Notations

All relations in this paper are binary. We use juxtaposition for composition of relations, represent union by $+$, and denote the inverse, n-fold composition, transitive closure, reflexive closure and reflexive-transitive closure of relation E by E^-, E^n, E^+, E^ε and E^*, respectively. Let I be the identity relation.

A relation E on a finite or infinite set of elements V may be viewed as a directed graph with vertices V and edges E. So, there is a (directed) path from s to t, for $s, t \in V$, if, and only if, $s\ E^*\ t$.

The *immortal* elements of a set V equipped with a relation E are those $t \in V$ that initiate infinite E-chains of (not necessarily distinct) elements of V,

$$t\ E\ t'\ E\ t''\ E \cdots .$$

We will also say that a step $t\ E\ t'$ is *immortalizing* if t' is immortal. Define

$$E^\infty = \{\langle u, v\rangle \mid u, v \in V,\ u \text{ is immortal for } E\} ,$$

relating immortal elements to all vertices, as is commonly done in relational program semantics (e.g. [13]). Thus,

$$E^\infty C \subseteq E^\infty , \tag{1}$$

for any relation C.

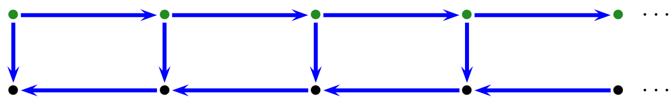


Fig. 1. Mortal (black) nodes on bottom and immortal (green) nodes on top

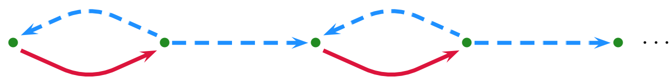


Fig. 2. Mortal in each alone (dashed **Azure** or solid **Bordeaux**), but immortal in their union

In Fig. 1, all the nodes on the upper row are immortal, while all those on the lower row are mortal. Taking a "wrong" turn can precipitate mortality, but, in contrast, once mortal, always mortal.

We compose relations, using notation like $s\ BA\ t$ to mean that there is two-step path from s to t, taking B followed by A. Similarly, $s \in AB^\infty$ means that s has an A-neighbor that is immortal for B. We say that a relation E is *well-founded* (regardless of whether E is transitive) if it admits no cases of immortality, that is, if $E^\infty = \varnothing$.

This paper explores properties of the (non-disjoint) union $E = A + B$ of two relations, A and B. Throughout, we will regularly use E, without elaboration, to denote the union $A + B$.

In Fig. 2, the individual relations A (think Azure in the illustrations) and B (Bordeaux) do not engender immortality, but their union does.

The following two main properties will claim our attention:

Definition 1 (Finite Separation). *Two relations A and B are* finitely separable *if*

$$(A + B)^* = A^* B^* \ .$$

Definition 2 (Infinite Separation). *Two relations A and B are* infinitely separable *if*

$$(A + B)^\infty = (A + B)^* (A^\infty + B^\infty) \ . \tag{2}$$

Nota bene. Finite separation is not a symmetric property: we want the A's before the B's. Infinite separation, on the other hand, is symmetric.

Any two points in the graph depicted in Fig. 3 can be connected by A's followed by B's, so the graph is finitely separable. Furthermore, every node is immortal in B alone, so it is infinitely separable, as well.

In terms of the bus routes (for which there are no simple [non-looping] infinite paths), infinite separability means that at least one company must provide a circular excursion, whenever it is possible to go in circles using both companies.

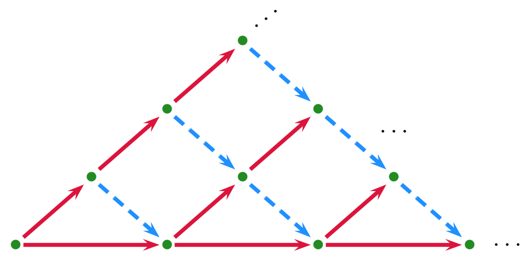


Fig. 3. The two relations are both finitely and infinitely separable

It is easy to see [33, Ex. 1.3.5(ii)] that finite separability (called "postponement" in [33]) is equivalent to the (global) "commutation" property

$$B^* A^* \subseteq A^* B^* \ . \tag{3}$$

Note 3. This is analogous to the equivalence of the Church-Rosser and global confluence properties of abstract rewriting [27].

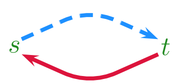
Another trivial induction also shows that (3) is equivalent to an even simpler property:

Proposition 4 ([30]). *Two relations A and B are finitely separable if, and only if,*

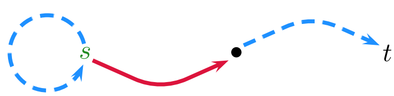
$$B^+ A \subseteq A^* B^* \ .$$

The point of infinite separability is that it ensures that the union $E = A + B$ is well-founded if each of A and B is, even without finite separability.

Example 5. An easy example of finite, but not infinite, separability is $s\ A\ t\ B\ s$.



Example 6. A simple example of infinite, but not finite, separability is $s\ A\ s\ BA$ t.



