Infinite Normal Forms*
(Preliminary Version)

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Abstract

We continue here a study of properties of rewrite systems that are not necessarily terminating, but allow for infinite derivations that have a limit. In particular, we give algebraic semantics for theories described by such systems, consider sufficient completeness of hierarchical systems, suggest practical conditions for the existence of a limit and for its uniqueness, and extend the ideas to conditional rewriting.

1. Introduction

Rewrite systems are sets of directed equations used to compute by repeatedly replacing equal terms in a given formula, as long as possible. A key property for rewrite system is "canonicity", i.e. that every term rewrites to a unique normal form. Canonicity is usually decomposed into two requirements: "termination", which ensures that at least one normal form always exists; and "confluence", which ensures that there can never be more than one normal form. For surveys of the theory of rewriting, see [Huet-Oppen-80], [Klop-87], or [Dershowitz-Jouannaud-89].

In [Dershowitz-Kaplan-89], an investigation was begun into analogous properties of systems that have *infinite* terms as normal forms. Such systems are not terminating in the classical sense; instead one is interested in establishing "ω-termination", i.e. that any (infinite) derivation has a limit, and "ω-confluence", which ensures uniqueness of limits. Together, these two properties imply the existence of a (potentially infinite) unique normal form for any input term.

Let $\rightarrow$ denote any binary relation; we use $\leftarrow$ to indicate its inverse; by $\overset{*}{\rightarrow}$ we mean the reflexive-transitive closure of $\rightarrow$. A relation $\rightarrow$ over a set S is said to be *finitely terminating* if there exist no infinite chains $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n \rightarrow \cdots$ of elements s_i in S; it is *(finitely) confluent*

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if for any s, t, and u in S such that $u \xrightarrow{*} s$ and $u \xrightarrow{*} t$, it is the case that $s \xrightarrow{*} v$ and $t \xrightarrow{*} v$ for some v in S. Confluence may be expressed as the set-theoretic inclusion $\xleftarrow{*}\circ\xrightarrow{*} \subseteq \xrightarrow{*}\circ\xleftarrow{*}$, where $\circ$ denotes composition of relations.

The following are the basic definitions for infinite rewriting:

Definition 1. Given a binary relation $\rightarrow$ on a topological space S, its *α-iterate* $\rightarrow^{\alpha}$ (for given ordinal α) is defined as follows:
(a) if $\alpha = 0$, then $\rightarrow^{\alpha}$ is the identity relation;
(b) if α is a successor ordinal $\beta+1$, then $\rightarrow^{\alpha} = \rightarrow^{\beta} \cup (\rightarrow^{\beta}\circ\rightarrow)$.
(c) if α is a limit ordinal, then $s_0 \rightarrow^{\alpha} s'$ if $s_0 \rightarrow^{\beta} s'$ for some $\beta < \alpha$ or if there exist elements $(s_\gamma)_{\gamma<\alpha}$ forming a transfinite $\rightarrow$-chain such that $\lim_{\gamma<\alpha} s_\gamma = s'$.[1]

In particular, $s \xrightarrow{\omega} t$, for s and t in S (as in [Dershowitz-Kaplan-89]), iff $s \xrightarrow{*} t$ or there exists a chain $s = s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n \rightarrow \cdots$ such that $\lim_{n<\omega} s_n$ is t.

Definition 2. A binary relation $\rightarrow$ over S is *ω-terminating* if for any infinite chain $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n \rightarrow \cdots$ of elements s_n of S, the limit $\lim_{n<\omega} s_n$ exists.

Definition 3. A binary relation $\rightarrow$ over S is *ω-confluent* if $\xleftarrow{\omega}\circ\xrightarrow{\omega} \subseteq \xrightarrow{\omega}\circ\xleftarrow{\omega}$.

Relations that are both ω-terminating and ω-confluent will be called *ω-canonical.*

We are particularly interested in relations over *terms*. Let $T_\Sigma(X)$ (or just $T(X)$) denote a set of *finite* (first-order) terms containing function symbols and constants from some finite vocabulary (signature) Σ and variables from some denumerable set X. Let $T^\infty_\Sigma(X)$ (or just $T^\infty(X)$) denote the set of finite and *infinite* terms over the same vocabulary and variable set. The set of finite *ground* (variable-free) terms is T_Σ (or just plain T); the set of finite *and* infinite ground terms is $T^\infty_\Sigma(\emptyset)$ (or just T^∞). A *distance d* is defined on T^∞ as follows: Denote by $v(t,t')$ the smallest depth of a position at which t and t' differ (with the convention that $v(t,t) = +\infty$). Let $d(t,t') = 2^{-v(t,t')}$. The terms T^∞, with this distance, form a complete ultra-metric space [Nivat-75].

A *rewrite system R* is a *finite* family of pairs (l,r) of (finite) terms of $T(X)$, each written in the form $l \rightarrow r$, such that all variables appearing on a right-hand side r also appear in l. A system R defines a *rewrite relation* $\rightarrow_R$ over $T^\infty(X)$ as follows: For $t \in T^\infty(X)$, we say that t *rewrites* via R to t', and write $t \rightarrow_R t'$ (or simply $t \rightarrow t'$), iff there exists a rule $l \rightarrow r$ in R, a "context" (term) c in $T^\infty(X)$ with a "position" (occurrence) p in c, and a substitution $\sigma: X \rightarrow T^\infty(X)$ such that $t = c[l\sigma]_p$ (the subterm of t at p is an instance of the left-hand side l) and $t' = c[r\sigma]_p$ (t' is the result of replacing the subterm at p with the corresponding instance $r\sigma$ of the right-hand side).

Definition 4. An *ω-normal form* for a rewrite system R is a term t that is minimal for $\rightarrow$, i.e. if $t \rightarrow t'$, then $t' = t$. An ω-normal form *of* a term s in $T^\infty(X)$ is a term t in $T^\infty(X)$, such that $s \xrightarrow{\omega} t$ and t is an ω-normal form for R.

Note that an ω-normal form need not be irreducible.

