Abstract Canonical Inference

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Abstract. We apply an abstract framework of canonical inference to explore how different proof orderings induce different variations of saturation and completeness. We relate completion, paramodulation, saturation, redundancy elimination, and rewrite system reduction to proof orderings.

Keywords: Inference, canonicity, saturation, completion, redundancy

They are not capable to ground a canonicity of universal consistency.

—Alexandra Deligiorgi (ΠΑΙΔΕΙΑ, 1998)

1. Introduction

Well-founded orderings of proofs, as developed in (Bachmair and Dershowitz, 1994), are used to distinguish between cheap "direct" proofs, those that are of a computational flavor (e.g. rewrite proofs), and expensive "indirect" proofs, those that require search to find (e.g. equational proofs). Accordingly, we suggest that proof orderings, rather than formula orderings, take center stage in theorem proving with contraction (simplification and deletion of formulæ). Then completeness of a set of formulæ, what we will call a *presentation*, means that all theorems enjoy a minimal proof, while completeness of an inference system means it has the ability to generate all formulæ needed as premises in such ideal proofs. This formalism is very flexible, since it allows small proofs to use large premises. Given a formula ordering, one can, of course, choose to compare proofs by simply comparing the multiset of their premises.

This proof-ordering based approach to deduction suggests generalizations of the concepts of "redundancy" and "saturation." Saturated, for us, will mean that *all* cheap proofs are supported, which demands more than completeness. By considering different orderings on proofs, one gets different kinds of saturated sets. The notion of saturation in

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theorem proving, in which superfluous deductions are not necessary for completeness, was suggested by (Rusinowitch, 1989, 1991) in the context of a Horn-clause resolution calculus. In our terminology: A presentation was said to be saturated when all inferrible formulæ are syntactically subsumed by formulæ in the presentation.

We also define redundancy in terms of the proof ordering, as propounded by Bonacina and Hsiang (1991, 1995): A sentence is redundant if adding it to the presentation does not decrease any minimal proof. (See Bonacina, 1992, Chap. 2.) The definition of redundancy in (Bachmair and Ganzinger, 1991, 1994)—an inference is redundant if its conclusion can be inferred from smaller formulæ—coincides with ours when proofs are measured first by their maximal premises. In (Bachmair and Ganzinger, 1991, 2001; Nieuwenhuis and Rubio, 2001), saturated means every possible inference is redundant.

The present work continues the development of an abstract theory of "canonical inference," sketched in (Dershowitz and Kirchner, 2003b; Dershowitz, 2003), and expanded in (Dershowitz and Kirchner, 2003a), which, in turn, grew out of the theory of rewriting (see, for example, Dershowitz and Plaisted, 2001; Terese, 2003) and deduction (see, for example, Bonacina, 1999; Bachmair and Ganzinger, 2001; Nieuwenhuis and Rubio, 2001). Although we will use ground equations as an illustrative example, the framework applies equally well in the first-order setting, whether equational or clausal. Though our motivation is primarily æsthetic; our expectation is that practical applications will follow.

The next section sets the stage with basic notions and notations, and introduces a running example. To make this paper self-contained, Section 3 recapitulates relevant definitions and results from (Dershowitz and Kirchner, 2003a, 2003b).[1] Specifically, the *canonical* basis of an abstract deductive system is defined in three equivalent ways: (1) formulæ appearing in minimal proofs; (2) minimal trivial theorems; (3) non-redundant lemmata. Section 4 exemplifies the abstract framework. Sections 5, 6 and 7 carry out the study of derivation and completion processes. Finally, Section 8 closes with some discussion.

[1] The study in (Dershowitz and Kirchner, 2003a) is concerned with defining abstract properties of sets of formulæ; this paper is about deducing sets enjoying those properties. We apply the abstract theory developed in the former paper, and extend it with notions, such as *fairness*, that describe properties of *derivations*. Whereas that paper is about properties of *objects* (presentations), here we study properties required of *processes* (derivations) so as to generate the desired presentations.

2. Proof systems

Let $\mathbb{A}$ be the set of all formulæ (ground equations and disequations, in our examples) over some fixed vocabulary. Let $\mathbb{P}$ be the set of all (ground equational) proofs. These sets of abstract objects are linked by two functions: $\Gamma : \mathbb{P} \to 2^{\mathbb{A}}$ gives the premises in a proof, and $\Delta : \mathbb{P} \to \mathbb{A}$ gives its conclusion. Both are extended to sets of proofs in the usual fashion.

The framework proposed here is predicated on two *well-founded* partial orderings over $\mathbb{P}$: a *proof ordering* $\geq$ and a *subproof relation* $\trianglerighteq$. They are related by a monotonicity requirement given below (6). We assume for convenience that the proof ordering only compares proofs with the same conclusion ($p \geq q \Rightarrow \Delta\, p = \Delta\, q$), rather than mention this condition each time we have cause to compare proofs.

We will use the term *presentation* to mean a set of formulæ, and *justification* to mean a set of proofs. We reserve the term *theory* for deductively-closed presentations. Let ΘA denote the *theory* of presentation A, that is, the set of conclusions of all proofs with premises A:[2]

$$\Theta A \quad \overset{!}{=} \quad \Delta\, \Gamma^{-1} A$$

The *pre-image* Γ^{-1} of A are those proofs with exactly A as premises: $\Gamma^{-1} A = \{p \colon p \in \mathbb{P},\ \Gamma\, p = A\}$. We assume the following three standard properties of Tarskian consequence relations:

$$\Theta A \subseteq \Theta(A \cup B) \tag{1}$$
$$A \subseteq \Theta A \tag{2}$$
$$\Theta\Theta A = \Theta A \tag{3}$$

Thus, Θ is a closure operation. It follows from (1) that

$$\Theta A = \{\Delta\, p \colon p \in \mathbb{P},\ \Gamma\, p \subseteq A\}$$

We say that presentation A is a *basis* for theory C if $\Theta A = C$. Presentations A and B are *equivalent* ($A \equiv B$) if their theories are identical: $\Theta A = \Theta B$.

As a very simple running example, let the vocabulary consist of the constant 0 and unary symbol s. Abbreviate tally terms $s^i 0$ as numeral i. The set $\mathbb{A}$ consists of all *unordered* equations $i = j$; so symmetry is built into the structure of proofs. (We postpone dealing with disequations for the time being.) An equational inference system (with this vocabulary)

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[2] We use $\overset{!}{=}$ for definitions.

might consist of the following five inference rules, where boxes surround assumptions:

$$\frac{\square}{0=0}\,\mathbf{Z} \qquad \frac{\boxed{i=j}}{i=j}\,\mathbf{I}_{i=j}$$

$$\frac{i=j}{si=sj}\,\mathbf{S} \qquad \frac{a \quad c}{c}\,\mathbf{P} \qquad \frac{i=j \quad j=k}{i=k}\,\mathbf{T}$$

where **Z** is an axiom, **I** introduces assumptions, and **S** infers $i+1=j+1$ from a proof of $i=j$. Proof tree branches of the transitivity rule **T** are *unordered*. Projection **P** allows irrelevant assumptions to be ignored and is needed to accommodate monotonicity (1).