Definition 7 (Full Separation). *Two relations A and B are* fully (infinitely) separable *if*

$$\begin{aligned}(A+B)^\infty &= A^* B^* (A^\infty + B^\infty) \\ &= A^* B^* A^\infty + A^* B^\infty \ .\end{aligned}$$

Table 1. Types of separation ("coefficients" of right-hand side terms)

<table>
<tr><th></th><th>$\subseteq$</th><th>A^*B^*</th><th>B^*</th><th>A^∞</th><th>B^∞</th></tr>
<tr><td>Weak (Def. 44)</td><td rowspan="2">E^*</td><td rowspan="2" colspan="2">I</td><td>A^*B^*</td><td>A^*</td></tr>
<tr><td>Finite (Def. 1)</td><td rowspan="2" colspan="2"></td></tr>
<tr><td>Productive (Def. 28)</td><td>E^+</td><td>A</td><td>B</td></tr>
<tr><td>Infinite (Def. 2)</td><td rowspan="4">E^∞</td><td rowspan="4" colspan="2"></td><td rowspan="2">E^*</td><td>E^*</td></tr>
<tr><td>Nice (Def. 9)</td><td rowspan="3">A^*</td></tr>
<tr><td>Full (Def. 7)</td><td>A^*B^*</td></tr>
<tr><td>Neat (Def. 8)</td><td>I</td></tr>
</table>

With finite separability, infinite separability (2) is equivalent to full separability.

A pleasant special case is the following:

Definition 8 (Neat Separation). *Two relations A and B are* neatly (infinitely) separable *if*

$$(A + B)^\infty = A^\infty + A^*B^\infty \ .$$

A nice in-between (asymmetric) notion of separability is the following:

Definition 9 (Nice Separation). *Two relations A and B are* nicely (infinitely) separable *if*

$$(A + B)^\infty = (A + B)^*A^\infty + A^*B^\infty \ .$$

Clearly, then, If relations A and B are nicely, or fully, separable, and A is well-founded, then

$$E^\infty = A^*B^\infty \ .$$

Table 1 summarizes the various types of separability we will be using, from weaker down to stronger. For example, the penultimate row defines a relatively strong form of infinite separability (see Definition 7) and should be read as

$$E^\infty \subseteq A^*B^*A^\infty + A^*B^\infty \ .$$

It implies nice separation, and, a fortiori, (plain) infinite separation, but not neat separation.

3 Warmup

We begin with relatively simple cases of separability. We are looking for *local* conditions on double-steps BA that help establish separability, finite or infinite. The point is that these local conditions suggest how to eliminate BA patterns, which are exactly what are prohibited in separated sequences.

3.1 Finite Separation

Note that $CB^\varepsilon = CB + C$, and recall the following:

Proposition 10 ([20]). *If, for relations A and B,*

$$BA \subseteq A^* B^\varepsilon \tag{4}$$

then A and B are finitely separable.

Note 11. When B is the inverse A^- of A, Eq. (4) becomes (one half of) Huet's [18] strong-confluence condition.

Proof. Repeatedly rewriting the last occurrence of BA in an E-chain with instances of $A^* B^\varepsilon$ must terminate in a sequence without any instance of BA, since either the number of B's decreases, or else the rightmost B moves rightward. □

By symmetry (of A and B and left and right):

Proposition 12. *If, for relations A and B,*

$$BA \subseteq A^\varepsilon B^* \ , \tag{5}$$

then A and B are finitely separable.

Proof. An alternate proof to that of the previous proposition proceeds as follows: First, an easy induction shows that (5) implies that $B^n A \subseteq A^\varepsilon B^*$, for all n:

$$B^n BA \subseteq B^n A^\varepsilon B^* = B^n B^* + B^n AB^* = B^* + A^\varepsilon B^* = A^\varepsilon B^* \ .$$

Then, an application of Proposition 4, on account of $B^* A \subseteq A^\varepsilon B^*$, gives finite separation. □

As an aside, in direct analogy to Newman's Lemma [27], we have the following:

Proposition 13. *If, for relations A and B, $A + B^-$ is well-founded and*

$$BA \subseteq A^* B^* \ , \tag{6}$$

then A and B are finitely separable.

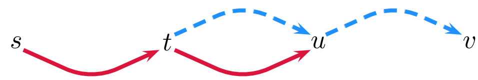
This means that *local (weak) commutation* (6) implies separability, but only with a strong side condition of well-foundedness.

Proposition 14 ([17,5]). *If relation A is well-founded and*

$$BA \subseteq A^+ B^* \ , \tag{7}$$

for relation B, then A and B are finitely separable.

Note 15. The necessity of either well-foundedness of $A + B^-$ or the limiting right-hand sides of inclusions (4) and (5) can be seen from the following example (adapted from [27]): $s\ B\ t\ B\ u\ A\ v$, and $t\ A\ u$. Pictorially:



Note that the edge $t\ E\ u$ in this graph belongs to both relations, A and B.

More generally, we have the following definition and result:

Definition 16 (Quasi-commutation [2]). *Relation A* quasi-commutes over *relation B when*

$$BA \subseteq A(A+B)^* \ . \tag{8}$$

The right-hand side $A(A+B)^*$ is equivalent to $(AB^*)^+$.

Obviously (by induction):

Lemma 17. *Relation A quasi-commutes over relation B if, and only if,*

$$B^+A \subseteq A(A+B)^* \ .$$

Theorem 18. *If relation A quasi-commutes over relation B, then*

$$(A+B)^* \subseteq A^*B^* + A^\infty \ . \tag{9}$$

(Compare Definition 44 below.)