By $NF^\omega_R(S)$ we denote the set of ω-normal forms, for R, of terms in the set S. For example, for $R = \{a \rightarrow s(a)\}$, $NF^\omega_R(\{a, s(a)\}) = \{s^{(\infty)}\}$, where $s^{(\infty)}$ is the infinite term $s(s(s(\cdots)))$, composed just of the unary symbol s.

[1] I.e. for any neighborhood V of s', there exists a $\beta<\alpha$ such that s_γ is in V if $\beta \le \gamma < \alpha$.

We say that a rewrite system R is ω-terminating if its rewrite relation $\rightarrow$ is, that it is ω-confluent if $\rightarrow$ is, and that it is ω-canonical if it is both. For instance, the system $\{x \rightarrow f(x)\}$ is ω-terminating; $\{f(x) \rightarrow g(f(x)),\ g(x) \rightarrow f(g(x))\}$ is not. A rewrite system R is said to be *left-linear* if the left-hand side l of each rule $l \rightarrow r$ in R has at most one occurrence of any variable. A term t is said to *overlap* a term t' if t unifies with a non-variable subterm of t'; a system is *non-overlapping* if no left-hand side overlaps another. A system that is both left-linear and non-overlapping is said to be *regular* [O'Donnell-77].

In the remainder of this section, we summarize the main results of [Dershowitz-Kaplan-89]. Related work is in [Ehrig-88].

Theorem 1. *For any left-linear rewrite system and ordinal* $\alpha \geq \omega$,

$$\rightarrow^{\alpha} \; = \; \rightarrow^{\omega}.$$

For non-left-linear systems, such as

$$\begin{aligned} if(x,x,y) &\rightarrow y \\ g(x,0) &\rightarrow g(s(x),0) \\ g(x,s(n)) &\rightarrow if(g(a,n),g(b,n),g(s(x),s(n))) \end{aligned}$$

this is not the case. Indeed, for this example, $g(x,s^{(i)}(0)) \rightarrow^{\omega \times i} g(s^{(\infty)},s^{(i)}(0))$. Note that $d(\lim_{i<\omega} s^{(i)}(a), \lim_{j<\omega} s^{(j)}(b)) = 0$.

Infinite normal forms can be considered the "value" of a term, when they are unique and lend themselves to approximation. A *fair* computation is a derivation for which no redex (position at which a rewrite is applicable) persists forever. More precisely, if $t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n \rightarrow \cdots$ is a fair derivation, and $t_n|_p$ (the subterm of t_n at some position p) is an instance of a left-hand side of a particular rule $l \rightarrow r$ in R for all n past some N, then (at least) one of the rewrites $t_i \rightarrow t_{i+1}$ $(i \geq N)$ must be at or above p (i.e. the redex is at a superterm of $t|_p$). For left-linear systems (only), fair derivations compute ω-normal forms at the limit:

Theorem 2. *Let R be a left-linear rewrite system.*
(a) If t *admits an* ω*-normal form* t'*, then there exists a fair derivation* $t = t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n \rightarrow \cdots$, *with* $\lim_{n<\omega} t_n = t'$.
(b) For any fair derivation $t = t_0 \rightarrow t_1 \rightarrow \cdots \rightarrow t_n \rightarrow \cdots$, *such that there exists a limit* $t' = \lim_{n<\omega} t_n$, t' *is an* ω*-normal form of* t.

If a system R is finitely terminating and confluent, then any finite term t has exactly one finite normal form. Regular systems are always confluent [Huet-80], but since they need not be terminating, there may be terms with no normal form. For infinite rewriting, the following two results hold [Dershowitz-Kaplan-89]:

Theorem 3. *If R is left-linear and* ω*-canonical, then each term has a unique* ω*-normal form.*

Theorem 4. *If R is regular, then it is* ω*-confluent.*

Thus, left-linearity is crucial and, throughout this paper, we deal exclusively with left-linear systems.

The next section provides semantics for infinite rewriting. Section 3 addresses the issue of sufficient completeness. Section 4 describes methods of establishing ω-canonicity. In the final section, the notion of infinite rewriting is extended to conditional systems.

2. Algebraic Semantics

In this section, we consider algebraic aspects of infinitary theories — i.e. their models — and their connection to operational aspects (viz. ω-rewriting). Since we are interested in infinite computations, it is natural to work with *continuous* models. (We refer to [Scott-76, Stoy-77] for general references on the topic.) It is also natural to use a *completion* process. Alternative notions of completion have been studied in the algebraic framework, leading to different initial models, with their own abstract properties. (see, e.g., [ADJ-77, Moeller-84, Tarlecki-Wirsing-86]).

In this paper, to define our class of models, we use the additional assumption that the systems under consideration are ω-terminating. That class admits an initial model, representing precisely the chains of ω-rewriting. Thus, our approach extends — in a natural fashion — the classical, finite approach, using intuition about rewriting.

Definition 5. Given a signature Σ, a *continuous* Σ*-algebra* consists of:
(a) a partially ordered universe $(M,\leq)$, such that each nonempty increasing sequence has a *least upper bound* (lub) in M, and
(b) an interpretation $f^M : M^{arity(f)} \to M$, for each $f \in \Sigma$, that is continuous.

Given a continuous Σ-algebra M, any assignment $\sigma : X \to M$ extends to a morphism $\sigma : T_\Sigma(X) \to M$, as follows:

- if $t = f(t_1, \ldots, t_n)$ for some $f \in \Sigma$, then $\sigma(t) = f^M(\sigma(t_1), \ldots, \sigma(t_n))$;
- if $t = x$ for some $x \in X$, then $\sigma(t) = \sigma(x)$.

Definition 6. Given an ω-terminating rewrite system R over $T_\Sigma(X)$, an *R-model* is a continuous Σ-algebra M that satisfies:
(1) for any rule $l \to r$ in R, assignment σ: $X \to M$, and context c in $T_\Sigma(X)$, the inequality $\sigma(c[l]) \leq \sigma(c[r])$ holds;
(2) for any $\sigma : X \to M$, lub $\sigma(u_n)$ = lub $\sigma(u'_n)$, for sequences (u_n) and (u'_n) over $T_\Sigma(X)$ such that $(u_n) \xrightarrow{\omega} w \xleftarrow{\omega} (u'_n)$ for some w.