For example, if $A=\{4=2, 4=0\}$, then

$$\Theta A = \{i=j : i \equiv j \pmod 2\}$$

Consider the proof schemata:

$$\cfrac{\cfrac{\cfrac{\cfrac{\square}{0=0}}{1=1}}{\vdots}}{i=i} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\boxed{4=2}}{4=2}}{5=3}}{\vdots}}{i+4=i+2} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\boxed{4=0}}{4=0} \quad \cfrac{\boxed{4=2}}{4=2}}{2=0} \quad \cfrac{\cfrac{\cfrac{\vdots}{i-j-2=0}}{i-j-1=1}}{i-j=2}}{i-j=0}}{\cfrac{\vdots}{i=j}}$$

Let's use proof terms for proofs, denoting the above three trees by $p = S^{i}Z$, $q = S^{i}I(4,2)$, and $r = S^{j}T(T(I(4,0), I(4,2)), SS(\cdots))$, respectively. Thus, $\Gamma p = \emptyset$, $\Gamma q = \{4=2\}$, and Δr is the formula $i=j$.

With a recursive path ordering (Dershowitz, 1982) to order proofs, a precedence $Z < S < T < I < P < 0 < 1 < 2 < \cdots$, on proof combinators and signature symbols, and multiset "status" for I, the minimal proof of a theorem in ΘA takes one of the forms

$$S^{j}\,(\nabla_{4k=0}) \qquad S^{j}\,(\nabla_{4k=2})$$

The subproofs $\nabla_{4k=0}$ and $\nabla_{4k=2}$ are defined recursively:

$$\begin{array}{lcl@{\qquad}lcl}
\nabla_{0=0} & = & Z & \nabla_{0=2} & = & T(\nabla_{4=0}, \nabla_{4=2}) \\
\nabla_{4=0} & = & I(4,0) & \nabla_{4(k+1)=0} & = & T(S^{4k}\nabla_{4=0}, \nabla_{4k=0}) \\
\nabla_{4=2} & = & I(4,2) & \nabla_{4(k+1)=2} & = & S^{2}T(\nabla_{0=2}, S^{2}\nabla_{4k=0})
\end{array}$$

We call a proof *trivial* when it proves only itself and has no subproofs other than itself, that is, if $\Gamma p = \{\Delta p\}$ and $p \trianglerighteq q \Rightarrow p = q$. We denote by $\widehat{a}$ such a trivial proof of $a \in \mathbb{A}$ and by $\widehat{A}$ the set of trivial proofs of each $a \in A$. For example, $\widehat{4{=}0} = I(4,0)$.

We assume that proofs use their premises (4), that subproofs do not use non-existent assumptions (5), and that proof orderings are monotonic with respect to (replacement of) subproofs (6). Specifically, for all proofs p, q, r and formulæ a:

$$a \in \Gamma\, p \;\Rightarrow\; p \trianglerighteq \widehat{a} \tag{4}$$

$$p \trianglerighteq q \;\Rightarrow\; \Gamma\, p \supseteq \Gamma\, q \tag{5}$$

$$p \triangleright q > r \;\Rightarrow\; \exists v \in \Pi\,(\,\Gamma\, p \cup \Gamma\, r).\; p > v \triangleright r \tag{6}$$

We make no other assumptions regarding proofs or their structure.

Obviously, the *Replacement Postulate* (6) implies:

$$p \trianglerighteq q > r \;\Rightarrow\; \exists v \in \Pi\,(\,\Gamma\, p \cup \Gamma\, r).\; p > v \trianglerighteq r \tag{7}$$

It states that $>$ (which we have restricted to proofs with the same conclusion) and $\triangleright$ commute. In other words, "replacing" a subproof q of a proof p with a smaller proof r "results" in a proof v that is smaller than the original p, and which does not involve extraneous premises. All proof orderings in the literature obey this monotonicity requirement.

Every formula a admits a trivial proof $\widehat{a}$ by (2,4). Let $\Sigma p = \{q : p \trianglerighteq q\}$ denote the subproofs of p, and likewise $\Sigma P = \cup_{p \in P} \Sigma p$. This way, (4) can be abbreviated $\widehat{\Gamma p} \subseteq \Sigma p$. On account of (4,6), proofs are also monotonic with respect to any inessential premises they refer to, should the latter admit smaller than trivial proofs.

It may be convenient to think of a proof-tree "leaf" as a subproof with only itself as a subproof; other subproofs are the "internal nodes." There are two kinds of leaves: trivial proofs $\widehat{a}$ (such as inferences **I**), and vacuous proofs $\bar{a}$ with $\Gamma\, \bar{a} = \emptyset$ and $\Delta\, \bar{a} = a$ (such as **Z**). By well-foundedness of $\trianglerighteq$, there are no infinite "paths" in proof trees. It follows from Replacement that the transitive closure of $> \cup \triangleright$ is also well-founded.

3. Canonical presentations

The results in this section are extracted from (Dershowitz and Kirchner, 2003a, 2003b), which should be consulted for proofs.

Denote the set of all proofs using premises of A by:

$$\Pi\, A \quad \stackrel{!}{=} \quad \{p \in \mathbb{P} \colon\, \Gamma\, p \subseteq A\}$$

and define the *minimal* proofs in a set of proofs as:

$$\mu P \quad \stackrel{!}{=} \quad \{p \in P \colon \neg \exists q \in P.\; q < p\}$$

On account of well-foundedness, minimal proofs always exist.

Note that Γ, Δ, Θ, and Π are all monotonic with respect to set inclusion, but $\mu\Pi$ is not.

We say that presentation A is *reduced* when $A = \Gamma\,\mu\Pi\,A$, that is, A contains precisely the premises of minimal proofs from A. By a "normal-form proof," we mean a proof in $\mu\Pi\,\Theta A$, the minimal proofs allowing any theorem as a lemma (that is, as a premise). This leads to our main definition:

DEFINITION 1 (Canonical Presentation). *The* canonical presentation *contains those formulæ that appear as premises of normal-form proofs:*

$$A^\sharp \quad \stackrel{!}{=} \quad \Gamma\,\mu\Pi\,\Theta A$$

So, we will say that A is canonical *if $A = A^\sharp$.*

The following proposition gives a second characterization of canonical presentation: as minimal trivial theorems.

PROPOSITION 1.

$$\begin{aligned} A^\sharp &= \Delta\,(\mu\Pi\,\Theta A \cap \widehat{\Theta A}) \\ \widehat{A^\sharp} &= \mu\Pi\,\Theta A \cap \widehat{\Theta A} \end{aligned}$$

THEOREM 1. *The function $\_^\sharp$ is "canonical" with respect to the equivalence of presentations. That is:*

$$\begin{array}{rcll} A^\sharp & \equiv & A & \textit{(Consistency)} \\ A \equiv B & \Leftrightarrow & A^\sharp = B^\sharp & \textit{(Monotonicity)} \\ A^{\sharp\sharp} & = & A^\sharp & \textit{(Idempotence)} \end{array}$$

By lifting proof orderings to justifications and presentations, the canonical presentation can be characterized in terms of the ordering directly. First, proof orderings are lifted to sets of proofs, as follows:

DEFINITION 2.