Proof. Consider any sequence $E^* = A^*(B^*A)^*$. Using the above lemma, repeatedly replace the first occurrence of B^+A with AE^*. If this process continues forever, then the initial A-segment grows ever larger. Otherwise, we are left with a sequence A^*B^* sans occurrences of BA. □

Corollary 19. *If relation A is well-founded, then quasi-commutation of A over a relation B implies their finite separability.*

Since A is well-founded, one may assume that a property in question holds for every t such that $s\ A\ t$, in an inductive proof that it holds for s. This suggests the following alternative proof of this corollary:

Proof. We show that every sequence E^* that is not in the separated form A^*B^* can be rearranged to be "more separated". By quasi-commutation (Eq. 8),

$$E^* \setminus A^*B^* = A^*BAE^* \subseteq A^*AE^* \ ,$$

and by induction on A

$$A^*AE^* \subseteq A^+B^* \ .$$

So, in all cases, $E^* \subseteq A^*B^*$. □

Note 20. Quasi-commutation applies to combinatory logic and to orthogonal (left-linear non-overlapping) term-rewriting systems, where A is a leftmost (outermost) step and B is anything but. This means that leftmost steps may precede all non-leftmost ones. Combined with the fact that $A^-B \subseteq B^*A^-$, this gives *standardization* (leftmost rewriting suffices for computing normal forms). See [23].

3.2 Infinite Separation

We start with some obvious aspects of infinite chains:

Proposition 21. *For all relations A and B,*

$$\begin{aligned}(A+B)^\infty &= (B^*A)^*B^\infty + (B^*A)^\infty \\ &= (A^*B)^*A^\infty + (A^*B)^\infty \ .\end{aligned} \tag{10}$$

It follows from (10) that if infinitely many interspersed A-steps are precluded, then $A + B$ is well-founded if, and only if, B is:

Corollary 22. *If relation B is well-founded, then*

$$(A+B)^\infty = (B^*A)^\infty \ . \tag{11}$$

for any relation A.

In other words, in any E-chain, there are either infinitely many A's, in which case E^∞ must be an infinite sequence of segments of the form B^*A, or else from some point on, there are only B's, and B is not well-founded. Cf. [2, Lemma 1] and [6, Lemma 12B].

The simplest local condition guaranteeing infinite separability is quasi-commutation (8). We have the following analogue of Theorem 18, based on the proof of infinite separation in [2]:

Theorem 23. *If relation A quasi-commutes over relation B, that is, if*

$$BA \subseteq A(A+B)^* \ ,$$

then A and B are neatly separable, that is,

$$(A+B)^\infty = A^\infty + A^*B^\infty \ .$$

The proof is also analogous to that of that earlier proposition:

Proof. Consider any E-chain, $E^\infty = E^*B^\infty + A^*(B^*A)^\infty$, as decomposed in (10). In the first instance (E^*B^∞), apply Theorem 18 to E^* and get $E^\infty \subseteq E^*B^\infty \subseteq (A^*B^* + A^\infty)B^\infty \subseteq A^*B^\infty + A^\infty$. Otherwise, repeatedly replace the first occurrence of B^*A in $A^n(B^*A)^\infty$ with AE^*, giving $A^{n+1}E^*(B^*A)^\infty = A^{n+1}(B^*A)^\infty$. This process continues forever, there being an infinite supply of A's, and the initial A-segment grows ever larger, yielding A^∞. □

Note 24. This form of separability was used to show that "forward closure" termination suffices for orthogonal term rewriting, where B are "residual" steps (at redexes already appearing *below the top* in the initial term) and A are at "created" redexes (generated by earlier rewrites). See [7].

Theorem 25. *If, for relations A and B,*

$$BA \subseteq A^* B \ , \tag{12}$$

then A and B are fully separable, that is,

$$(A+B)^\infty = A^* B^* A^\infty + A^* B^\infty \ .$$

In what follows (Theorem 72), this condition, equivalent to $BA \subseteq A^+ B + B$, will be weakened to laziness, the condition that $BA \subseteq AE^* + B$ (Definition 52), to give separation. That way, each occurrence of BA need not always leave a B, as in this theorem, nor always an A, as in the previous, but might sometimes leave one and sometimes the other.

Proof. If the number of A's or B's in an infinite E-chain is finite, then by virtue of Proposition 10, we have either $A^* B^* A^\infty$ or $A^* B^\infty$. In any case, easy inductions show that $BA^* \subseteq A^* B$ and that $B^n A^* \subseteq A^* B^n$, for all n. Thus, a sequence $A^m B^n A^* B E^\infty$ turns into either $A^{m+k} B^{n+1} E^\infty$, $k > 0$, in which case the number of initial A's increases, or else into $A^m B^{n+1} E^\infty$, in which case the number of following B's increases. In the final analysis, one gets either A^∞, or else $A^* B^\infty$. □

Note 26. This form of separability was used to show that "forward closure" termination suffices for right-linear term rewriting, where B are "created" steps and A are "residual" ones. See [7].

Note 27. When B is the subterm relation, condition (12) is a form of quasi-monotonicity (weak monotonicity) of A. Compare [8,1].