Notes:

- The existence of the least upper bounds in (2) comes from the fact that the two sequences (u_n) and (u'_n) are increasing.
- A model need not satisfy *equality* of left- and right- hand sides, as in the classical case, but, rather, an *inequality*.
- The class Mod_R^ω (or just Mod^ω) of R-models, with *continuous* Σ-morphisms, is a non-empty category.

Definition 7. Suppose R is an ω-terminating rewrite system. The model $\hat{T}_R$ (or just $\hat{T}$) has a universe $T \cup NF_R^\omega(T)$, consisting of all finite ground terms and all their (possibly infinite) ω-normal forms, partially ordered by $\xrightarrow{\omega}$. Similarly, the model $\hat{T}_R(X)$ (or just $\hat{T}(X)$) is $T(X) \cup NF_R^\omega(T^\infty(X))$, ordered via $\xrightarrow{\omega}$.

Theorem 5. *If R is ω-terminating, then $\hat{T}$ is initial in Mod^ω.*

Proof. Clearly, $\hat{T}$ satisfies both conditions for R-models. Let $M \in Mod^{\omega}$; we wish to define a morphism ϕ: $\hat{T} \to M$. For t finite, we must take $\phi[t] = t^M$. For an infinite ω-normal form t_∞, we have $t_n \twoheadrightarrow^{\omega} t_\infty$, for finite terms (t_n). We must have, then: $\phi[t_\infty] = \text{lub } \phi[t_n] = \text{lub } (t_n{}^M)$. This shows that, if such a ϕ is suitable, it is unique. Let $\phi[f(\overline{t_\infty})] = \text{lub} f(\overline{t_n}) = f^M(\text{lub } \overline{t_n}) = f^M(\overline{\phi[t_\infty]})$. Thus, ϕ is a Σ-morphism. Lastly, ϕ is continuous, by construction. □

Similarly, it can be seen that $\hat{T}(X)$ is a model of Mod^{ω}. We may now define $\sigma(t)$, for any model M in Mod^{ω} and $t \in \hat{T}(X)$, as follows:

- if t is finite, $\sigma(t)$ is as above;
- if t is infinite, it is the limit of a chain (t_n), and we let $\sigma(t) = \text{lub } \sigma(t_n)$.

Thanks to condition (2), the lub does not depend on the choice of (t_n).

Definition 8. Given t, t' in $\hat{T}(X)$, and $M \in Mod^{\omega}$, we say that M *satisfies the inequality* $t \le t'$ if for every assignment $\sigma : X \to M$ we have $\sigma(t) \le \sigma(t')$. In that case, we write $M \models t \le t'$.

As usual, we say that Mod^{ω} satisfies $t \le t'$, and write $Mod^{\omega} \models t \le t'$, if $M \models t \le t'$, for every $M \in Mod^{\omega}$.

Theorem 6. *Let R be ω-terminating and $t,t' \in \hat{T}(X)$. Then*

$$Mod^{\omega} \models t \le t' \quad \textit{iff} \quad t \twoheadrightarrow^{\omega} t'.$$

This result may be seen as extending Birkhoff's theorem to the validity of *inequalities* in the algebra of continuous models.

Proof. Suppose that $Mod^{\omega} \models t \le t'$. In particular, $\hat{T}(X) \models t \le t'$, which means $t \twoheadrightarrow^{\omega} t'$. Conversely, suppose that $t \twoheadrightarrow^{\omega} t'$. Let $M \in Mod^{\omega}$ and let $\sigma : X \to M$. The term t is finite, since otherwise it would be a normal form. There exists a chain (t_n) of finite terms such that $t = t_0 \to \cdots \to t_n \to \cdots \to^{\omega} t'$. For any n, we have $\sigma(t) \le \sigma(t_n)$. Thus, $\sigma(t) \le \text{lub } (\sigma(t_n)) = \sigma(t')$, i.e. $M \models t \le t'$. □

Definition 9. The class Eq^{ω} is the subclass of Mod^{ω} for which $\sigma(c[l]) = \sigma(c[r])$, for any rule $l \to r$ in R, substitution σ: $X \to T_\Sigma$, and context c in $T(X)$.

Note that M is in Eq^{ω} iff $M \models l \le r$ and $M \models r \le l$.

Definition 10. Suppose that R is ω-canonical. NF^{ω} is the model whose universe $NF_R^{\omega}(T)$ consists of the ω-normal forms of the finite terms, ordered in a discrete fashion (i.e. $\le_{NF^{\omega}}$ is equality).

Theorem 7. *If R is ω-canonical, then NF^{ω} is initial in Eq^{ω}.*

Proof. Let $M \in Eq^{\omega}$. We wish to define a morphism ψ: $NF^{\omega} \to M$. Denote by ϕ_{NF} and ϕ_M the above morphisms $\hat{T} \to NF^{\omega}$ and $\hat{T} \to M$, respectively. We must have: $\phi_{NF} \circ \psi = \phi_M$. Thus, ϕ must be defined as follows: for any ω-normal form t_∞, for any sequence of finite terms $(t_n) \twoheadrightarrow^{\omega} t_\infty$, $\psi[t] =_{def}$ lub $(t_n{}^M)$. One can check that such a ψ is well-defined. It is clear that NF^{ω} satisfies the conditions on R-models. Moreover, as previously, ψ is a morphism and is continuous. □

Corollary. *Let R be ω-canonical. Let $t,t' \in T_\Sigma$.*

$$Eq^{\omega} \models t \le t' \wedge t' \le t \quad \textit{iff} \quad t \twoheadrightarrow^{\omega} \circ \twoheadleftarrow^{\omega} t'.$$

As above, the last result may be seen as extending Birkhoff's theorem to the validity of *equations* in the algebra of continuous models.

