A Gap Tree Theorem for Quasi-Ordered Labels [1]

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Abstract

Given a quasi-ordering of labels, a labelled ordered tree s is *embedded with gaps* in another tree t if there is an injection from the nodes of s into those of t that maps each edge in s to a unique disjoint path in t with greater-or-equivalent labels, and which preserves the order of children. We show that finite trees are well-quasi-ordered with respect to gap embedding when labels are taken from an arbitrary well-quasi-ordering provided every tree path can be partitioned into a bounded number of subpaths of comparable nodes. This extends Kříž's combinatorial result [7] to partially ordered labels.

Key words: Well-quasi-ordering, Gap embedding, Homeomorphic embedding, Kruskal's Tree Theorem, Termination, Term orderings, Non-simplifying rewrite orderings

1 Introduction

Kruskal's Tree Theorem [8], establishing that finite trees are well-quasi-ordered under homeomorphic embedding, and its extensions have played an important rôle in both logic and computer science. In proof theory, it was shown to be independent of certain logical systems by exploiting its close relationship with ordinal notation systems [13], while in computer science it provides a common tool for proving the well-foundedness of term orderings, such as the *recursive path ordering* [1], used to prove termination of rewrite systems and other forms of symbolic computation.

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Kruskal's theorem applies to finite ordered trees with well-quasi-ordered labels, and states that any infinite sequence of such trees contains a pair of trees s, t, such that s can be homeomorphic embedded into t. Harvey Friedman ([2]; see [13]) proposed a variation in which homeomorphic embedding is limited by further stipulations regarding the labels of the paths pertaining to the embedding tree, but labels were bounded integers.

In [3,4] Gordeev extended Friedman's variation of the homeomorphic embedding, by considering ordinal labels, while modifying slightly the conditions regarding the labels of the paths in the embedding tree. Křížs [7] extension verifies a conjecture of Friedman's that finite trees labelled by ordinals are well-quasi-ordered under gap embedding. The embeddability conditions in [7] are stronger than the embeddability conditions in [4]. That is, the conditions regarding the labels of the paths in the embedding tree are stronger. Nevertheless, it is claimed in [5] that in the *proof-theoretical sense*, the statement that finite trees are well-quasi-ordered under the embeddability conditions of [7] is not stronger than the statement that finite trees are well-quasi-ordered under the embeddability conditions of [4]. (This means that both statements are provable in the same formal theory.)

Our work is of a purely combinatorial nature. It extends the result by Křížs to finite trees with well-quasi-ordered labels, provided each tree path contains only comparable labels. In that case, the well-quasi-order property of the set of trees is preserved. By simple induction, this extends also to the case where every path in the tree can be partitioned into some bounded number of subpaths with comparable labels. Moreover, since the absence of such a bound yields a bad sequence with respect to gap embedding, this is actually the canonical counterexample: every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

A term ordering is said to have the *subterm* property if terms are always bigger than all their subterms, and to have the *replacement* property if reducing subterms always reduces the whole term. Orderings with both properties are called *simplification orderings* [1]. In the context of rewriting, the tree-label ordering corresponds to a precedence ordering of the function symbols belonging to a given signature. Simplification orderings perforce include the homeomorphic embedding relation and are well-quasi-orderings whenever the labelling is. For demonstrating termination of rewriting, it is beneficial to use a *partial* (or *quasi-*) ordering on labels, rather than a total one. It is, however, sometimes necessary to prove termination of rewrite systems that are not "simplifying" in the sense of always causing a reduction in some simplification ordering. Gap embedding allows one to go beyond simplification orderings.

In [14], it was shown that many important order-theoretic properties of the well-partial-ordered precedence relations on function symbols carry over to the induced termination ordering. This is done by defining a general framework for precedence-based termination orderings via so-called *relativized ordinal notations*. Based on a few examples, it is further conjectured that every such application of a partial-order to an ordinal notation system carries the order-theoretic properties of the partial-order to the relativized notation system. An example of such a construction, using Takeuti's ordinal diagrams, is introduced in [12] under the name *quasi-ordinal-diagrams*. The definition of these diagrams is the only result we have encountered that deals with gap embedding of trees and *quasi*-ordered labels.

2 Preliminaries

A *quasi-ordering* is a set Q together with a reflexive and transitive binary relation $\precsim$. Given a quasi-ordering $(Q, \precsim)$ and two elements $a, b \in Q$, we say that a and b are *comparable* if either $a \precsim b$ or $b \precsim a$; otherwise we say that they are *incomparable*, symbolized $a\#b$. We denote by $\prec$ the strict part of $\precsim$.