4 Productive Separation

With finite separability, a non-empty looping sequence (as in Example 5) can be "reordered" to give an empty one. To preclude an empty reordering, one can use the following notions:

Definition 28 (Productive Separation). *Relations A and B are* (finitely) productively separable *if*

$$(A+B)^+ = A^+ B^* + B^+ \ ,$$

or, expressed more symmetrically, if

$$(A+B)^+ = A^+ B^* + A^* B^+ \ .$$

Note 29. If, for example, there is no "mixed" loop, that is, if

$$(A+B)^+ \cap I \ \subseteq \ A^+ + B^+ \ ,$$

then finite separation implies productive separation.

Theorem 30. *If relation A quasi-commutes over relation B and A is well-founded, then A and B are productively separable.*

Proof. By Corollary 19, the relations are finitely separable. So, we have:

$$E^+ = AE^* + BE^* \subseteq AE^* + BA^*B^* = AE^* + B^+ + BAE^*$$
$$\subseteq AE^* + B^+ + AE^*E^* \subseteq A^+B^* + B^+ \ . \qquad \square$$

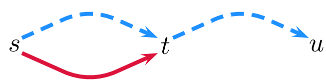
Definition 31 (Promotion). *Relation B* promotes *relation A if*

$$BA \subseteq AB^* + B^+ \ , \tag{13}$$

or, expressed more symmetrically, if

$$BA \subseteq (A+B)B^* \ .$$

Note 32. One can have productivity without promotion: $s \; A \; t \; A \; u$ with $s \; B \; t$.



An easy induction shows the following:

Lemma 33. *Relation B promotes relation A if, and only if,*

$$B^+A \subseteq AB^* + B^+ \ .$$

Theorem 34. *If relation B promotes relation A, then A and B are productively separable.*

Note 35. Obviously, promotion (13) cannot be weakened to allow the erasure of both the A and B (cf. 5), as can easily be seen from $t \; BA \; t$ (Example 5), which is finitely, but not productively, separable.

Proof. By Proposition 12, promotion (13) gives finite separability, $E^* = A^*B^*$. By the previous lemma, $B^*A \subseteq AB^* + B^+$, whence the proposition follows:

$$E^+ = E^*(A+B) = A^*B^*A + A^*B^*B = A^+B^* + A^*B^+ \ . \qquad \square$$

In symmetry with promotion, we also have the following:

Proposition 36. *If, for relations A and B,*

$$BA \subseteq A^*B + A^+ \ ,$$

or, equivalently,

$$BA^* \subseteq A^*B + A^+ \ ,$$

then A and B are productively separable.

5 Constriction

Let V and E be the vertices and edges of a graph, respectively, and let $U = V^2$ be all pairs of vertices, so $E \subseteq U$. For any relation $R \subseteq U$ and property $P \subseteq V$, $RP = \{v \in V \mid v\ R\ w,\ \text{for some } w \in P\}$ are all the vertices from which R takes one to a vertex satisfying P. The kinds of properties P we are interested are:

- a specific goal vertex;
- a set of vertices along some path; and
- the set of immortal vertices.

The point is to consider the impact of a preference for edges $A \subseteq E$ over others en route to P.

Given some property P, it will be convenient throughout this section to restrict E to only those edges allowing one to get to P. Let $\underline{V} = E^*P$ be those vertices from which P is reachable, let $\underline{U} = \underline{V}^2$, and let $\underline{R} = R \cap \underline{U}$ be the restriction of any relation R to vertices leading to P.

For any $B \subseteq E$, let

$$B_\sharp = \underline{B} \setminus \underline{AU}\ ,$$

that is, a useful B-step at a point such that no useful A-step can lead to P. Such $B_\sharp$-steps will be called *constricting*.

Example 37. Fair termination of $E = A + B$ with respect to A, à la the work of Porat and Francez [29], means that the union E is well-founded, except for the possibility of an infinite sequence that "unfairly" avoids taking an A-step infinitely often. So, the only way E can diverge is if from some point on A-steps were possible infinitely many times, but were never taken. This property can be expressed as

$$(B^*A)^\infty = B_\sharp^\infty = \varnothing\ ,$$

where the property P that defines $B_\sharp$ is always satisfied. It means that there are no infinite sequences with infinitely many A's (which certainly are fair), nor are there infinite sequences of B's, at every point of which no A-step was possible (these are the $B_\sharp$).

Accordingly:

Definition 38 (Constricting Chains). *An E-chain $t_0\ E\ t_1\ E\ \cdots$ is* constricting in A with respect to P *if its non-A steps are constricting, in the sense that there is no case of $t_i\ AE^*P$, for any $t_i\ (E \setminus A)\ t_{i+1}$.*

This means that the steps are all in $A + B_\sharp$. Our notion generalizes the "constricting" sequences of [28].

For any property, one can always build a constricting sequence. The idea is to ignore B-steps whenever an A-step is available:

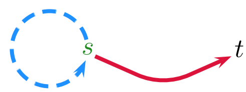
Theorem 39. *For any relations A and B and property P,*

$$\begin{aligned}(\underline{A}+\underline{B})^* &\subseteq (\underline{A}+B_\sharp)^* + (\underline{A}+B_\sharp)^\infty \\ &= (\underline{A}+B_\sharp)^* + (\underline{A}+B_\sharp)^*\underline{A}^\infty + (\underline{A}^*B_\sharp)^\infty \ ,\end{aligned}$$

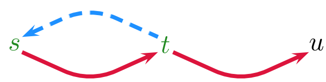
where $\underline{A}$ and $\underline{B}$ are A and B steps, respectively, from which P is reachable and $B_\sharp$ is a $\underline{B}$-step from whose origin there is no outgoing $\underline{A}$.