- *Justification Q is* better *than justification P if:*

$$P \sqsupseteq Q \quad \stackrel{!}{\equiv} \quad \forall p \in P.\, \exists q \in Q.\; p \geq q$$

- *It is* much better *if:*

$$P \sqsupset Q \quad \stackrel{!}{\equiv} \quad \forall p \in P.\, \exists q \in Q.\; p > q$$

– *Two justifications are* similar *if:*

$$P \simeq Q \quad \stackrel{!}{\equiv} \quad P \sqsupseteq Q \sqsupseteq P$$

Recall that only proofs with the same conclusion are comparable by the proof ordering $\geq$.

Transitivity of these three relations follows from the definitions. They are compatible: $(\sqsupseteq \circ \sqsupset) \subseteq \sqsupset$, $(\sqsupseteq \circ \simeq) \subseteq \sqsupseteq$, etc. Since it is also reflexive, $\sqsupseteq$ is a quasi-ordering.

The next proposition says that subproofs of minimal proofs are minimal, bigger presentations may offer better proofs, and minimal proofs are the best proofs. It follows from (6) and Definition 2 that:

PROPOSITION 2.

– *For all presentations A and B:*

$$\begin{aligned} \Sigma\mu\Pi\, A &= \mu\Pi\, A \\ \Pi\, A &\sqsupseteq \Pi\,(A \cup B) \end{aligned}$$

– *For all justifications P:*

$$P \sqsupseteq \mu P$$

This "better than" quasi-ordering on proofs is lifted to a "simpler than" quasi-ordering on (equivalent) sets of formulæ, as follows:

DEFINITION 3.

– *Presentation B is* simpler *than an equivalent presentation A when B provides better proofs than does A:*

$$A \gtrsim B \quad \stackrel{!}{\equiv} \quad \Theta A = \Theta B \wedge \Pi\, A \sqsupseteq \Pi\, B$$

– *Presentations are* similar *if their proofs are:*

$$A \approx B \quad \stackrel{!}{\equiv} \quad \Pi\, A \simeq \Pi\, B$$

Similarity $\approx$ is the equivalence relation associated with $\gtrsim$.

These relations are also compatible.

Canonicity may be characterized in terms of this quasi-ordering:

THEOREM 2. *The canonical presentation is the simplest:*

$$A \gtrsim A^\sharp$$

Recalling that all subproofs of normal-form proofs are also in normal form (Prop. 2), we propose the following definitions:

DEFINITION 4 (Saturation and Completeness).

- *A presentation A is* saturated *if it supports all possible normal form proofs:*

$$\mu\Pi\, A \;=\; \mu\Pi\,\Theta A$$

- *A presentation A is* complete *if every theorem has a normal form proof:*

$$\Theta A \;=\; \Delta\,(\Pi\, A \cap \mu\Pi\,\Theta A)$$

It can be shown that:

LEMMA 1. *A presentation A is saturated if and only if*

$$\Pi\, A \;\supseteq\; \mu\Pi\,\Theta A$$

A presentation is complete if it is saturated, but for the converse, we need a further hypothesis: *minimal proofs are unique* if, for all theorems $c \in \Delta\,\Pi\, A$, there is exactly one proof in $\mu\Pi\,\Theta A$ with conclusion c. In particular, this holds for proof orderings that are total (on proofs of the same theorem).

PROPOSITION 3.

1. *A presentation is complete if it is saturated.*
2. *If minimal proofs are unique, then a presentation is saturated if and only if it is complete.*

For example, suppose all rewrite (valley) proofs are minimal but incomparable. Then any Church-Rosser system is complete, since every identity has a rewrite proof, but only the full deductive closure is saturated.

The next theorem relates canonicity and saturation.

THEOREM 3.

1. *A presentation A is saturated if and only if it contains its own canonical presentation:* $A \supseteq A^\sharp$.
   - *In particular,* $A^\sharp$ *is saturated.*

2. *Moreover, the canonical presentation $A^\sharp$ is the smallest saturated set:*

   – *No equivalent proper subset of $A^\sharp$ is saturated.*
   – *If A is saturated, then every equivalent superset also is.*

PROPOSITION 4.

1. *Presentation A is saturated if and only if $\Theta A \approx A$.*
2. *Similar presentations are either both saturated or neither is.*
3. *Similar presentations are either both complete or neither is.*

The following definition sets the stage for the third characterization of canonical presentation, as non-redundant lemmata. Formulæ that can be removed from a presentation—without making proofs worse—are deemed "redundant":

DEFINITION 5 (Redundancy).

– *A set R of formulæ is* (globally) redundant *with respect to a presentation A when:*

$$A \cup R \succsim A \setminus R$$

– *The set of all* (locally) redundant *formulæ of a given presentation A will be denoted as follows:*

$$\rho A \quad \stackrel{!}{=} \quad \{r \in A \colon A \succsim A \setminus \{r\}\}$$

– *A presentation A is* irredundant *if*

$$\rho A = \emptyset$$

Intuitively, the notion of global redundancy captures the redundancy of a whole set of formulæ within or without the presentation, whereas local redundancy captures the redundancy of formulæ within the presentation, one at a time. If $R \subseteq A$, then $A \cup R \succsim A \setminus R$ reduces to $A \succsim A \setminus R$. Operationally, this means formulæ in R can be removed from A. If $R \cap A = \emptyset$, then $A \cup R \succsim A \setminus R$ reduces to $A \cup R \succsim A$: operationally, this means it is redundant to add the formulæ in R to A.

It is thanks to the well-foundedness of $>$ that the set of all *locally* redundant formulæ in ρA is *globally* redundant:

PROPOSITION 5. *For all presentations A:*

$$A \approx A \setminus \rho A$$

Thus, it can be shown that A is *reduced* (i.e., $A = \Gamma\,\mu\Pi\,A$) if and only if it is *irredundant* ($\rho A = \emptyset$).

The third characterization of the canonical set is central for our purposes:

THEOREM 4. *A presentation is canonical if and only if it is saturated and reduced.*

Informally, A is reduced if it is the set of premises of its minimal proofs; it is saturated if minimal proofs in A are exactly the normal-form proofs in the theory; it is canonical if it is the set of premises of normal-form proofs. Hence, saturated and reduced is equivalent to canonical.

4. Variations

The idea we are promoting is that, given a set of axioms, A, one is interested in the (unique) set of lemmata, $A^\sharp \subseteq \Theta A$, which—when used as premises in proofs—supports *all* the *minimal* proofs of the theorems ΘA. These lemmata form *the* "canonical basis" of the theory.

Returning to our simple example, we can add three inference rules for disequalities:

$$\frac{i = j \quad j \neq k}{i \neq k}\ \mathbf{T} \qquad \frac{i \neq i}{j = k}\ \mathbf{F}_{j=k} \qquad \frac{\boxed{i \neq j}}{i \neq j}\ \mathbf{I}_{i\neq j}$$

With them, one can infer, for example, $0 \neq 0$ from $1 \neq 1$. If F is smaller than other proof combinators, and I nodes are incomparable, then the canonical basis of any inconsistent set is $\{i \neq j : i, j \in \mathbf{N}\}$. All positive equations are redundant, because $\mathbf{F}_{j=k}$ is a smaller proof than $\mathbf{I}_{j=k}$.