The relationship between the classes Mod^{ω}, Eq^{ω}, and *Alg* (the class of the finite, usual models, ordered by inclusion), is are lattices. Their relationship, for ω-canonical systems, is illustrated in Figure 1.

Example: Let the signature Σ consist of two constants, *a* and *b*, and a unary operator *s*. Let $R = \{a \to s(a)\ ,\ b \to s(b)\}$. The model $\hat{T}_R$ has as universe

$$\{a,\ s(a),\ \cdots,\ s^{(n)}(a),\ \cdots,\ b,\ s(b),\ \cdots,\ s^{(n)}(b),\ \ldots,\ s^{(\infty)}\}$$

ordered by:

$$a \le s(a) \le\ \cdots\ \le s^{(n)}(a) \le\ \cdots\ \le s^{(\infty)}$$
$$b \le s(b) \le\ \cdots\ \le s^{(n)}(b) \le\ \cdots\ \le s^{(\infty)}.$$

The universe NF^{ω} is reduced to $\{s^{(\infty)}\}$. Notice that Mod^{ω} does not satisfy $a \le b$ or $b \le a$, and that *Alg* does not satisfy $a=b$, whereas Eq^{ω} validates all these.

Definition 11. The class of the ω-*reachable models* is the subclass of the models *M* of Mod^{ω} such that the canonical morphism $\phi_M : \hat{T} \to M$ is surjective.

The ω-reachable models form a non-empty, complete sublattice of Mod^{ω} (containing at least $\hat{T}$).

Theorem 8. *For any ω-reachable model M, there exists a continuous congruence $\equiv_M$ on $\hat{T}$ such that M is isomorphic to $\hat{T} / \equiv_M$.*

By "continuous", we mean that for a rewrite chain $t_0 \to \cdots \to t_n \to \cdots$ such that $\lim_{n<\omega} t_n = t$ and $t_n \equiv t'$ for each $n \ge 0$, it is the case that $t \equiv t'$. The proof is as in the finite case, with $t \equiv_M t'$ iff $\phi_M(t) = \phi_M(t')$.

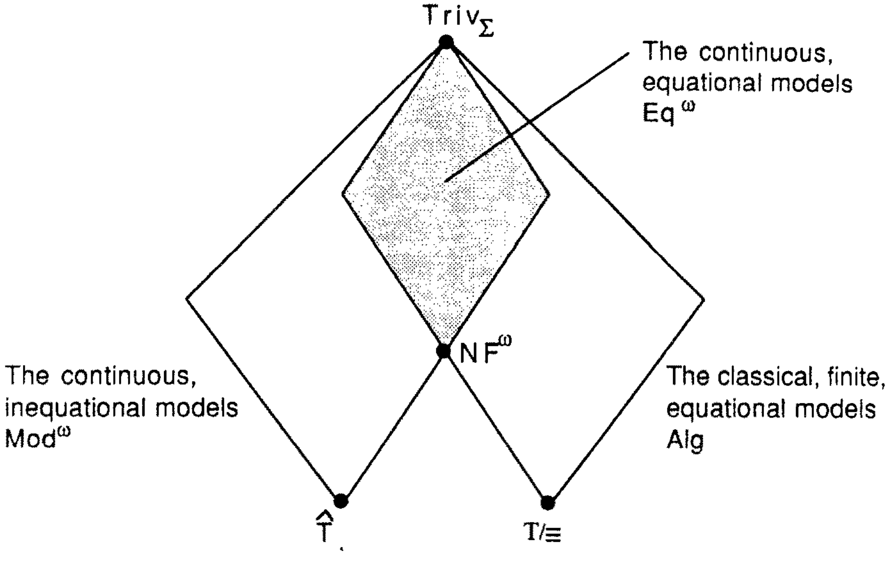

Figure 1. Classes of models.

3. Hierarchical Systems

In this section, we consider *typed* systems (cf. [ADJ-77], [Huet-Oppen-80]). A signature is now a pair (S,Σ) where S stands for a finite family of *sort* names and Σ is a finite family of operators on S. All the definitions given so far extend to the sorted case.

Definition 12. A *hierarchical specification* is a triple $(S_0 \cup S_1, \Sigma_0 \cup \Sigma_1, R_0 \cup R_1)$, where R_0 and $R_0 \cup R_1$ are ω-terminating rewrite systems on (S_0,Σ_0) and $(S_0 \cup S_1, \Sigma_0 \cup \Sigma_1)$, respectively. The class $HMod^{\omega}_{R_0 \cup R_1}$ of the *hierarchical models* is the class of models M of $Mod^{\omega}_{R_0 \cup R_1}$ such that the restriction $M_{|S_0,\Sigma_0}$ of M to the signature (S_0,Σ_0) is isomorphic to $\hat{T}_{S_0,\Sigma_0,R_0}$.

In the sequel, we suppose that the left-hand sides of the inequalities of R_1 always contain a symbol of Σ_1 (otherwise, the next condition of hierarchical consistency could not be satisfied — except in trivial cases).

Definition 13. A hierarchical specification is *sufficiently complete* iff for every $t \in T_{S_0,\Sigma_0 \cup \Sigma_1}$, there exists $t' \in \hat{T}_{S_0,\Sigma_0,R_0}$ such that $Mod^{\omega}_{R_0 \cup R_1} \models t \leq t'$.

Definition 14. A hierarchical specification is *hierarchically consistent* iff for any $t \in T_{S_0,\Sigma_0}$ and for any $t' \in \hat{T}_{S_0,\Sigma_0,R_0}$, $Mod^{\omega}_{R_0 \cup R_1} \models t \leq t'$ iff $Mod^{\omega}_{R_0} \models t \leq t'$.

Note: An infinite term $t' \in \hat{T}_{S_0,\Sigma_0,R_0}$ is by definition a normal form for R_0. It is also a normal form for $R_0 \cup R_1$, due to the above hypothesis about the left-hand sides of the inequalities of R_1. Thus, the above two definitions are equivalent to the existence of a t' such that $t \rightarrow^{\omega}_{R_0 \cup R_1} t'$, and to $t \rightarrow^{\omega}_{R_0 \cup R_1} t'$ iff $t \rightarrow^{\omega}_{R_0} t'$, respectively.