Proof. Just take an A-step whenever possible, that is, whenever there is one that leads to P. Only take B if there is no choice, so the B-steps are constricting. Then, every A-step is actually $\underline{A}$ and every B-step is $B_\sharp$ (as well as $\underline{B}$). This process either terminates, or else goes on forever, with either infinitely many B steps, or finitely many. □

Note 40. If there is an infinite A-path, then preferring A can be a bad idea. Witness: $s\ A\ s\ B\ t$, that is,



Note 41. Even if the sole infinite path involves both relations, A's can preclude reaching the goal, as in $t\ A\ s\ B\ t\ B\ u$:



6 The Selection Property

In this section we deal with non-local properties for which constriction yields separation.

Definition 42 (Selection). *We say that relation B* selects *relation A if*

$$BA^+ \subseteq A(A+B)^* + B^+ \ . \tag{14}$$

Definition 43 (Weak Selection). *We say that relation B* weakly selects *relation A if*

$$BA^+ \subseteq A(A+B)^* + B^* + A^*B^*(A^\infty + B^\infty) \ . \tag{15}$$

Clearly, weak selection is weaker than selection.

Quasi-commutation gave a special case (Eq. 9) of the following property:

Definition 44 (Weak Separation). *Two relations A and B are* weakly (finitely) separable *if*

$$\begin{aligned}(A+B)^* &\subseteq A^*B^* + A^*B^*A^\infty + A^*B^\infty \\ &= A^*B^*(I + A^\infty + B^\infty) \ .\end{aligned}$$

It is straightforward to see that

Proposition 45. *Infinite separability plus weak separability give full separability.*

Restricting attention to constricting sequences teaches the following:

Theorem 46. *If relation B weakly selects relation A, then A and B are weakly separable.*

Proof. Suppose $s\ E^*\ t$ and let $P(x)$ hold only for $x = t$. The premise (15) means that

$$B_\sharp \underline{A}^* \subseteq B^* + B^*(A^\infty + B^\infty) = B^*(I + A^\infty + B^\infty) ,$$

since there can be no (relevant) outgoing A-step where there is a constricting B-step. Theorem 39 asserts the (algebraic) equivalent of

$$\underline{E}^* \subseteq \underline{A}^*(B_\sharp \underline{A}^*)^* + \underline{A}^*(B_\sharp \underline{A}^*)^\infty + \underline{A}^*(B_\sharp \underline{A}^*)A^\infty ,$$

and, replacing $B_\sharp \underline{A}^*$ as reasoned above,

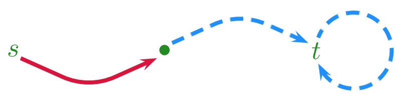
$$\begin{aligned} E^* &\subseteq A^*B^*[(I + A^\infty + B^\infty) + (I + A^\infty + B^\infty)^\infty + (I + A^\infty + B^\infty)A^\infty] \\ &\subseteq A^*B^*(I + A^\infty + B^\infty) . \end{aligned}$$

□

Note 47. An alternative weak version of selection,

$$BA^+ \subseteq A(A+B)^* + B^* + (A+B)^*(A^\infty + B^\infty) ,$$

does not yield weak separability. For example, $s\ BA\ t\ A\ t$ is not weakly separable, though $BA^+ \subseteq B^*A^\infty$:



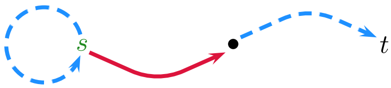
Theorem 48. *If relation B selects relation A and both are well-founded, then A and B are productively separable.*

Proof. Selection implies weak selection, which, per the previous theorem, gives weak separation, which, with well-foundedness, is finite separation. So selection, itself, becomes

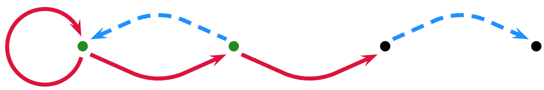
$$BA^+ \subseteq AE^* + B^+ \subseteq A^+B^* + B^+ ,$$

as required by Definition 28. □

Note 49. Without well-foundedness of A, one does not have separation, as may be seen from the following selecting, but unseparable example: $s\ A\ s\ BA\ t$. That is,



Note 50. Well-foundedness of B is also needed. To wit:



A somewhat analogous way of obtaining a "productive" version of weak separability, similar in flavor to promotion as used in Theorem 34, is the following:

Theorem 51. *If, for relations A and B,*

$$B^+A \subseteq A(A+B)^* + B^+ + A^*B^*A^\infty \; , \tag{16}$$

then

$$(A+B)^+ \subseteq A^+ + A^*B^+ + A^*B^*A^\infty \; .$$

Proof. If a nonempty E-chain is not of the form A^*B^*, then it must be of the form $A^mB^+AE^n$ $(m,n \geq 0)$. The premise (16) is then used to replace the segment B^+A. The situation devolves into one of three possibilities:

1. We get A^mAE^*, in which case the initial run of A's grows by at least one, and the process may be repeated.
2. We get $A^mB^+E^n$, in which case the AE^*-tail shrinks to length less than $n+1$, and the process is repeated.
3. We get $A^mA^*B^*A^\infty$, in which case we are done.