If projection P is the most expensive type of inference, then no minimal proof includes it. And if proofs are compared in a simplification ordering (subproofs are always smaller than their superproofs), then minimal proofs will never have superfluous transitivity inferences of the form

$$\frac{u = t \quad t = t}{u = t}\ \mathbf{T}$$

Suppose we are using something like the recursive path ordering for proof terms and consider these inference rules for ground equality and disequality: S, T, F, I, Z, with S extended to apply to all function symbols of any arity.

Refutation. If the inference rule F is the cheapest in the proof ordering, $T < I$, and $I(i,j)$ nodes are measured by the values of i and j, then the canonical basis of any inconsistent presentation is a (smallest) trivial disequation $\{t \neq t\}$. Indeed, all positive equations can be obtained by applying F to $t \neq t$, and all negated equations can be obtained by two applications of T:

$$\frac{\dfrac{n = t \quad t \neq t}{n \neq t} \quad t = m}{n \neq m}$$

for all numerals m, n and t.

Deduction. If the proof ordering prefers introduction I of assumptions over all other inferences (including Z), then trivial proofs are best. In that case, $\rho\Theta A = \emptyset$ and the canonical basis includes the whole theory: $A^\sharp = \Theta A$. In other words, everything is needed, because it is the smallest proof of itself.

Paramodulation. If the proof ordering makes functional reflexivity S smaller than I ($S < T < I$), but the only ordering on leaves is $I(u,t) \leq I(c[u], c[t])$ for any context c, then the canonical basis will be the congruence closure, as generated by paramodulation: $\rho A = \{f(u_1, \ldots, u_n) = f(t_1, \ldots, t_n) \colon u_1 = t_1, \ldots, u_n = t_n \in \Theta A\}$. The theory ΘA is the closure under functional reflexivity of the basis $A^\sharp$. If A is as in our first example (i.e., $A = \{4 = 2, 4 = 0\}$), then $A^\sharp = \{2j = 0 \colon j > 0\}$. The other equalities in $\Theta A = \{i = j \colon i \equiv j \pmod 2\}$ are obtained from those in $A^\sharp$ by applying S (e.g., $8 = 4$ is derived from $4 = 0$ by applying S^4 to both sides).

Completion. On the other hand, if the ordering on leaves compares terms in some simplification ordering $\gtrsim$, then the canonical basis will be the fully reduced set, as generated by (ground) completion: $\rho A = \{u = u\} \cup \{u = t \colon t = v \in \Theta A,\ t \gg v,\ v \text{ is not } u\}$. Operationally, $u = t$ can be reduced to $u = v$. For our first example, $A^\sharp = \{2 = 0\}$, as all equations in $\{2j = 0 \colon j > 0\}$ reduce to $2 = 0$. For another example, if $A = \{a = c, sa = b\}$ and $sa \gg sb \gg sc \gg a \gg b \gg c$, then $I(sa, b) > T(S(I(a,c)), I(sc, b))$, and hence $A^\sharp = \{a = c, sc = b\}$.

Superposition. If one distinguishes between T steps based on the weight of the shared term j, making $T > I$ when j is the greatest, and $T < I$ otherwise, then the canonical basis is also closed under paramodulation into the larger side of equations. Indeed, assume we have $k = j$ and $j = i$. If the shared term j is the greatest, the transitivity proof is a peak $k \leftarrow j \rightarrow i$, and $k \leftarrow j \rightarrow i\ (T) > k = i\ (I)$ means that adding $k = i$ by superposition provides a smaller proof. If the shared term j is the smallest, the transitivity proof is a valley $k \rightarrow j \leftarrow i$, and $k \rightarrow j \leftarrow i\ (T) < k = i\ (I)$ means that valley proofs are the smallest.

5. Inference and derivations

There are two basic applications for saturation-based inference: constructing a finite canonical presentation when such exists, and searching for proofs by forward reasoning from axioms, avoiding inferences that do not help saturate. Inference steps are defined by deduction mechanisms.

In general, a *(one-step) deduction mechanism* $\rightsquigarrow$ is a binary relation over presentations, and we call a pair $A \rightsquigarrow B$, a *deduction step.* A deduction mechanism is *functional* if for any A there is a unique B (possibly A) such that $A \rightsquigarrow B$. We consider only functional mechanisms here, using δA to refer to that unique B deducible from A, so $A \rightsquigarrow \delta A$, always.

Practical mechanisms are functional (and usually operate deterministically); they are obtained by coupling a (nondeterministic) inference system with a *search plan*, or search strategy, to yield a *completion procedure* or *proof procedure.* Specific procedures may impose additional structure, such as singling out a formula as the *target* theorem or *goal*, in which case the deduction mechanism applies to pairs or tuples; see (Bonacina, 1999) for examples. In this paper, we consider only functional mechanisms that apply to presentations, and take the notion of a deduction mechanism as a whole. This entails no loss of generality, since the abstract set $\mathbb{P}$ may be limited on the concrete level to proofs and subproofs of a specific goal.

A sequence of presentations $A_0 \rightsquigarrow A_1 \rightsquigarrow \cdots$ is called a *derivation.* (We do not consider transfinite derivations in this paper.) Let $A_* = \cup_i A_i$ be all formulæ appearing anywhere in the derivation. The *result* of the derivation is, per Huet (1981), its *persisting* formulæ:

$$A_\infty \quad \stackrel{!}{=} \quad \liminf_{j\rightarrow\infty} A_j \;=\; \bigcup_j \bigcap_{i\geq j} A_i$$

We say that a proof p *persists* when $\Gamma p \subseteq A_\infty$. Thus, if a proof persists, so do its subproofs (by Postulate 5). By Proposition 2, we have $\Pi A_i \sqsupseteq \Pi A_*$ for all i.

DEFINITION 6 (Soundness and Adequacy).

– *A deduction mechanism δ is* sound *if $\delta A \subseteq \Theta A$.*

– *It is* adequate *if $A \subseteq \Theta\delta A$.*

– *It is both if $A \equiv B$ whenever $A \rightsquigarrow B$.*

DEFINITION 7 (Goodness).

– *A deduction step $A \rightsquigarrow B$ is* good *if $A \succsim B$.*

– *A deduction mechanism δ is* good *if proofs only get better: $A \succsim \delta A$, for all presentations A. That is, if $\rightsquigarrow$ is sound and adequate, and $\Pi A \sqsupseteq \Pi B$ whenever $A \rightsquigarrow B$.*

– *A derivation $A_0 \rightsquigarrow A_1 \rightsquigarrow \cdots$ is* good *if $A_i \succsim A_{i+1}$ for all i.*

We are only interested in good derivations. From here on in, only good (hence, also sound and adequate) derivations will be considered.

DEFINITION 8 (Finiteness and Compactness).