A quasi-ordering $(Q, \precsim)$ is a *well-quasi-ordering* (*wqo*) if for every infinite sequence $a_0, a_1, a_2, \dots$ from Q there exist $i < j \in \mathbf{N}$ such that $a_i \precsim a_j$. (An "infinite" sequence is always an ω-sequence in this paper.) An infinite sequence from Q is referred to as *bad* if for all $i < j$, $a_i \not\precsim a_j$ holds; otherwise it is *good*. If, for all $i, j \in \mathbf{N}$, $a_i \# a_j$, the sequence is an *antichain*. Every partial ordering extending a well-quasi-ordering is well-founded (has no infinite strictly decreasing chains).

The following proposition lists some well-known equivalent definitions of well-quasi-orderings:

Proposition 1 *Given a set Q and a binary relation $\precsim$ over Q, the following conditions are equivalent:*

(1) $(Q, \precsim)$ is a well-quasi-ordering.
(2) $(Q, \precsim)$ has no infinite strictly decreasing chains and no infinite antichains.
(3) Any infinite Q-sequence contains an infinite increasing $\precsim$-chain.

The equivalence of these statements follows from the infinite version of Ramsey's Theorem. (See [9] for more on the theory of well-quasi-orderings.)

Let t be a tree, we denote by $t^\bullet$ its root. For a pair of nodes u, v in a rooted tree, we denote by $u \sqcap v$ the *closest common ancestor* of u and v; we write $u \sqsubset v$ if u is to the left (descendent of elder sibling of ancestor) of v. The following defines (homeomorphic) tree embedding:

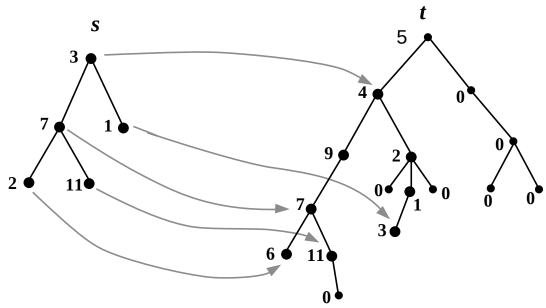

Fig. 1. Gap embedding of s into t.

Definition 2 (Tree embedding) *For two labelled ordered trees s, t we say that s is* embedded *in t (with respect to $\precsim$) if there is an injection $\iota : s \to t$ such that:*

- *Label increasing: for all nodes x in s, $x \precsim \iota(x)$;*
- *Ancestry preserving: for all nodes x, y in s, $\iota(y \sqcap x) = \iota(y) \sqcap \iota(x)$;*
- *Sibling order preserving: for all nodes x, y in s, $x \sqsubset y$ implies $\iota(x) \sqsubset \iota(y)$.*

Next, we supply definitions for the *gap subtree* and *gap embedding* relations:

Definition 3 (Gap embedding) *For two labelled ordered trees s, t, we say that s is* embedded with gaps *in t and write $s \hookrightarrow t$, if there is a* tree embedding *$\iota : s \to t$ satisfying the following additional conditions (see Fig. 1):*

- *Edge gap condition: for all edges $\langle x, y\rangle$ in s (x is the parent of y) and for all nodes z in the path from $\iota(x)$ to $\iota(y)$ in t, $z \succsim y$;*
- *Root gap condition: $x \succsim s^\bullet$ for all nodes x in the path from $t^\bullet$ to $\iota(s^\bullet)$.*

Definition 4 (Gap subtree) *For two trees s, t in $\mathcal{T}$, we say that t is a* gap subtree *of s, and write $s \trianglerighteq t$, if t is a subtree of s and the path $P = [s^\bullet : t^\bullet]$ from $s^\bullet$ to $t^\bullet$ in s meets the following condition:*

- $\min_{\precsim} P \in \{s^\bullet, t^\bullet\}$.

In particular, immediate subtrees are gap subtrees (t is an immediate subtree *of s if $t^\bullet$ is an immediate child of $s^\bullet$).*

Remark 5 *A more intuitive definition of gap embedding (given in the abstract) can be stated for trees with labels on* edges *instead of nodes. Denote by $s \hookrightarrow' t$ an embedding of an edge-labelled tree s in a likewise labelled tree t, such that each edge of s is mapped to a path in t all labels of which are greater than or equivalent to (with respect to the node ordering $\precsim$) the label of the edge in s. It is not hard to show that, if ordered rooted trees with labels on nodes is wqo under the gap embedding of Definition 3, then also the set of edge-labelled* *trees is a wqo under this edge-based embedding (cf. [7, Sect. 1.3]).*

A set of trees is well-quasi-ordered under a gap embedding relation $\hookrightarrow$ if every infinite sequence of trees contains a pair of trees s, t, where s precedes t, such that $s \hookrightarrow t$. Our main theorem will provide sufficient conditions on trees for them to be well-quasi-ordered under gap embedding.