These definitions are consistent with the standard ones in the finitary case (cf. [Wirsing-89]), and with those of [Tarlecki-Wirsing-86] for their notion of continuous specifications. Sufficient completeness means that any finite term t of an old sort, built with old and (possibly) new operators, is smaller than a (possibly infinite) term t' built with old operators only. Hierarchical completeness means that for two terms t and t' built with old operators only, $t \leq t'$ holds in the new specification iff it holds in the old one. Note also that the above definitions extend, as is true for finitary specifications, to the case where no new sort is introduced (i.e. $S_1=\emptyset$); operators of Σ_0 are then called *constructors*, and operators of Σ_1 are called *derived* operators (or simply "non-constructors"). A *constructor term* is a term containing only constructors. A *non-constructor term* is a term containing at least one non-constructor. For instance, the specification:

constructors :	a: $\rightarrow$ *elem*, cons: *elem* $\times$ *elem* $\rightarrow$ *elem*
derived operator :	b: $\rightarrow$ *elem*
law :	b $\leq$ cons(a,b)

is (ω-) sufficiently complete. Note that in the classical, finitary framework, it would simply be rejected as not being (finitely) complete.

Now, the main result is that, as in the finitary case, a hierarchically consistent and sufficiently complete specification satisfies its hierarchical constraints — in the following sense:

Theorem 9. *If $(S_0 \cup S_1, \Sigma_0 \cup \Sigma_1, R_0 \cup R_1)$ is sufficiently complete and hierarchically consistent, then $HMod^{\omega}_{R_0 \cup R_1}$ is a non-empty, complete sub-lattice of $Mod^{\omega}_{R_0 \cup R_1}$. Its initial model is $\hat{T}_{R_0 \cup R_1}$.*

Proof. The proof is essentially as in the finite case. The main difference is in showing that $\hat{T}_{R_0 \cup R_1}$ is actually in $HMod^{\omega}_{R_0 \cup R_1}$, which we establish as follows:

The restricted model $(\hat{T}_{R_0 \cup R_1})_{|S_0,\Sigma_0}$ may be canonically embedded into $\hat{T}_{R_0}$. If $t \in (\hat{T}_{R_0 \cup R_1})_{|S_0,\Sigma_0}$ is finite, then t is in fact in T_{S_0,Σ_0} and therefore also in $\hat{T}_{R_0}$. If t is infinite, we may write $t_0 \rightarrow_{R_0 \cup R_1} t_1 \rightarrow_{R_0 \cup R_1} \cdots \rightarrow^{\omega}_{R_0 \cup R_1} t$, where the (t_n) are finite terms of $T_{S_0,\Sigma_0 \cup \Sigma_1}$. Using sufficient completeness, there exists $t' \in \hat{T}_{S_0,\Sigma_0,R_0}$ such that $t \rightarrow_{R_0 \cup R_1} t'$. Since t is a normal form for $R_0 \cup R_1$, this proves that $t=t'$, i.e. that t actually belongs to $\hat{T}_{S_0,\Sigma_0,R_0}$.

Thus, $(\hat{T}_{R_0 \cup R_1})_{|S_0,\Sigma_0}$ may be seen as a subset of $\hat{T}_{S_0,\Sigma_0,R_0}$. Now, hierarchical consistency shows that it is actually equal to the whole set, and that the orderings induced by R_0 and by $R_0 \cup R_1$ are identical. This finally establishes that $\hat{T}_{R_0 \cup R_1}$ is actually in $HMod^{\omega}_{R_0 \cup R_1}$. □

Define on $\hat{T}_{R_0 \cup R_1}$ a quasi-ordering $\leq^{obs}$ as follows:[2] its restriction to the sorts of S_0 is $\rightarrow^{\omega}_{R_0}$ (or equivalently $\rightarrow_{R_0 \cup R_1}$, because of the hierarchical consistence property), and for a sort s of S_1, for t,t' in $(\hat{T}_{R_0 \cup R_1})^s$, $t \leq^{obs} t'$ iff for any context c with result in a sort of S_0, then $c[t] \rightarrow^{\omega}_{R_0} c[t']$. As usual, we let $\equiv^{obs} = \leq^{obs} \cap \leq^{obs^{-1}}$; now, $\hat{T}_{R_0 \cup R_1}/\equiv^{obs}$ is *ordered* by $\leq^{obs}$.

Theorem 10. *$\hat{T}_{R_0 \cup R_1}/\equiv^{obs}$ is terminal among the ω-generated models of $HMOD^{\omega}_{R_0 \cup R_1}$.*

The proof is classical.

4. ω-Normal Forms

In this section, we aim for sufficient conditions for existence and for uniqueness of ω-normal forms. We concentrate on special cases that are of practical importance. Recall that a rewrite system is ω-terminating if every infinite (ω) rewrite chain has a limit.

Definition 15. A rewrite system R over T is *top-terminating* if there are no infinite rewrite sequences $t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \cdots$ of terms in T with infinitely many rewrites at the topmost position.

Note that a top terminating system need not be finitely terminating.

Theorem 11. *A rewrite system R is ω-terminating for finite terms in T if it is top terminating for T.*

The converse is wrong, as demonstrated by the system $\{x \rightarrow f(x)\}$.

Proof. If R is top terminating, then after a finite number of rewrites, no more rewrites are applied at the top. The same argument can then be applied to the subterms to show that the rewrites must occur deeper and deeper, and so R is ω-terminating. □

We define ω-termination orderings for proofs of ω-termination that are analogous to the well-founded quasi-orderings used to show finite termination [Dershowitz-87]. We will say that a quasi-ordering $\gtrsim$ over a set S is *well-founded* if it admits no infinite strictly descending sequences $s_1 > s_2 > \cdots$ of elements s_i in S. As usual, $s > t$ means $s \gtrsim t$ and $t \not\gtrsim s$, $s \sim t$ means $s \gtrsim t$ and $t \gtrsim s$.