So, either the process continues ad infinitum and infinitely many A's get pushed frontwards, giving A^∞, or else eventually the process ends happily. It can end either in immortality, when the third case transpires, or else in separation – with no subsequences B^+A to power the process. □

7 Lazy Commutation

In [13,14], the following local property is defined and examined:

Definition 52 (Lazy Commutation). *Relation A* commutes lazily over *relation B if*

$$BA \subseteq A(A+B)^* + B \; . \tag{17}$$

Figure 3 has lazily commuting relations; Fig. 2 does not.

Back to busses: Lazy commutation means that a Red-Blue trip can be substituted by a single ride on Red or by a junket beginning with a Blue segment.

Example 53. The relations $s\ B\ t\ A\ t$ commute lazily:

Note 54. Lazy commutation is noticeably weaker than quasi-commutation (8), which was shown in [2] to separate well-foundedness of the union into well-foundedness of each. (See [31] for an automated proof using Kleene algebras.) Corollary 76 below is stronger.

Note 55. As pointed out in [13], lazy commutation is also much weaker than transitivity of the union, which – by a direct invocation of a very simple case of the infinite version of Ramsey's Theorem – gives separation. (See [17,4] for some of the history of this idea.)

Note 56. Note that replacing BA with $AABB$ is a process that can continue unabated: BBA, $BAABB$, $AABBABB$, $AABAABBBB$, ..., with infinitely many paths through the graph.

The following is known:

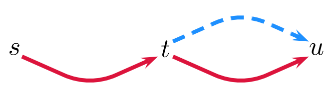
Proposition 57 ([14, Eq. 4.5]). *If A commutes lazily over B, then*

$$BA^* \subseteq A(A+B)^* + B \ , \tag{18}$$

and – in particular – B selects A.

In short, laziness is stronger than selection.

Note 58. An example of selection without laziness is $s\ B\ t\ B\ u$ with $t\ A\ u$, the graph of which looks like this:



It is convenient to make use of the following notion:

Definition 59. *Relation X* absorbs *relation A if*

$$XA \subseteq X \ .$$

For example, A^*, E^*, B^∞, and $\varnothing$ all absorb A. (Recall Eq. 1.) Obviously, if X absorbs A, then it absorbs any number of A's, and CX absorbs A whenever X does.

Proposition 60. *If*

$$CA \subseteq C + X$$

for a relation X that absorbs A, then

$$CA^* \subseteq C + X \ .$$

Proof. By induction on n,

$$CA^n \subseteq C + X \ ,$$

since, trivially, $C \subseteq C + X$, while, by the inductive hypothesis and absorption,

$$CAA^n \subseteq (C + X)A^n = CA^n + XA^n \subseteq C + X + X = C + X \ . \qquad \square$$

Proposition 57 is a corollary, with $C = B$ and $X = AE^*$:

$$BA^+ \subseteq B + AE^* \subseteq B^+ + AE^* \ .$$

A weaker version of laziness, which allows for infinite exceptions, is the following:

Definition 61 (Lackadaisical Commutation). *Relation A* commutes lackadaisically over *relation B if*

$$BA \subseteq A(A + B)^* + B + A^*B^*(A^\infty + B^\infty) \ . \tag{19}$$

Theorem 62. *If relation A commutes lackadaisically over relation B, then B selects A weakly.*

Proof. Applying the previous proposition (E^*, A^∞, and B^∞ each absorb A), the premise (19) implies

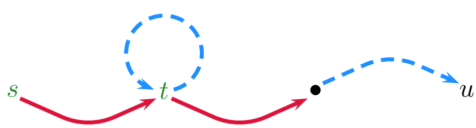
$$BA^+ \subseteq AE^* + B + A^*B^*(A^\infty + B^\infty) \ ,$$

which is more than weak selection. $\square$

Now, by Theorem 46:

Theorem 63. *If relation A commutes lackadaisically over relation B, and – in particular – if A commutes lazily over B, then A and B are weakly separable.*

Note 64. That an infinite A-chain cannot be ruled out can be seen from $s \ B \ t \ A \ t \ BA \ u$:



So lackadaisicalness gives only weak separability, rather than finite separability.

With well-foundedness, weak separation turns into finite separation and lackadaisical commutation into lazy commutation.

Corollary 65. *If relation A commutes lazily over relation B, and both are well-founded, then in fact*

$$BA \subseteq A^+B^* + B \ .$$

Proof. Since laziness with well-foundedness gives finite separation,

$$BA \subseteq AE^* + B \subseteq AA^*B^* + B \ . \qquad \square$$

Moreover, we have productive separability (Definition 28):

Theorem 66. *If relation A commutes lazily over relation B, and both are well-founded, then A and B are productively separable.*

Proof. A simple induction – using the previous lemma – yields

$$BA^+ \subseteq A^+B^* + B \ .$$

Thus, using finite separation,

$$E^+ = EA^*B^* = A^+B^* + BA^+B^* + B^+ \subseteq A^+B^* + B^+ \ . \qquad \square$$

Note 67. Well-foundedness of A and of B are necessary, even in the presence of lazy commutation. Refer to the examples in Notes 49 and 50.