3 The Main Theorem

Let $\mathcal{T}$ be the set of *ordered* (rooted, planted-plane) finite trees, with nodes well-quasi-ordered by $\precsim$, and such that *every node is comparable with all its ancestor nodes*, and let $\hookrightarrow$ and $\trianglelefteq$ be the gap embedding and gap subtree relations as defined by Definition 3 and 4, respectively. A *sequence* s is a *partial* function $s : \mathbf{N} \to \mathcal{T}$. If $s(i)$ is not defined we shall write $s(i) = \perp$.

We first lay out five simple properties involving the embedding and subtree relations over $\mathcal{T}$ that we use in the construction of a minimal bad sequence. We then show the construction itself, required in order to apply the usual Nash-Williams [10] method (for an illustrated proof, see [11]).

The *gap subtree* relation over $\mathcal{T}$, as defined in 4 conforms to the following conditions:

$$s \trianglerighteq t \trianglerighteq u \wedge t^\bullet \succsim u^\bullet \Rightarrow s \trianglerighteq u \tag{A}$$
$$s \trianglelefteq t \Rightarrow s^\bullet \precsim t^\bullet \vee t^\bullet \precsim s^\bullet \tag{B}$$

We denote by $\triangleright$ the proper gap subtree relation. The *gap embedding* quasi-ordering $\hookrightarrow$ over $\mathcal{T}$, as defined in Definition 3, conforms to the following properties:

$$s \hookrightarrow t \trianglelefteq u \wedge t^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u \tag{C}$$
$$s \hookrightarrow t \trianglelefteq u \wedge s^\bullet \precsim u^\bullet \Rightarrow s \hookrightarrow u \tag{D}$$

An additional condition is:

$$\text{The relation } \trianglerighteq \text{ is noetherian;} \tag{E}$$

In other words, there is no infinite sequence of trees $t_1, t_2, t_3, \ldots$ from $\mathcal{T}$, such that $t_1 \trianglerighteq t_2 \trianglerighteq t_3 \trianglerighteq \ldots$

It is very convenient to extend the subtree relation and node ordering to empty positions of a sequence, so that: $t \trianglerighteq \perp$ and $t^\bullet \precsim \perp^\bullet$ for all $t \in \mathcal{T}$.

Let Seq be the set of infinite sequences of trees from $\mathcal{T}$. Define:

$$\begin{aligned} Ds &:= \{i \in \mathbf{N} \mid s(i) \neq \bot\} \\ \text{Bad} &:= \{s \in \text{Seq} \mid \forall i < j \in Ds.\ s(i) \not\hookrightarrow s(j)\} \\ \text{Sub } h &:= \{s \in \text{Seq} \mid \forall i \in Ds.\ h(i) \rhd s(i)\} \\ \text{Inc } k &:= \{s \overset{\infty}{\subseteq} k \mid \forall i < j \in Ds.\ s^\bullet(i) \precsim s^\bullet(j)\} \end{aligned}$$

where $s \overset{\infty}{\subseteq} k$ denotes that s is an infinite subsequence of k. A sequence s is *infinite* when its domain of definition, Ds, is. Thus, Bad is the set of infinite bad sequences; Sub h is the set of all infinite subsequences of gap subtrees of h.

Since $\succsim$ is a well-quasi-ordering, Inc k (the set of infinite increasing subsequences of k) is nonempty, as long as k is infinite, by Proposition 1.

Our goal, then, is to prove the following:

Theorem 6 (Main Theorem) Bad $= \emptyset$.

This means that the set $\mathcal{T}$ is wqo under $\hookrightarrow$. In other words, for every $s \in$ Seq there exist $i < j \in Ds$ such that $s(i) \hookrightarrow s(j)$. This extends the result of Kříž [7] for well-orderings to quasi-ordered labels.

3.1 Construction of the Minimal Bad Sequence

3.1.1 The approach

The Nash-Williams proof method is centered around the concept of *minimal bad sequence*. An infinite bad sequence of trees is said to be *minimal* if no subsequence of its subtrees is also bad. The existence of such a sequence yields a contradiction when dealing with tree embedding (without the gap conditions): Since no subsequence of its subtrees is bad, each such sequence is *good*. Hence, in particular, the set of the minimal bad sequence's immediate subtrees forms a well-quasi-ordering under the embedding relation. By Higman's Lemma [6], we know that the finite sequences of a well-quasi-ordered set forms a well-quasi-ordering under the natural extension of the order relation to finite sequences of elements (see Section 3.3). Therefore, we conclude that there ought to be a pair of trees s, t in the minimal bad sequence, where each child of s is embedded amongst unique children of t. By mapping the root of s to that of t we get a tree embedding of s into t, which contradicts the badness of the minimal bad sequence. For details, see Section 3.3 and, for the original application of that method, see [10,11].