– *An ordered proof system has* finitely-based proofs*, if its proofs use only a finite number of premises:*

$$\forall p \in \mathbb{P}. \quad |\Gamma p| < \infty$$

– *It is* compact *if minimal proofs use only a finite number of premises:*[3]

$$\forall A \in \mathbb{A}.\ \forall p \in \mu\Pi A. \quad |\Gamma p| < \infty$$

DEFINITION 9 (Continuity). *(Minimal) Proofs are* continuous *if*

$$\liminf_{i\to\infty} \mu\Pi A_i \ = \ \mu\Pi A_\infty \ (= \mu\Pi \liminf_{i\to\infty} A_i)$$

for any chain $A_0 \succsim A_1 \succsim \cdots$.

[3] We call this *compactness* (of proofs), because it is used traditionally to infer compactness (of a logic), namely, that a set A of formulae is unsatisfiable if and only if it has a finite unsatisfiable subset A', from completeness (viz. a set A is unsatisfiable if and only if it is inconsistent). Indeed, if A is unsatisfiable, there is a proof of $\bot$ (falsehood) in ΠA (unsatisfiable implies inconsistent). Take a minimal proof p of $\bot$, i.e., $p \in \mu\Pi A$, and let A' be the finite set Γp; since $p \in \Pi A'$, A' is unsatisfiable (inconsistent implies unsatisfiable), and it is a finite subset of A.

LEMMA 2. *For good derivations, compactness implies continuity.*

Proof. Continuity requires $\bigcup_j \bigcap_{i\geq j} \mu\Pi\, A_i = \mu\Pi \bigcup_j \bigcap_{i\geq j} A_i$.

- $\mu\Pi \cup_j \cap_{i\geq j} A_i \subseteq \cup_j \cap_{i\geq j} \mu\Pi\, A_i$: Let $p \in \mu\Pi \cup_j \cap_{i\geq j} = \mu\Pi\, A_\infty$. Let $a \in \Gamma\, p$. By compactness, there are only finitely many such a. Let j be the smallest index in the derivation such that all $a \in \Gamma\, p$ are in A_j. Then $p \in \Pi\, A_j$. Second, $p \in \mu\Pi\, A_j$, because $p \in \mu\Pi\, A_\infty$ and the derivation is good. Third, $p \in \cap_{i\geq j} \mu\Pi\, A_j$, because all $a \in \Gamma\, p$ persist, since $\Gamma\, p \subseteq A_\infty$. It follows that $p \in \cup_j \cap_{i\geq j} \mu\Pi\, A_j$.

- $\cup_j \cap_{i\geq j} \mu\Pi\, A_i \subseteq \mu\Pi \cup_j \cap_{i\geq j} A_i$: Let $p \in \cap_{i\geq j} \mu\Pi\, A_i$ for some j. It follows that for all $a \in \Gamma\, p$, $a \in \cap_{i\geq j} A_i$, whence $a \in \cup_j \cap_{i\geq j} A_j = A_\infty$. This means that $p \in \Pi\, A_\infty$. Because p is minimal at all stages $i \geq j$, and the derivation is good, $p \in \mu\Pi\, A_\infty$.

Since the proof ordering is well-founded:

LEMMA 3. *If a deduction mechanism is good then*

$$\Pi\, A_i \sqsupseteq \Pi\, A_\infty$$
$$\Theta A_i \subseteq \Theta A_\infty$$

for all presentations A_i in a derivation $\{A_i\}_i$.

Let $\Pi_c\, A \stackrel{!}{=} \{p \in \Pi\, A : \Delta\, p = c\}$ signify the proofs of formula c.

Proof. Let $p_i \in \mu\Pi_c\, A_i$. Since the derivation is good, there are proofs $p_j \in \Pi_c\, A_j$, $j > i$, such that $p_i \geq p_{i+1} \geq \cdots$. By well-foundedness, from some point on these are all the same proof q. Thus, $\Gamma\, q \subseteq A_\infty$, $q \in \Pi\, A_\infty$, and $\Pi\, A_i \sqsupseteq \Pi\, A_\infty$. That $\Theta A_i \subseteq \Theta A_\infty$ follows from the definitions.

NOTE 1. *For bad (i.e. non-good) derivations this is not the case. To wit, let*

$$\mathbb{P} = \left\{ \frac{a}{b}, \frac{b}{a} \right\}$$

and consider $a \rightsquigarrow b \rightsquigarrow a \rightsquigarrow b \rightsquigarrow \cdots$. As the derivation oscillates perpetually between deriving b from a and a from b, at the limit $A_\infty = \emptyset$ and $\Theta A_\infty = \emptyset$, whereas $\Theta A_i = \{a, b\}$ for all finite i.

LEMMA 4. *If proofs are compact, then any good derivation $\{A_i\}_i$ is sound and adequate. That is,*

$$A_i \equiv A_\infty$$

for all i.

Proof. Lemma 3 gives one direction (adequacy), namely $\Theta A_i \subseteq \Theta A_\infty$. By Lemma 2, compactness and goodness imply continuity. Suppose $c \in \Theta A_\infty$. Then there is a $p \in \mu\Pi_c A_\infty$. By continuity, $p \in \cup_j \cap_{i\geq j} \mu\Pi A_i$, whence $p \in \cap_{i\geq j}\mu\Pi A_i$ for some j. Thus, $c \in \Theta A_i$ for all $i \geq j$. That $c \in \Theta A_i$ for all $i < j$, follows from goodness, since $A_i \succsim A_j$ implies $A_i \equiv A_j$ (see Definition 3).

NOTE 2. *This does not necessarily hold for infinitary systems that violate the compactness hypothesis. Let all proofs be incomparable, including:* $\widehat{a_i}$ *(for all* i*),* $\frac{\widehat{a_j}}{a_i}$ *(for all* i, j*),* $\frac{\widehat{a_0}, \widehat{a_1}, \ldots}{c}$*, and* $\frac{}{c}$*. The derivation* $\{a_j : j \leq i\}_i$ *is good, but only its limit includes the infinitary proof.*

LEMMA 5. *For all presentations* A *and* B*:*

$$A \succsim B \;\Rightarrow\; B \cap \rho A \subseteq \rho B$$

That is, "once redundant, always redundant."

Proof. Consider a proof $p \in \Pi B$ that uses a premise $a \in \rho A \subseteq A$. Since (by Reflexivity) $\widehat{a} \in \Pi A$, a must also have an alternative (nontrivial) proof $q \in \Pi_a (A \setminus \{a\})$, such that $\widehat{a} > q$. By assumption, there is an $r \in \Pi B$ such that $q \geq r$. By the postulates of subproofs, $p \trianglerighteq \widehat{a} > r$ implies the existence of a proof $p' \in \Pi (B \cup \{a\}) = \Pi B$ such that $p > p'$. If $a \in \Gamma p'$, then this process continues. It cannot continue forever, so we end up with a strictly smaller proof not involving a, establishing a's redundancy vis-à-vis B.