Note 68. Lazy commutation cannot be weakened to

$$BA \subseteq AE^* + B^+ \ .$$

Even

$$BA \subseteq AA + BB$$

causes trouble, as can be seen from the non-separable example in Note 15.

Note 69. Lazy commutation also cannot be weakened to

$$BA \subseteq AE^* + B^\varepsilon \ ,$$

as can be seen from the following non-separable graph:

8 Endless Commutation

We will see below that selection (Definition 42) suffices for the main consequences of lazy commutation, namely finite and infinite separation. But, first, observe the following:

Lemma 70. *If*

$$BA^+ \subseteq A(A+B)^* + B^+ + (A+B)^*(A^\infty + B^\infty) , \tag{20}$$

for relations A *and* B*, then they are infinitely separable.*

This will be our main tool for separation.

Note 71. The condition (20) clearly does not suffice for full separation. See the example in Note 47.

Proof. We use constriction (Sect. 5), letting $P(x)$ hold for immortal x. The premise (20) implies that

$$B_\sharp \underline{A}^* = B^+ + E^*(A^\infty + B^\infty) ,$$

since $B_\sharp$ precludes there being an initial immortalizing A-step. By Theorem 39:

$$E^* \subseteq A^*(B_\sharp \underline{A}^*)^* + A^*(B_\sharp \underline{A}^*)^\infty + A^*(B_\sharp \underline{A}^*)^* A^\infty .$$

Infinite composition of this means that either the first possibility, $A^*(B_\sharp \underline{A}^*)^*$, repeats forever, or else it repeats finitely often and then one of the two infinite cases takes over:

$$\begin{aligned} E^\infty &= (A^*(B_\sharp \underline{A}^*)^*)^\infty \\ &\quad + (A^*(B_\sharp \underline{A}^*)^*)^* A^*(B_\sharp \underline{A}^*)^\infty + (A^*(B_\sharp \underline{A}^*)^*)^* A^*(B_\sharp \underline{A}^*)^* A^\infty \\ &= A^*(B_\sharp \underline{A}^*)^\infty + A^*(B_\sharp \underline{A}^*)^\infty + A^*(B_\sharp \underline{A}^*)^* A^\infty . \end{aligned}$$

By the premise, this is

$$A^*B^\infty + A^*B^*A^\infty + E^*(A^\infty + B^\infty) = E^*(A^\infty + B^\infty) ,$$

as desired. □

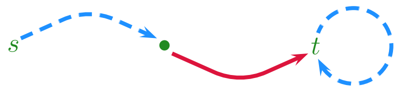
The net result is that

Theorem 72. *If relation* B *selects relation* A*, and – in particular – if* A *commutes lazily over* B*, then* A *and* B *are fully separable.*

Proof. Selection (14) implies the condition (20) of the previous lemma, giving infinite separation. Theorem 46 gives weak separation; Proposition 45 gives full separation; Proposition 57 gives the particular, lazy case. □

The fact that laziness implies infinite separability is the main result of [13].

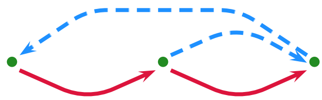
Example 73. The graph $s\ AB\ t\ A\ t$ is lazily commuting, since $BA \subseteq B$, and in fact $s\ ABA^\infty$, as can be seen in the following diagram:



So, the relations are fully separable, but not neatly separable.

Note 74. Obviously, sans lazy commutation, immortality cannot necessarily be separated. For example, with $s\ AB\ s$, only in the union is s immortal. See Example 5.

Note 75. Lazy commutation cannot be weakened to include $BA \subseteq AA + BB$. To see that, wrap the counterexample of Note 68 around itself, as follows:



Combining this theorem with Theorem 48, we can summarize by saying the following:

Corollary 76. *If relations A and B are both well-founded, and, furthermore, B selects A, or – in particular – A commutes lazily over B, then the union $A+B$ is also well-founded and A and B are (finitely) productively separable.*

When the relations are finite, as for the bus companies of the introduction, any infinite tours are circular.

Corollary 77. *If relation B promotes relation A, then A and B are fully (infinitely) separable, as well as (finitely) productively separable.*

Proof. By Lemma 33, $B^+A \subseteq AB^* + B^+$, so A commutes lazily over B^+. Clearly, $E^\infty = (A+B^+)^\infty$. The result follows from Theorems 72 and 34. □

Note 78. Promotion (Eq. 13) cannot be weakened to allow the erasure of both the A and B, as can be seen from $t\ BA\ t$, which only cycles in the union. Compare Note 35.

Actually:

Theorem 79. *Whenever relations A and B are (finitely) productively separable, they are also fully (infinitely) separable.*

Proof. Productive separability ($E^+ \subseteq A^+B^* + B^+$) implies that A^+ commutes lazily over B^+ (i.e. that $B^+A^+ \subseteq A^+E^* + B^+$) which means, thanks to Theorem 72, that A^+ and B^+ are fully separable, which is the same as A and B themselves being fully separable, since

$$E^\infty = (A^+ + B^+)^\infty = (A^+)^*(B^+)^*(A^+)^\infty + (A^+)^*(B^+)^\infty = A^*B^*A^\infty + A^*B^\infty .$$

□

Alternatively, one could have agreed directly from constriction.