Assuming Theorem 6 is false, and there are bad sequences of trees, the proof constructs a minimal counterexample, that is, a bad sequence $h \in \text{Bad}$, which is minimal in the sense that no infinite sequence of proper gap subtrees of its elements is also bad:

$$\text{Bad} \cap \text{Sub}\, h = \emptyset$$

When dealing with *gap embedding*, in contrast with the regular tree embedding, the task of constructing a minimal bad sequence is not trivial. In what follows we describe the strategy of the construction, following in the footsteps of Kříž [7].

3.1.2 Overview of the construction

As explained above, our goal is to construct a minimal bad sequence, in the sense that no subsequence of subtrees is bad. However, when dealing with gap embedding, the construction is substantially more difficult than in the original proof of Nash-Williams [10]. With ordinary homeomorphic embedding, the construction is obvious: just take a bad sequence that is lexicographically the smallest in size. That is, take the smallest tree t_1 (in the number of nodes — if there are several with the same size, chose one of them) which is the first element in some bad sequence; then take the smallest tree t_2 which is the second element in some bad sequence that starts with t_1, and so forth. However, this method does not work for gap embedding (because of the root gap condition). For this reason, a different tack is taken: One needs to show directly that such a *minimal* bad sequence h, i.e. a bad sequence with no bad children subsequences, exists. This is in contrast to the usual way of taking the minimal sequence lexicographically in the sizes of trees (which is then shown to disallow bad subsequences of children).

To construct the minimal bad sequence for gap embedding, the strategy is to build — by transfinite induction — an infinite table H of ostensible size $\omega_1 \times \omega$: Each row h_α is a bad sequence of length ω. By induction, we build the next row $h_{\alpha+1}$ from the gap subtrees of h_α. We then show that this inductive process of constructing bad sequences of subtrees must terminate eventually, via a cardinality argument: one cannot proceed with taking proper subtrees more than a countable number of times (see property (E) above). Therefore, the inductive construction terminates with a bad sequence h having no bad subsequence of subtrees.

However, it is not sufficient to take at each step a bad subsequence of (gap) subtrees from the previous sequence. The reason is that when reaching a limit ordinal λ the rows would converge to the *empty sequence* (i.e. $h_\alpha = \emptyset$), as otherwise it would mean taking infinite proper subtrees from some finite tree.

To avoid this situation, we append a (prefix of) the preceding sequence to the new one. For this to result in a *bad* sequence at each induction step, there is a need to maintain special invariants throughout the construction: root labels of trees must weakly increase in each row and in the columns

3.1.3 Formal description of the construction

We now give a formal description of the construction of the minimal bad sequence by ordinal induction. We present it as a transfinite 'algorithm' to simplify both the presentation and the proof. The function $H(\alpha)$, for any countable ordinal α, either returns a minimal bad sequence, or else constructs a new bad sequence h_α. There are three cases: the base case $H(0)$; the successor case $H(\alpha+1)$, where h_α is used to construct $h_{\alpha+1}$; and the limit case, $H(\lambda)$, for limit ordinal λ. These are:

$H(0)$:	$h :\in \text{Bad}$
	if $\text{Bad} \cap \text{Sub}\ h = \emptyset$ **then return** h
	$h_0 :\in \text{Inc lex}(h)$

$H(\alpha+1)$:	**if** $\text{Bad} \cap \text{Sub}\ h_\alpha = \emptyset$ **then return** h_α
	$k := \text{lex}(h_\alpha)$
	$\forall i \in \mathbf{N}.\ f(i) := \begin{cases} k(i) \ \textbf{if}\ h^\bullet_\alpha(i) \precsim k^\bullet(i) \\ \bot \quad \textbf{otherwise} \end{cases}$
	$g :\in \text{Inc}\ f$
	$\forall i \in \mathbf{N}.\ h_{\alpha+1}(i) := \begin{cases} h_\alpha(i) \ \textbf{if}\ i < \min Dg \\ g(i) \quad \textbf{otherwise} \end{cases}$
$H(\lambda)$:	$\forall i \in \mathbf{N}.\ \ell(i) := \lim_{\gamma \to \lambda} h_\gamma(i)$
	if $\text{Bad} \cap \text{Sub}\ \ell = \emptyset$ **then return** ℓ
	$h_\lambda :\in \text{Inc lex}(\ell)$

where the construct $s :\in S$ chooses an arbitrary s from S (and $s = \bot$ if $S = \emptyset$). The function $\text{lex} : \text{Bad} \to \text{Bad}$ chooses a bad sequence of subtrees (that is,

$\mathrm{lex}(h) \in \mathrm{Bad} \cap \mathrm{Sub}(h))$ with (lexicographically) minimal labels:

$\mathrm{lex}(h)$:	$K := \mathrm{Bad} \cap \mathrm{Sub}\ h$
	for $i := 1$ **to** ∞ **do**
	$\quad t :\in \arg\min\{s^\bullet(i) \mid s \in K\}$
	$\quad K := \{s \in K \mid s(i) = t(i)\}$
	$k :\in K$
	return k

where $\arg\min\{s^\bullet(i) \mid s \in K\}$ denotes the set of those $s \in K$ for which $s^\bullet(i)$ is minimal.