PROPOSITION 6. *If a derivation* $\{A_i\}_i$ *is good, then the limit supports the best proofs:*

$$A_* \;\approx\; A_\infty$$

Proof. One direction, namely $\Pi A_\infty \sqsupseteq \Pi A_*$, follows by Proposition 2 from the fact that $A_\infty \subseteq A_*$. To establish that $\Pi A_* \sqsupseteq \Pi A_\infty$, we show that $\mu\Pi A_* \sqsupseteq \Pi A_\infty$ and rely on Proposition 2. Suppose $p \in \mu\Pi A_*$. It follows from (Eq. 4 and Prop. 2) that $\widehat{\Gamma p} \subseteq \Sigma p \subseteq \mu\Pi A_*$. By goodness, each $a \in \Gamma p$ persists from some A_i on. Hence, $\Gamma p \subseteq A_\infty$, and $p \in \Pi A_\infty$.

DEFINITION 10 (Canonical Derivations).

- *A derivation* $\{A_i\}_i$ *is* completing *if its limit is complete.*
- *It is* saturating *if its limit is saturated.*

– *It is* clean *if its limit is reduced.*

– *It is* canonical *if it is both saturating and clean.*

LEMMA 6.

– *A derivation* $\{A_i\}_i$ *is completing if every theorem of* A_0 *eventually admits a persistent normal-form proof:*

$$\Theta A_0 \;\subseteq\; \Delta\,(\Pi\, A_\infty \cap \mu\Pi\,\Theta A_0)$$

– *It is saturating if all normal-form proofs emerge eventually:*

$$\mu\Pi\,\Theta A_0 \;\subseteq\; \Pi\, A_\infty$$

– *It is clean if no formula remain persistently redundant:*

$$\rho A_* \cap A_\infty \;=\; \emptyset$$

Proof. By Lemma 4, we know that $A_\infty \equiv A_0$ and $\Delta\,(\Pi\, A_\infty \cap \mu\Pi\,\Theta A_0) \subseteq \Theta A_0$. Hence, $\Theta A_\infty = \Delta\,(\Pi\, A_\infty \cap \mu\Pi\,\Theta A_\infty)$.

Similarly, by Lemma 1, the condition $\mu\Pi\,\Theta A_0 \subseteq \Pi\, A_\infty$ gives saturation.

Suppose that some $r \in \rho A_\infty \subseteq A_\infty \subseteq A_*$. Consider $\widehat{r}$, and compare it to a smaller proof $p \in \Pi_r\, A_\infty$, which must exist because r is redundant. Let q be any proof in $\mu\Pi\, A_*$. Were $r \in \Gamma\, q$, then replacing $\widehat{r}$ as a subproof of q with p, would by (6) result in a smaller proof than q, contradicting the fact that q is minimal. Thus, r cannot be premise of any minimal proof of A_*. It follows that $r \in \rho A_*$, which contradicts cleanness.

LEMMA 7. *A sufficient condition for a good derivation* $\{A_i\}_i$ *to be completing is that each non-normal-form proof eventually becomes much better:*

$$\bigcup_i \mu\Pi\, A_i \setminus \mu\Pi\,\Theta A_0 \;\sqsupset\; \bigcup_i \Pi\, A_i$$

Proof. By Lemma 3, if $p_i \in \mu\Pi_c\, A_i$ then $q \in \Pi_c\, A_\infty$, for some q. If $q \in \mu\Pi\,\Theta A_0$ then $c \in \Delta\,(\Pi\, A_\infty \cap \mu\Pi\,\Theta A_0)$ and we are done. Otherwise, the sufficient condition implies that for some k, there is a proof $q_k \in \Pi\, A_k$ of c such that $p_i \geq q > q_k$. Completeness follows by induction on proofs.

LEMMA 8. *A good derivation* $\{A_i\}_i$ *is canonical if and only if*

$$A_\infty = A_0^\sharp$$

Proof. Assume the derivation is canonical, that is, saturating and clean. Saturating means $\mu\Pi\,\Theta A_0 \subseteq \Pi\, A_\infty$, hence $\Gamma\,\mu\Pi\,\Theta A_0 \subseteq A_\infty$, hence $A_0^\sharp \subseteq A_\infty$. Clean means $\rho A_\infty = \emptyset$, from which it follows that $A_\infty \subseteq A_0^\sharp$ (by way of contradiction, if there were an $x \in A_\infty$, but $x \notin A_0^\sharp$, this x would be redundant, contradicting cleanness). Together, $A_0^\sharp \subseteq A_\infty$ and $A_\infty \subseteq A_0^\sharp$ give $A_0^\sharp = A_\infty$. The other direction is trivial.

In summary, the limit of a derivation is *complete*, *reduced*, *saturated*, if the derivation is *completing*, *clean*, *saturating*, respectively, where saturated is stronger than complete, and saturated and clean together mean *canonical.*

6. Completion procedures and proof procedures

The central concept underlying completion (Knuth and Bendix, 1970) is the existence of critical proofs. Completion alternates "expansions" that infer the conclusions of critical proofs with "contractions" that remove redundancies. More generally, theorem proving with simplification (e.g., Dershowitz, 1991b; Bonacina and Hsiang, 1991; Bachmair and Ganzinger, 1991) entails two processes: **Expansion**, whereby any sound deductions (anything in ΘA) may be added to the set of derived theorems; and **Contraction**, whereby any redundancies (anything in ρA) may be removed. The inference-rule interpretation of completion, accommodating both expansion and contraction, was developed in (Bachmair and Dershowitz, 1994).

DEFINITION 11 (Expansion and Contraction).

- *A deduction step* $A \rightsquigarrow A \cup B$ *is an* expansion *provided* $B \subseteq \Theta A$*.*
- *A deduction step* $A \cup B \rightsquigarrow A$ *is a* contraction *provided* $A \cup B \succsim A$*.*

It is easy to see that:

PROPOSITION 7.

- *Expansions and contractions are good.*
- *Derivations, the steps of which are expansions and/or contractions, are good.*

DEFINITION 12 (Critical Proof). *A minimal proof* $p \in \mu\Pi\, A$ *is* critical *if it is not in normal form, but all its subproofs are:*

$$p \in \mu\Pi\, A \setminus \mu\Pi\, \Theta A$$
$$\forall q.\ p \rhd q \ \Rightarrow\ q \in \mu\Pi\, \Theta A$$

We use $C(A)$ to denote the set of all such critical proofs in A, and use the following notation for their conclusions:

DEFINITION 13 (Critical Formulæ).

$$\nabla A \quad \overset{!}{=} \quad \Delta\, C(A)$$

DEFINITION 14 (Fairness).

– *A good derivation* $\{A_i\}_i$ *is* fair *(w.r.t.* C*) if*

$$C(A_\infty) \ \sqsupset\ \Pi\, A_*$$

– *It is* uniformly fair *if*

$$\widehat{A_\infty} \setminus \widehat{A^\sharp} \ \sqsupset\ \Pi\, A_*$$

Critical obligations are proofs that are not in normal form but all of whose proper subproofs are already in normal form. Fairness means that all persistent obligations are eventually “subsumed” by a strictly smaller proof.

THEOREM 5. *Presentation* A *is complete if and only if* $C(A) \sqsupset \Pi\, A$*.*

Proof. Recall that A complete means $\Theta A = \Delta\,(\Pi\, A \cap \mu\Pi\, \Theta A)$.