Definition 16. A well-founded quasi-ordering $\gtrsim$ is an ω-termination ordering if $s \gtrsim t$ implies $f(\cdots s \cdots) \gtrsim f(\cdots t \cdots)$ for all $f \in \Sigma$ and $s,t \in T(X)$.

[2] A *quasi-ordering* is a reflexive and transitive binary relation.

Theorem 12. *A rewrite system R is top-terminating if there exists an ω-termination ordering $\geq$ such that $l\sigma > r\sigma$ for all rules $l \to r$ in R and substitutions σ of finite terms for variables.*

Proof. Suppose $l\sigma > r\sigma$ for an ω-termination ordering $\geq$. For an infinite rewrite sequence $t_0 \to t_1 \to t_2 \to \cdots$, if $t_i \to t_{i+1}$ at the topmost position, then $t_i > t_{i+1}$, and if $t_i \to t_{i+1}$ at an inner position, then $t_i \geq t_{i+1}$. An infinite number of *top* rewrites $t_i \to t_{i+1}$ would contradict the well-foundedness of $>$. □

ω-termination orderings are not that hard to devise. What is significant is what happens near the top of the term. For example, one can define an ω-termination ordering $\geq^*$ on terms that is induced by a given quasi-ordering $\geq$ on operators in Σ: if $f > g$ in the operator ordering, then $f(s_1, \ldots, s_n) >^* g(t_1, \ldots, t_n)$; while if $f = g$, the two are equivalent.

ω-termination orderings can be applied to hierarchical systems to give methods of establishing *ground* ω-confluence, that is, ω-confluence on ground terms, and also sufficient completeness, that is, any ground term has an ω-normal form that is a constructor term. In the remainder of this section, we deal only with the special case in which the following "constructor condition" is satisfied: no two terms of the form $u[f(s_1 \cdots s_n)]$ and $u[g(t_1 \cdots t_n)]$ are provably equal (by replacement of equals) from R, for distinct constructors $f,g \in \Sigma_0$ and for a context u in $T_{\Sigma_0}(X)$.

Theorem 13. *Let R be a left-linear rewrite system. Suppose there is a well-founded quasi-ordering $\geq$ with the following properties:*
(a) If $s \to t$ by a rewrite that is not inside a constructor of s and t does not have a constructor at the top level, then $s > t$ in the quasi-ordering; if t does have a constructor at the top, then $s \geq t$ in the quasi-ordering.
(b) If f is a constructor, then $f(\cdots s \cdots) \geq s$ in the quasi-ordering; if f is not, then $f(\cdots s \cdots) > s$.
(c) All finite ground non-constructor terms are reducible.
Then R is ground ω-canonical. Moreover, all ω-normal forms of ground terms are constructor terms.

Note that we are not requiring regularity; R may have overlapping left-hand sides.

Proof. We show that, for any ground term s and for any integer k, there is a term t such that $s \xrightarrow{*} t$ and t has no non-constructors at depths less than or equal to k. This is by induction on s, using the quasi-ordering, and for equivalent terms, by induction on k.

Suppose $s = f(s_1, \ldots, s_n)$ and f is a constructor. Then $s \geq s_i$ and the argument below shows that (for each i) there is a t_i such that $s_i \xrightarrow{*} t_i$ and t_i has no non-constructors at the top $k-1$ levels.

Suppose, without loss of generality, that f is not a constructor. Then the s_i are less than s in the quasi-ordering, so we can apply the induction hypothesis to them, and get arbitrarily many constructors at the top levels of those subterms. Thus, $s_i \xrightarrow{*} t_i$, where the t_i have no constructors at the top m levels, for m larger than the maximum depth of a left-hand side in R. Thus, $s \xrightarrow{*} f(t_1, \ldots, t_n)$. By assumption (c), there must be a top-rewrite $f(t_1, \ldots, t_n) \to t$, since all rules in R have depth smaller than m and are left-linear, so cannot depend on anything lower down. If t has a constructor at the top level, $f(t_1, \ldots, t_n) \geq t$ in the quasi-ordering. The subterms of t are smaller than t by properties of the ordering, so we can use the induction hypothesis on them as before. If t does not have a constructor at the top, then $s > t$ in the quasi-ordering, and we can apply the induction hypothesis to t.

This implies that the limit of the rewrite sequence is a constructor term. Confluence follows because no two distinct constructor terms can be equal, by the constructor property. □

We next give a non-obvious application of the preceding results. The *nesting level* $N(t)$ of non-constructors in a term t is defined as follows:

i) If t is a constructor constant or a variable, then $N(t) = 0$.

ii) If t is a non-constructor constant, then $N(t) = 1$.

iii) If t is $f(t_1, \ldots, t_n)$ and f is a constructor, then $N(t) = \max_{1 \le i \le n} N(t_i)$.

iv) If t is $f(t_1, \ldots, t_n)$ and f is not a constructor, then $N(t) = 1 + \max_{1 \le i \le n} N(t_i)$.

We say that a system R *does not increase* the nesting level of non-constructors if, for all rules $l \rightarrow r$ in R and substitutions σ, $N(l\sigma) \ge N(r\sigma)$. This condition can be checked syntactically by noting the nesting of function symbols above each variable.

Theorem 14. *Suppose that R is a left-linear top-terminating rewrite system that does not increase the nesting level of non-constructors. Suppose further that all finite ground non-constructor terms are reducible. Then R is ground ω-canonical and all ω-normal forms of ground terms are constructor terms.*

Proof. Let $s_1 \rightarrow s_2 \rightarrow s_3 \ldots$ be an infinite rewrite sequence. Since R does not increase the depth of nesting of non-constructors, $N(s_1) \ge N(s_2) \ge N(s_3) \cdots$. It can be shown that the non-constructors must get farther and farther apart in the terms $s_1, s_2, s_3, \cdots$, and so the limit exists and is a constructor term. □

For instance, let a be a constructor constant, c be a unary constructor, and f, a unary non-constructor. The system:

$$\begin{array}{rcl} f(a) & \rightarrow & a \\ f(c(x)) & \rightarrow & c(f(c(x))) \end{array}$$

is ω-convergent by the above theorem. On the other hand, if, instead of the second rule, we have the "increasing" rule

$$f(c(x)) \rightarrow f(f(c(x))),$$

then there are non-constructor ω-normal forms, e.g., $f^{(\infty)}$.