3.2 Termination and Correctness of the Construction

By construction we have (for all α and i):

$$Dh_\alpha \supseteq Dh_{\alpha+1} \tag{1}$$
$$h_\alpha(i) \trianglerighteq h_{\alpha+1}(i) \tag{2}$$
$$h^\bullet_\alpha(i) \precsim h^\bullet_{\alpha+1}(i) \tag{3}$$

Also by construction, for every countable *limit* ordinal α and for all $i < j \in Dh_\alpha$, we have:

$$h_\alpha(i) \not\hookrightarrow h_\alpha(j) \tag{4}$$
$$h^\bullet_\alpha(i) \precsim h^\bullet_\alpha(j) \tag{5}$$

For α a *successor* ordinal, (4,5) are proved by induction: The only interesting case is $i < \min Dg \leq j$, where we append two bad sequences together, $h_\alpha[0 : (\min Dg) - 1]$ and g. Therefore, we need to take care that no tree in the first part is embedded into some tree in the second part. We have:

$$h^\bullet_{\alpha+1}(i) = h^\bullet_\alpha(i) \precsim h^\bullet_\alpha(j) \precsim k^\bullet(j) = f^\bullet(j) = g^\bullet(j) = h^\bullet_{\alpha+1}(j)$$

from which $h_\alpha(i) \not\hookrightarrow h_\alpha(j)$ follows using (D).

By (4) we have that every sequence h_α is bad. By considering the limit case and (1) above, it can be seen also that for all $\alpha < \beta$:

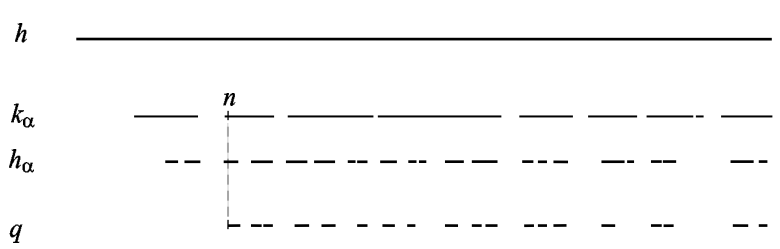

Fig. 2. The bad sequences of the proof of the main theorem. The (dotted) lines represent the domains of the sequences, which get sparser as the induction proceeds.

$$Dh_\alpha \supseteq Dh_\beta \tag{6}$$

To complete the proof of the construction, it remains only to establish three additional aspects:

Aspect 1 *The constructed sequences h_α are all infinite.*

Aspect 2 *The constructed sequences h_α are each distinct.*

Aspect 3 *The construction eventually terminates with a minimal bad sequence.*

Proof of Aspect 1. We consider the successor and limit cases separately.

Lemma 7 *For $\alpha + 1$ a successor ordinal, $|\mathrm{D}h_{\alpha+1}| = \infty$.*

PROOF. We prove that $|\mathrm{D}f| = \infty$ in the $\alpha + 1$ case. By construction this implies that $|\mathrm{D}h_{\alpha+1}| = \infty$.

Suppose, by way of contradiction, that f is finite at stage $\alpha + 1$. Let k be the bad sequence of subtrees of h_α constructed by lex at stage $\alpha + 1$; Let k_α be the one constructed at the prior step α from subtrees of some sequence h (in case $\alpha = 0$, this k_α is the output of $\operatorname{lex} h$ at the $H(0)$ stage), and let g_α be the sequence g constructed at the α stage. Let $q := k \setminus f$ be those elements of k that have *smaller* root symbols than h_α (see Fig. 2).

By supposition and condition (B), q is infinite and bad. Let $n := \min(\mathrm{D}k_\alpha \cap \mathrm{D}g_\alpha \cap \mathrm{D}q)$ and consider

$$p := k_\alpha[0 : n-1] \cup (q \restriction \mathrm{D}k_\alpha)$$

Clearly, $\mathrm{D}p \subseteq \mathrm{D}k_\alpha$. We also have that $\mathrm{D}k_\alpha \cap \mathrm{D}g_\alpha \cap \mathrm{D}q \neq \emptyset$, since by the induction hypothesis $g_\alpha \neq \emptyset$ and by construction $g_\alpha \subseteq k_\alpha$, $g_\alpha \subseteq h_\alpha$ and $\mathrm{D}q \subseteq \mathrm{D}h_\alpha$.