9 Lifting Relations

For various applications of commutation arguments, the well-foundedness of one relation is dependent on that of the other.

First, note the following:

Proposition 80. *If, for relations A, B, and C, one has*

$$B \subseteq A^+B + C \ ,$$

then

$$B \subseteq A^\infty + A^*C \ .$$

Proof. By unending application of the premise, we get (the greatest pre-fixpoint) $A^\infty + A^*C$ (for B). □

In particular, if $B^\infty \subseteq A^+B^\infty + A^\infty$, then $B^\infty \subseteq A^\infty$, meaning that B is well-founded if A is.

Theorem 81. *If relations A and B are infinitely separable and if*

$$B^\infty \subseteq A^+B^\infty + (A+B)^*A^\infty \ ,$$

then

$$(A+B)^\infty = (A+B)^*A^\infty \ .$$

Proof. By the premise and the above proposition,

$$B^\infty \subseteq E^*A^\infty \ .$$

By infinite separation,

$$E^\infty = E^*A^\infty + E^*B^\infty \subseteq E^*A^\infty + E^*E^*A^\infty = E^*A^\infty \ . \qquad \square$$

Definition 82 (Lifting). *Relation A* lifts to *relation B if*

$$B^\infty \subseteq A(A+B)^\infty \ . \tag{21}$$

Regarding lifting (21), see [22,16,19]. It implies that if there is an infinite E-chain, then there is one with infinitely many interspersed A-steps. That is:

Lemma 83. *If relation A lifts to relation B, then*

$$(A+B)^\infty = (B^*A)^\infty \ .$$

Compare Eq. (11).

Proof. By Proposition 21 and lifting,

$$\begin{aligned} E^\infty &= (B^*A)^*B^\infty + (B^*A)^\infty \\ &\subseteq (B^*A)^*AE^\infty + (B^*A)^\infty = (B^*A)^+E^\infty + (B^*A)^\infty . \end{aligned}$$

By the previous proposition, we obtain the following:

$$E^\infty \subseteq (B^*A)^\infty + (B^*A)^*(B^*A)^\infty = (B^*A)^\infty . \qquad \square$$

Moreover, B steps need only be taken when no A step leads to immortality.

Lemma 84. *If relations A and B are nicely separable and A lifts to B, then*

$$(A+B)^\infty = (A+B)^*A^\infty .$$

Proof. Combining niceness and lifting, we have that

$$E^\infty \subseteq E^*A^\infty + A^*B^\infty \subseteq A^+E^\infty + E^*A^\infty .$$

By Proposition 80, E^∞ gives either A^∞ or else $A^*E^*A^\infty$. □

In other words:

Theorem 85. *If relations A and B are nicely separable and A lifts to B, then $A+B$ is well-founded if (and only if) A is.*

Finally, we have the following:

Theorem 86. *If relation A commutes lazily over relation B and A lifts to B, then $A+B$ is well-founded if, and only if, A is.*

Proof. Use Theorems 72 and 85, bearing in mind that full separation is more than just nice separation. □

10 Discussion

Table 2 summarizes many of the more significant dependencies derived in the preceding sections.

The claims of Sects. 3 and 4 are amenable to automated proofs [32]. More work is needed, however, to automate the more advanced results of Sects. 6–8. This interest of ours in mechanization is why we have indulged in alternative proofs of some results.

In future work, we plan to use the ideas developed herein to show how laziness and constriction can contribute to proving well-foundedness of an abstract path ordering – in the style of [19,6] – which includes the nested multiset ordering [12], multiset path ordering [8], lexicographic path ordering [22], and recursive path ordering [24,9] as special cases. We seem to need a weaker alternative to lifting [16,19], in which lifting need only take place eventually.

Table 2. Implication graph (Legend: **D**efinition; **N**ote; **P**roposition; **T**heorem)

Quasi-Commutation		$\Longrightarrow$ T23		Neat Separation		Promotion
$\Downarrow$**N54**				$\Downarrow$**D8**		$\Downarrow$**T34**
Lazy Commutation	$\Longrightarrow$ P57	Selection	$\Longrightarrow$ T72	Full Separation	$\Longleftarrow$ T79	Productive Separation
$\Downarrow$**D61**		$\Downarrow$**D43**				$\Downarrow$**D28**
Lackadaisical Commutation	$\Longrightarrow$ T62	Weak Selection	$\Longrightarrow$ D44	Weak Separation	$\Longleftarrow$ T46	Finite Separation

As pointed out in [10], there is an analogy between the use of reducibility predicates and the use of constricting derivations in proofs of well-foundedness. We are optimistic that the commutation-based approach taken here will likewise help for advanced path orderings, like the general path ordering [11] and higher-order recursive-path-ordering [15,21,3], without recourse to reducibility/computability predicates.

We plan to analyze minimal bad sequence arguments for well-quasi-orderings (pioneered in [26]) in a similar fashion. See [25]; compare [19]. We also hope to apply the results of the previous sections to analyze the dependency-pair method of proving termination. See [1]; compare [10].

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