Theorem 15. *If R is top-terminating and left-linear, no rewrite sequence has unbounded nesting of non-constructors, and all finite ground non-constructor terms are reducible, then R is ground ω-canonical.*

In particular, one can prevent unbounded increase in non-constructor nesting depth in the following way: We say that a term is *constructor-based* if it is of the form $f(t_1, \ldots, t_n)$, where f is a non-constructor and the t_i are all constructor terms and that a system is *defined on constructors* if all its left-hand sides are constructor-based. Any system defined on constructors, for which there is a constant N such that t has a constructor at the top whenever $s \xrightarrow{*} t$ (for some finite term s) by a sequence of N or more rewrites, has constructor normal forms.

To conclude this section, we present some definitions and results leading up to a corollary to the above theorem.

Definition 17. A term t is a *k-constructor term* if all function symbols at depths less than or equal to k, are constructors. A rewrite is a *k-constructor rewrite* if the redex is of the form $f(s_1 \cdots s_n)$ where f is a non-constructor and the s_i are k-constructor terms.

Definition 18. *Parallel outermost k-constructor rewriting* is the rewriting strategy in which at each step all outermost k-constructor rewrites are done.

Lemma 1. *Suppose R is left-linear, defined on constructors, and ω-terminating. Suppose further that no R rewrite sequence has unbounded nesting of non-constructors, and all finite ground non-constructor terms are reducible. Then, the set of ω-normal forms of a term is the same as the set of normal forms obtained by parallel outermost k-constructor rewriting, for k greater than or equal to the maximum depth of a left-hand side in R.*

Proof. Any infinite fair R rewrite sequence can be rearranged to have a prefix that is a parallel outermost k-constructor rewrite sequence. □

Lemma 2. *If R is defined on constructors and left-linear, no R rewrite sequence has unbounded nesting of non-constructors, and all finite ground non-constructor terms are reducible, then R is ω-terminating iff all parallel outermost k-constructor rewrite steps produce a term with a constructor at the top level, with k as in the previous lemma.*

Definition 19. A rewrite system R has *bounded parallel outermost k-constructor rewriting* if there is an integer N such that for all constructor-based ground terms s, if $s \xrightarrow{*} t$ by a parallel outermost k-constructor rewrite sequence of length N or more, then t has a constructor at the top.

The intuition is that an "output" must be produced after a bounded amount of "computation time". A constructor at the top level is like an output of a computation.

If R is defined on constructors and left-linear, no R rewrite sequence has unbounded nesting of non-constructors, and all finite ground non-constructor terms are reducible, then it is semi-decidable whether R has bounded parallel outermost k-constructor rewrites. This is because such rewriting can only depend on the structure of s near the top, where there are only finitely many possibilities.

We are ready now for the promised corollary:

Corollary. *If R is left-linear, defined on constructors, has bounded parallel outermost k-constructor rewriting (for k no less than the maximum depth of a left-hand side in R), no rewrite sequence has unbounded nesting of non-constructors, and all finite ground non-constructor terms are reducible, then R is ground ω-canonical.*

Note that R may be top-terminating even if the set of rules $l \rightarrow r$ with r having a non-constructor at the top is non-terminating. For example, take $a \rightarrow f(a)$, with f a non-constructor.

A constructor-based programming language with infinite normal forms is described in [Narain-88].

5. Conditional ω-rewriting

In this section, we consider extensions of some of the results given so far to *conditional rewriting*. We presume the reader knows the basics about ordinary, finitary conditional rewriting (cf., e.g., [Kaplan-Remy-89]). A *conditional* rewrite system is a finite family of conditional rules of the form $\bigwedge_{i=1}^{m} u_i = v_i \Rightarrow l \rightarrow r$, where all variables of a rule also occur in the left-hand side. Operational semantics are defined as follows:

Theorem 16. *For any conditional rewrite system R, there exists a smallest binary relation* $\to$ *on* $T^\infty(X)$ *such that, for each rule* $\bigwedge_{i=1}^{m} u_i{=}v_i \Rightarrow l{\to}r$ *in R, context c, and substitution* σ*,*

$$c[l\sigma] \to c[r\sigma] \quad \text{if} \quad u_i\sigma \to^{\omega}\circ\leftarrow^{\omega} v_i\sigma \text{ for all } i = 1, \ldots, m. \qquad (*)$$

This stems from the fact if a family of predicates satisfies the condition (*), then their intersection also satisfies (*) and that the "always true" predicate satisfies (*). Now, the essential question is whether $\to^{\omega}$ may be finitely approximated, as is the case for unconditional rules (see Section 1). Somewhat surprisingly, a positive answer may be provided to this question, by means of what we will call "clausal rewriting".

We first note that the usual, finitary approach to conditional rewriting (cf. [Kaplan-84]) is unsatisfactory here. Indeed, the "limit"—in any reasonable sense—of the sequence of relations, $\to_0 = \varnothing$ and $\to_{n+1}$ such that $c[l\sigma] \to_{n+1} c[r\sigma]$ *iff* for all $i = 1, \ldots, n$, $u_i\sigma \to_n^* \circ_n^* \leftarrow v_i\sigma$, does not yield $\to$. For example, consider the system $\{a \to s(a),\ b \to s(b),\ a = b \Rightarrow c \to d\}$. The sequence of relations $(\to_n)$ is stationary, and thus certainly cannot approximate $\to$ (which satisfies $c \to d$).