We show now that $p \in \mathrm{Bad} \cap \mathrm{Sub}\ h$. Since $k^\bullet_\alpha(n) = h^\bullet_\alpha(n) \succ q^\bullet(n) = p^\bullet(n)$ also holds, this would contradict the picking of $k_\alpha(n)$, rather than $p(n)$, by lex at the α stage. Hence, it remains to prove the following two claims:

Claim 8 *$p \in$ Sub h .*

PROOF. Let $i \in \mathrm{D}p$. In case $i < n$, we have $p(i) = k_\alpha(i) \lhd h(i)$, by construction of k_α.

In case $i \geq n$

$$p(i) = q(i) = k(i) \lhd h_\alpha(i) \tag{7}$$

and $h_\alpha(i) = k_\alpha(i) \lhd h(i)$ (by construction $h_\alpha(i) = g_\alpha(i)$, for each $i \in \mathrm{D}g_\alpha$). Therefore, $p(i) \lhd h(i)$ follows from $p^\bullet(i) \prec k^\bullet_\alpha(i)$ and (A). Hence $p \in \mathrm{Sub}\ h$, concluding the proof of the claim. □

Claim 9 *$p \in$ Bad .*

PROOF. In case $i < j < n$ or $n \leq i < j$, we know that $k_\alpha(i) \not\hookrightarrow q(j)$ by the induction hypothesis, and construction, respectively.

Thus, assume that $i < n \leq j$. Note that for all j, $q(j)$ is a subtree of $k_\alpha(j)$. Hence, if $k_\alpha(i) \hookrightarrow q(j)$ then, by (C), $k_\alpha(i) \hookrightarrow k_\alpha(j)$, which is in contradiction to $k_\alpha \in \mathrm{Bad}$. This concludes the proof of the claim. □

This concludes the proof of Lemma 7 □

Lemma 10 *For λ a limit ordinal, h_λ is infinite.*

PROOF. For a successor ordinal α, let g_α be the g constructed at step α and $n_\alpha := \min Dg_\alpha$. Since trees have only finitely many subtrees, and g_α is built of proper subtrees of the prior bad sequence, we have

$$\liminf_{\alpha \to \lambda} n_\alpha \to \omega \tag{8}$$

(here we used property (E)). Otherwise, if $\liminf_{\alpha \to \lambda} n_\alpha \to c$ for some $c \in \mathbf{N}$, then by the pigeonhole principle, there would have been infinitely many *proper* subtrees taken from $h(c)$ which is clearly impossible, as the trees are finite.

Now, once $n_\alpha < n_\gamma$ for all γ such that $\alpha < \gamma < \lambda$, we get $h_\gamma[n_\alpha] = g_\alpha[n_\alpha] \neq \bot$ for all such γ. By (8) this happens for infinitely many such n_α's. $\square$

Proof of Aspect 2 Distinctness follows from the construction, since, as long as f is infinite, $\min Dg$ is defined and $h_{\alpha+1} \neq h_\alpha$.

Proof of Aspect 3 Termination follows from distinctness by a cardinality argument: There are only countably many sequences h_α, each corresponding to the pair $\langle i, j\rangle$, for the jth time a proper subtree is taken (by lex) in the ith index position.

3.3 Proof of Main Theorem

PROOF. Assume by way of contradiction that Bad $\neq \emptyset$. We saw in the previous subsection that there is a minimal bad sequence $h \in \text{Bad}$ such that $\text{Bad} \cap \text{Sub}\, h = \emptyset$.

We say that a tree s is an *immediate subtree* of t if s is a subtree of t rooted by an immediate child of $t^\bullet$. Recall that, by Definition 4, if u is an immediate subtree of t then $u \trianglelefteq t$. Let S be the set of all immediate subtrees of trees in h; then we have $S \in \text{Sub}\, h$. Since the labels are taken from a wqo set, there can be at most finitely many trees of only one vertex in h; therefore S is infinite.

For a tree $t \in \mathcal{T}$, denote by $\langle t_1, \ldots, t_n\rangle$ the finite ordered sequence consisting of its immediate subtrees, in the order they occur as children of $t^\bullet$; by $t^\bullet\langle t_1, \ldots, t_n\rangle$, we denote t itself.

Now, S must be a wqo under gap embedding, or else there would be a bad infinite sequence $\sigma \subseteq S$: Since, for each tree in h, the number of children of the root is finite, we can assume that σ contains at *most* one subtree for each tree in h. Therefore, $\sigma \in \text{Bad} \cap \text{Sub}\, h$, in contradiction to the construction of h.