– $C(A) \sqsupset \Pi\, A$ implies $\Theta A = \Delta\,(\Pi\, A \cap \mu\Pi\, \Theta A)$: Assume, by way of contradiction, that A is incomplete. Then there is a $c \in \Theta A$ such that $c \notin \Delta\,(\Pi\, A \cap \mu\Pi\, \Theta A)$, or there is no proof of c in $\Pi\, A \cap \mu\Pi\, \Theta A$. However, there are proofs of c in $\Pi\, A$: let’s take a minimal one, that is, let $p \in \mu\Pi_c\, A$. By the above, $p \notin \mu\Pi\, \Theta A$, or p is not in normal form. If p is not in normal form, it means that it has some subproof(s) that are not in normal form, that is, some $q \trianglelefteq p$ that is not in normal form. By the well-foundedness of $\trianglelefteq$, let q be a minimal (with respect to $\trianglelefteq$) such proof. Minimality with respect to $\trianglelefteq$ means that all subproofs of q are in normal form. Thus, we have a (possibly trivial) subproof q of p, which is not in normal form, but such that all its subproofs are. But this is the definition of critical proof: $q \in C(A)$. The hypothesis $C(A) \sqsupset \Pi\, A$ implies that there exists a proof $r \in \Pi\, A$ such that $r < q$. Since we have $p \trianglerighteq q > r$, by Replacement (7), there exists a $p' \in \Pi\, A$, such that $p' < p$, with r in place of q, i.e., $p > p' \trianglerighteq r$. This contradicts the fact that p is minimal.

– $\Theta A = \Delta\, (\Pi\, A \cap \mu\Pi\, \Theta A)$ implies $C(A) \sqsupset \Pi\, A$: Assume, by way of contradiction, that $C(A) \not\sqsupset \Pi\, A$: there exists a $p \in C(A)$ such that for no $q \in \Pi\, A$ do we have $p > q$. Let $c = \Delta\, p$. By completeness, there is a normal form proof q of c in $\Pi\, A \cap \mu\Pi\, \Theta A$ and q is smaller than p, precisely because it is in normal form, contradicting the above.

COROLLARY 1. *If a good derivation is fair, then its limit is complete.*

Proof. By the definition of fairness we have $C(A_\infty) \sqsupset \Pi\, A_*$. By Proposition 6, $A_* \approx A_\infty$, and $\Pi\, A_* \simeq \Pi\, A_\infty$, so that $C(A_\infty) \sqsupset \Pi\, A_\infty$. By Theorem 5, A_∞ is complete.

This result suggests completing an axiomatization A_0 by adding, step by step, what is needed to make for better proofs than the critical ones.

For example, suppose a proof ordering makes $\widehat{c} > \frac{\hat{b}}{c}$ and $\frac{\hat{c}}{b} > \widehat{b}$. Start with $A_0 = \{c\}$, and consider $\widehat{c}$. Were $\widehat{c}$ to persist, then by fairness a better proof would evolve, the better proof being $\frac{\hat{b}}{c}$. If $\widehat{b}$ is in normal form, then $b \in A_\infty$ and both minimal proofs persist.

Another example: $\mu\mathbb{P} = \{\widehat{b}, \widehat{c}, \frac{\hat{b}}{c}\}$ and $A = \{b\}$, then $A \rightsquigarrow A \rightsquigarrow \cdots$ is fair, since $A_\infty = A$ and $C(A_\infty) = \emptyset$. The result is complete but unsaturated (c is missing).

Clearly, a fair derivation is also completing. On the other hand, completing does not imply fair, because the limit could feature a normal form proof of some $c \in \Theta A_0$, without having reduced all persistent critical proofs of c. The two notions serve different purposes: completing is more abstract, and represents a precondition for getting a complete limit. Fair is stronger and more concrete, as it specifies a way to achieve completeness by reducing all persistent critical proofs.

A saturated limit is not necessarily reduced, unless it is also clean, in which case it is canonical:

THEOREM 6 (Fair Completion). *Clean, fair derivations are canonical, provided minimal proofs are unique.*

Proof. This follows from Lemma 6, Corollary 1, Proposition 3 and Theorem 4.

By Proposition 1, this also means that each $a \in A_\infty$ $(= A^\sharp)$ is its own ultimate proof $\widehat{a} \in \mu\Pi\,\Theta A$, so is not susceptible to contraction.

We are left with the task of identifying sufficient conditions for saturation, in case minimal proofs are not unique:

THEOREM 7. *Presentation A is saturated if and only if $\widehat{A} \setminus \widehat{A^\sharp} \sqsupset \Pi\,A$.*

Proof. Recall that A saturated means $\mu\Pi\,A = \mu\Pi\,\Theta A$.

- $\widehat{A} \setminus \widehat{A^\sharp} \sqsupset \Pi\,A$ implies $\mu\Pi\,A = \mu\Pi\,\Theta A$: Assume, by way of contradiction, that $\mu\Pi\,A \neq \mu\Pi\,\Theta A$. Then there is a theorem $c \in \Theta A$ for which a normal form proof p^* is absent from $\mu\Pi\,A$. Instead, there is a minimal non-normalized proof $p \in \mu\Pi\,A \setminus \mu\Pi\,\Theta A$. Then, there is some $x \in \Gamma\,p$ such that $x \in A$ but $x \notin A^\sharp$ (were $\Gamma\,p \subseteq A^\sharp$, then p would be in normal form). By the hypothesis, $\widehat{x} > r$ for some $r \in \Pi\,A$. By Replacement (7), there exists a $v \in \Pi\,A$, such that $p > v \trianglerighteq r$, and p is not minimal.

- $\mu\Pi\,A = \mu\Pi\,\Theta A$ implies $\widehat{A} \setminus \widehat{A^\sharp} \sqsupset \Pi\,A$: If $x \in A \setminus A^\sharp$, there exists a $p \in \mu\Pi_x\,A = \mu\Pi_x\,\Theta A$ such that $\widehat{x} > p$, because $\widehat{x} \notin \mu\Pi_x\,\Theta A$, since $x \notin A^\sharp$.

By the above theorem, if A is saturated, $A \setminus A^\sharp$ is redundant: $A \setminus A^\sharp \subseteq \rho A$ or $A \setminus A^\sharp = A \cap \rho A$.

COROLLARY 2. *If a good derivation is uniformly fair, then its limit is saturated.*

Proof. By uniform fairness we have $\widehat{A_\infty} \setminus \widehat{A^\sharp} \sqsupset \Pi\,A_*$. By Proposition 6, $A_* \approx A_\infty$, and $\Pi\,A_* \simeq \Pi\,A_\infty$, so that $\widehat{A_\infty} \setminus \widehat{A^\sharp} \sqsupset \Pi\,A_\infty$. By Theorem 7, A_∞ is saturated.