Let $C^\infty(X)$ stand for the set of finite or infinite conjunctions of equalities between terms of $T^\infty(X)$ modulo finitary and infinitary applications of the following identities:

$$\begin{aligned}
(P \wedge Q) \wedge R &\equiv P \wedge (Q \wedge R)\\
P \wedge Q &\equiv Q \wedge P\\
P \wedge P &\equiv P\\
P \wedge u{=}v &\equiv P \wedge v{=}u\\
P \wedge u{=}u &\equiv P\\
P \wedge u{=}v \wedge c[u]{=}c[v] &\equiv P \wedge u{=}v
\end{aligned}$$

Let $CT^\infty(X)$ stand for the Cartesian product $C^\infty(X) \times T^\infty(X)$. We call elements of $CT^\infty(X)$ *conditional terms*, and write them in the form $p{:}t$. The *distance* for $C^\infty(X)$, is just the distance between the conjunctions considered as trees, with an associative-commutative operator '$\wedge$' and a commutative operator '='. A distance is then finally defined on the conditional terms of $CT^\infty(X)$ as the *minimum* of the distances in $C^\infty(X)$ and $T^\infty(X)$.

The main idea of this section is to rewrite *conditional terms* so that (for instance):

$$true{:}c \twoheadrightarrow a{=}b{:}d \twoheadrightarrow sa{=}b{:}d \twoheadrightarrow sa{=}sb{:}d \twoheadrightarrow^{\omega} s^{(\infty)}{=}s^{(\infty)}{:}d \equiv true{:}d,$$

validating the (infinitary) rewrite $c{\to}d$. We call this "clausal rewriting".

Definition 20. The relation $\twoheadrightarrow$ of *clausal rewriting* is defined as the smallest relation on $CT^\infty(X)$ such that, for any rule $q \Rightarrow l{\to}r$, any context c and any substitution σ:

$$\begin{aligned}
p : c[l\sigma] &\twoheadrightarrow p \wedge q\sigma : c[r\sigma] &\quad (a)\\
p \wedge (c[l\sigma]{=}v) : t &\twoheadrightarrow p \wedge q\sigma \wedge (c[r\sigma]{=}v) : t &\quad (b)
\end{aligned}$$

Step (a) corresponds to conditionally rewriting the conclusion, and step (b) to rewriting inside the premise.

Definition 21. A conditional TRS is *multi-linear* if

- the left-hand side of each rule is linear,
- the terms occurring in the premises are linear,
- there is no equation between variables in the premises.

Theorem 17. *If R is multi-linear, then for any ordinal* $\alpha \geq \omega$, $\twoheadrightarrow^{\alpha} = \twoheadrightarrow^{\omega}$.

The proof is by transfinite induction. It is actually enough to show that $\twoheadrightarrow^{\omega+1} = \twoheadrightarrow^{\omega}$, which is easy by case analysis on the $(\omega+1)^{st}$ step — of type (a) or (b).

Definition 22. A conditional rewrite system R is ω-*terminating* if the relation $\twoheadrightarrow$ is, i.e. if for any chain $t_1 \twoheadrightarrow \cdots \twoheadrightarrow t_n \twoheadrightarrow \cdots$, there exists $\lim_{n<\omega} t_n$.

This extends the notions of Section 1. We then have the following fundamental theorem, linking the relations $\rightarrow$ and $\twoheadrightarrow$:

Theorem 18. *If R is multi-linear and* ω*-terminating, then*

$$t \rightarrow^{\omega} t' \text{ iff } true{:}t \twoheadrightarrow^{\omega} true{:}t'$$

Note: This means that the relation of interest $\rightarrow^{\omega}$ is approximated by the *finitary* iterates $\twoheadrightarrow^{n}$.

Proof. Define the relation $t \longrightarrow t'$ iff $true{:}t \twoheadrightarrow^{\omega} true{:}t'$. We first prove that $\longrightarrow$ satisfies Condition (*) (of Theorem 16). Suppose that for substitution σ and rule $\bigwedge_{i=1}^{m} u_i{=}v_i \Rightarrow l{\rightarrow}r$ in R, we have $u_i\sigma \longrightarrow^{\omega} \circ\, {}^{\omega}\!\longleftarrow v_i\sigma$ for each antecedent condition. We need to show that, for any context c, $c[l\sigma] \longrightarrow c[r\sigma]$. Notice that:

$$true{:}\ c[l\sigma] \twoheadrightarrow \bigwedge_{i=1}^{m} u_i\sigma{=}v_i\sigma : c[r\sigma]. \tag{**}$$

Now, for multi-linear systems (by the previous theorem), $true{:}u_i\sigma \twoheadrightarrow^{\omega} true{:}w_i \twoheadleftarrow^{\omega} true{:}v_i\sigma$, for some infinite term w_i, for each i $(1{\leq}i{\leq}n)$. Consider the first part: $true{:}u_i\sigma \twoheadrightarrow^{\omega} true{:}w_i$; it states that there exist terms $u_{i,n}$ and conjunctions $p_{i,n}$ such that:

$$true{:}u_i\sigma \twoheadrightarrow p_{i,1} : u_{i,1} \twoheadrightarrow \cdots \twoheadrightarrow^{\omega} true{:}\ u_{i,\omega}.$$

Now, the fact that $\lim_{n<\omega} p_{i,n}$ is equivalent to "true", implies that, actually, for n greater than some N_i, we have $p_{i,n}$ true. Similarly, for the $v_i\sigma$'s. Let N be the *maximum* of the N_i's. The above relation (**) now yields (for all $n \geq N$):

$$true{:}c[l\sigma] \twoheadrightarrow^{*} \bigwedge_{i=1}^{m} u_{i,n}{=}v_{i,n} : c[r\sigma] \twoheadrightarrow^{\omega} \bigwedge_{i=1}^{m} w_i{=}w_i\ true{:}\ c[r\sigma] \equiv true{:}\ c[r\sigma],$$

i.e. $c[l\sigma] \longrightarrow c[r\sigma]$. By minimality of $\rightarrow$, this actually proves that $\rightarrow \subseteq \longrightarrow$. Thus, $\rightarrow^{\omega} \subseteq \longrightarrow$, since $\longrightarrow$ is ω-closed. Since the reverse inclusion is clear, this concludes the proof. □

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