So, S is a wqo. Let $(s_i)_{i \in \mathrm{D}h}$ be the infinite sequence defined by:

$$\forall i \in \mathrm{D}h.\; s_i := \langle h(i)_1, \ldots, h(i)_{n_i}\rangle$$

where n_i is the number of children of $h^\bullet(i)$. Since S is a wqo, by Higman's Lemma [6], $(s_i)_{i \in \mathrm{D}h}$ is a good sequence with respect to the embedding relation on finite sequences of trees from $\mathcal{T}$ defined by:

$$
\begin{aligned}
&\langle s_1,\ldots,s_n\rangle \hookrightarrow \langle t_1,\ldots,t_m\rangle \ \text{ if} \\
&\quad \exists\iota:\{1,\ldots,n\} \to \{1,\ldots,m\}\,.\ \iota \text{ is strictly monotone} \wedge \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall j\,(1\le j\le n).\, s_j \hookrightarrow t_{\iota(j)}
\end{aligned}
$$

Therefore, as h is *increasing*, there exists a pair of trees s, t in h, such that s precedes t and $s = s^{\bullet}\langle s_1,\ldots,s_n\rangle \hookrightarrow t^{\bullet}\langle t_1,\ldots,t_m\rangle = t$, where the root is mapped to the root and the immediate subtrees of s are embedded in those of t, according to Higman's sequence embedding. Note that this embedding is actually a gap embedding (the fact that ι is strictly monotone is required so that the order of children denoted by $\sqsubset$ is preserved in the embedding). Thus, we arrive at a contradiction to the badness of h. This concludes the proof of Theorem 6. □

4 Comparable Subpaths

The condition that each node in a path is comparable with all its ancestors can be relaxed by allowing each path to be partitioned into a bounded number of comparable subpaths.

Definition 11 *A tree subpath from* $[u:v]$ *is* comparable *if its vertices have comparable labels, that is, for all nodes* $x, y \in [u:v]$ *either* $x \precsim y$ *or* $y \precsim x$*.*

Note that subpaths in the definition can begin and end in *internals* nodes. We have the following:

Theorem 12 *Let* $\mathcal{T}_n$ *be a set of finite trees with well-quasi-ordered nodes such that each path can be partitioned into* $n \in \mathbf{N}$ *or fewer comparable subpaths. Then* $\mathcal{T}_n$ *is a wqo under gap embedding.*

PROOF. Let us first slightly change the gap embedding relation $\hookrightarrow$ to allow trees to have leaves labelled by a possibly *distinct* node ordering: For two trees, the *gap embedding* of s into t is defined the same as before *except for leaves* (for which the gap condition is not applicable). That is, if $\langle u, v\rangle$ is an edge of s and v is a leaf, then we require that v be mapped to a node with greater or equivalent node, which could only be a leaf of t, since the leaf ordering is disjoint from that of internal nodes (by disjoint orderings we mean that the set of labels are disjoint). No additional condition on the path from $\iota(u)$ to $\iota(v)$ is required. For internal edges of s the conditions remain the same.

We prove Theorem 12 in two steps:

i. We show that putting an arbitrary well-quasi-ordering on leaves from $\mathcal{T}$

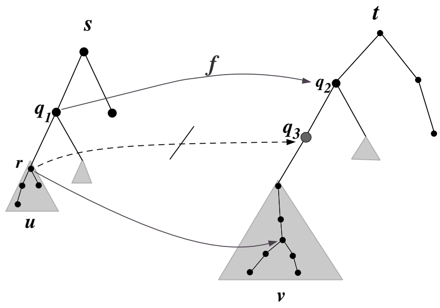

Fig. 3. u and v are two trees such that u is gap embedded in v. If we consider u and v as labels of leaves in s and t, respectively, then the mapping f is a gap embedding of s in t. However, if we 'unfold' u and v, that is, we consider u and v to be subtrees of s and t, respectively, then any extension of f, that includes also the nodes of u and v, is not a gap embedding of s in t, since q_3 and r are incomparable.

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maintains the wqo property of $\mathcal{T}$ with respect to gap-embedding.

ii. Since we can also put *trees* as labels of leaves, we can choose to label the leaves of $\mathcal{T}$ by some set of trees with nodes well-quasi-ordered by some ordering possibly disjoint from that of $\mathcal{T}$. Hence, if we could "unfold" the leaves of $\mathcal{T}$ into subtrees and still keep the set of trees well-quasi-ordered under gap embedding then, by induction on n, Theorem 12 would follow.