7. Instances of the framework

What has traditionally been called "completion" can be described as an inference system, wherein each step $A_i \rightsquigarrow A_{i+1}$ is the composition of an expansion, $A_i \rightsquigarrow A_i \cup \nabla A_i = B_i$ followed by a contraction, $B_i \rightsquigarrow B_i \setminus \rho B_i = A_{i+1}$:

DEFINITION 15 (Bulk Completion, Bachmair, 1991, pp. 28–29).
Bulk completion *is a sequence of steps:*

$$A \;\rightsquigarrow\; [A \cup \nabla A] \setminus \rho[A \cup \nabla A]$$

By Proposition 7:

LEMMA 9. *Bulk completion is good.*

LEMMA 10. *The canonical presentation has no critical formulæ.*

Proof. $\nabla A^\sharp = \mu\Pi\, A^\sharp \setminus \mu\Pi\, \Theta A^\sharp = \emptyset$.

COROLLARY 3. *The canonical presentation is stable under bulk completion:*

$$A^\sharp \leadsto A' \;\Rightarrow\; A' = A^\sharp$$

THEOREM 8. *Bulk completion is canonical, provided minimal proofs are unique:*

$$A_0^\sharp \;=\; A_\infty^{\text{bulk}}$$

This follows from Theorem 6, because derivations by bulk completion are (a) fair, since bulk completion derives all critical formulæ en masse, and (b) clean, since bulk completion also removes redundancies immediately.

Returning to the ground equation case, let $\gtrsim$ be a total simplification-ordering of terms, let $P > I > T > S > Z$ in the precedence, let proofs be greater than terms, and compare proof trees in the corresponding total recursive path simplification-ordering. *Ground completion* is an inference mechanism consisting of the following inference rules:

Deduce: $\quad E \cup \{w = t[u]\} \leadsto E \cup \{w = t[v]\} \quad$ if $u = v \in E$ and $u \gg v$

Delete: $\quad E \cup \{t = t\} \leadsto E$

Furthermore, operationally, completion implements these inferences "fairly": No persistently enabled inference rule is ignored forever.

COROLLARY 4 (Completeness of Completion). *Ground completion results—at the limit—in the canonical, Church-Rosser basis.*

Proof. Ground completion is good, since **Deduce** and **Delete** do not increase proofs ($\leadsto \subseteq \succsim$). In particular, $I(w, t[u]) > T(I(w, t[v]), S^n(I(u, v)))$ if $u \gg v$, since $t[u] \gg t[v]$ and $t[u] \gtrsim u \gg v$. Ground completion is fair and clean. For example, the critical obligation

$$\frac{w = t \quad t = v}{w = v}\ \mathbf{T}$$

when $t \gg w, v$, is resolved by **Deduce**. Also, since $T > S$, non-critical cases resolve naturally:

$$\frac{\dfrac{w = t}{fw = ft} \quad \dfrac{t = v}{ft = fv}}{fw = fv} \quad > \quad \frac{\dfrac{w = t \quad t = v}{w = v}}{fw = fv}$$

8. Discussion

Completion processes have been studied intensively since their discovery and application to automated theorem proving by Knuth and Bendix (1970) and Buchberger (1985). The fundamental role of proof orderings in automated deduction, and the interpretation of completion as nondeterministic application of inference rules, was conceived in (Bachmair et al., 1986; see Bachmair and Dershowitz, 1994). The (inference-rule based) completion principle can be applied in numerous situations (Dershowitz, 1989; Bonacina and Hsiang, 1995), including equational rewriting (Peterson and Stickel, 1981; Jouannaud and Kirchner, 1986; Bachmair and Dershowitz, 1989), Horn theories (Kounalis and Rusinowitch, 1987; Dershowitz, 1991a, 1991c), induction (Kapur and Musser, 1987; Fribourg, 1989), unification (Doggaz and Kirchner, 1991), and rewrite programs (Bonacina and Hsiang, 1992; Dershowitz and Reddy, 1993).

Our abstract framework can be applied to re-understand completion mechanisms in a fully uniform setting. Because we have been generic in our approach, the results here apply to any completion-based framework, including standard completion mechanisms, like ground completion (Snyder, 1989; Gallier et al., 1993), as illustrated herein, equational completion, or completion for unification, and also to derive new completion algorithms, such as for constraint solving.

In (Bachmair and Dershowitz, 1994), a completion sequence is deemed fair if all persistent critical inferences are generated. In (Nieuwenhuis and Rubio, 2001, fn. 8), an inference sequence is held to be fair if all persistent inferences are either generated or become redundant. In (Bonacina, 1992; Bonacina and Hsiang, 1995), the notion of fairness was formulated in terms of proof reduction with respect to a proof ordering, and made relative to the target theorem, suggesting for the first time that fairness should earn one a property weaker than saturation. The definition of fairness propounded here combines all these ideas. Fairness means that all persistent critical proofs are reduced, but it only earns completeness, not saturation. As we saw, a stronger version of fairness, namely *uniform fairness*, is needed for saturation when the proof ordering is partial.

Bulk completion, as investigated here, is an abstract notion. Concrete procedures are obtained by coupling the inference system with a search plan that determines the order in which expansion and contraction steps take place. ¿From a practical point of view, *fairness* and *cleanness* are two requirements for the search plan: it should schedule enough expansion steps to be fair, hence complete, and enough contraction steps to be clean. Specific search plans may settle for some approximation of these properties. The two are intertwined, as a basic control issue is how best to avoid performing expansion inferences from premises that can be contracted, because such expansion inferences would generate redundancies. This principle has led many to design search plans called by various authors *simplification-first*, *contraction-first* or *eager contraction* plans. Our definition of critical obligations also allows one to incorporate "critical pair criteria" (see, for example, Bachmair and Dershowitz, 1988).

On the other hand, making sure that contraction takes priority over expansion is not cost-free, because it involves keeping a potentially very large database of formulæ *inter-reduced*. In turn, this involves *forward contraction*, that is, contracting newly generated formulæ with respect to already existing ones, and *backward contraction*, that is, contracting formulæ already in the database with respect to the new formulæ that survived forward contraction. In practice, forward contraction is considered to be part and parcel of the generation of a formula, while backward contraction is considered to be a bookkeeping task for the database of formulæ. In our framework, the effort to implement contraction efficiently is the effort to make *clean* derivations efficient.

An observation that has helped streamline implementations of completion, and of theorem-proving strategies based on completion, was that backward contraction can be implemented by forward contraction. That is, it suffices to detect that a formula in the database is reducible, and then subject it to forward contraction, as if it were newly generated. This way, formulæ generated by backward contraction are treated like formulæ generated by expansion. This observation appeared in implementations since the late eighties, most notably in Otter (McCune, 1994). This kind of prover works by maintaining a list of formulæ *already selected* as expansion parents, and a list of formulæ *to be selected*, where new formulæ that survived forward contraction are added. Another major intuition in the implementation of completion-based strategies was to realize that, in addition to search plans that aim at keeping the union of the two lists inter-reduced, it is good to have search plans that inter-reduce only the list of selected formulæ. The E theorem prover (Schulz, 2002) features these search plans, while most of Otter's successors, such as SPASS (Weidenbach et al., 1999), Vampire (Riazanov and Voronkov, 2002) and Waldmeister (Hillenbrand, 2003), implement both kinds.

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