Step (i) stems easily from the proof of the main theorem: As before, we need a *minimal bad sequence* theorem for the set of trees; but now, we have distinct node orderings for internal nodes and leaves. The proof is identical, since the leaf ordering is a wqo, so, in every induction stage of the construction, there can only be finitely many single-node trees (that is, just leaves), and they are skipped when building f.

Step (ii) consists of showing that using a set of well-quasi-ordered trees to label the leaves yields again a wqo with respect to the original definition of gap embedding, even when we unfold these leaves to form a set of trees such that each path can be partitioned into *two* comparable subpaths. Note that if we have two trees s, t with all internal nodes comparable to their ancestor nodes, and leaves labelled by some set of *comparable paths trees*, such that s is embedded in t according to the relaxed definition above, then unfolding the leaves of s and t would *not* necessarily yield that the resulting trees have a gap embedding such that *all* the nodes preserve the gap conditions. The reason is that we did not require leaves to have a gap condition in the relaxed gap embedding (as they contain labels from a disjoint ordering, this would be impossible). See Fig. 3 for an example.

Therefore, the second step is achieved by *forcing the embedding to map each terminal edge to a terminal edge.* (This ensures that leaves trivially preserve the gap conditions.) We do this simply by introducing a new node as a parent of each leaf, labelled with a new maximum element, $\top$, of the internal node ordering. Since the new maximum element is comparable to all elements of the internal node ordering, the minimal bad sequence theorem of the previous paragraph applies to the resulting set of trees.

Now, any embedding of two trees from this set of trees ought to map a terminal edge to a terminal edge, since their new parent nodes, $\top$, can only map to each other. Therefore, by the above explanation, Theorem 12 follows. $\square$

5 Conclusion

We have showed how to extend the well-quasi-ordering of finite trees under gap embedding, as developed by Kruskal, Friedman and Kříž, to arbitrary well-quasi-ordered labels. This was achieved by requiring that each node is comparable to its ancestors. Then, we extended the class of trees, by induction, to the case where each path can be partitioned into a bounded number of comparable subpaths.

As noted earlier, a simple counterexample shows that if the paths of trees in $\mathcal{T}$ do not necessarily contain comparable nodes then our Main Theorem might fail, provided the number of comparable subpaths is unbounded. This can happen even for strings: Let a, b, c be three incomparable elements of the node ordering. The following is an antichain with respect to gap embedding:

c-a-c   c-b-a-c   c-a-b-a-c   c-b-a-b-a-c   ...

Consequently, Theorem 12 shows that the above counterexample is *canonical*: every bad sequence with respect to gap embedding must contain paths of unbounded incomparability.

References

[1] N. Dershowitz. Orderings for term-rewriting systems. *Theor. Comput. Sci.*, 17:279–301, 1982.

[2] H. Friedman. Beyond Kruskal's Theorem I-III. Unpublished reports, Ohio State University, Columbus, OH, June–July 1982.

[3] L. Gordeev. Generalizations of the one-dimensional version of the Kruskal-Friedman theorems. *Journal of Symbolic Logic*, 54(1):101–121, 1989.

[4] L. Gordeev. Generalizations of the Kruskal-Friedman theorems. *Journal of Symbolic Logic*, 55(1):157–181, 1990.

[5] L. Gordeev. Quasi-Ordinals and Proof Theory. In N. Robertson, P. Seymour, editors, *Graph Structure Theory*, pages 485–494. AMS, 1993.

[6] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Mathematical Society*, 2:326–336, 1952.

[7] I. Kříž. Well-quasiordering finite trees with gap-condition. Proof of Harvey Friedman's conjecture. *Ann. of Math.*, 130:215–226, 1989.

[8] J. B. Kruskal. Well-quasiordering, the Tree Theorem, and Vazsonyi's conjecture. *Trans. American Mathematical Society*, 95:210–223, 1960.

[9] E. C. Milner. Basic wqo- and bqo-theory. In *Graphs and order (Banff, Alta., 1984)*, volume 147 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 487–502. Reidel, Dordrecht, 1985.

[10] C. St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, 1963.

[11] C. St. J. A. Nash-Williams. On well-quasi-ordering trees. In: *A Seminar on Graph Theory*, F. Harary, ed., pages 79–82. Holt, Rinehart and Winston, New York, 1967.

[12] M. Okada and G. Takeuti. On the theory of quasi-ordinal diagrams. In: *Logic and Combinatorics* (Arcata, CA, 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 295–308.

[13] S. G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In L. A. Harrington, ed., *Harvey Friedman's Research on the Foundation of Mathematics*, pages 87–117. Elsevier, 1985.

[14] A. Weiermann. Proving termination for term rewriting systems. In *Computer Science Logic* (Berne, 1991), volume 626 of *Lecture Notes in Comput. Sci.*, pages 419–428. Springer, Berlin, 1992.