Canonical Inference for Implicational Systems*

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Abstract. *Completion* is a general paradigm for applying inferences to generate a canonical presentation of a logical theory, or to semi-decide the validity of theorems, or to answer queries. We investigate what *canonicity* means for *implicational systems* that are axiomatizations of *Moore families* – or, equivalently, of propositional *Horn theories*. We build a correspondence between implicational systems and associative-commutative rewrite systems, give deduction mechanisms for both, and show how their respective inferences correspond. Thus, we exhibit completion procedures designed to generate canonical systems that are "optimal" for forward chaining, to compute minimal models, and to generate canonical systems that are *rewrite-optimal*. Rewrite-optimality is a new notion of "optimality" for implicational systems, one that takes contraction by simplification into account.

1 Introduction

Knowledge compilation is the transformation of a knowledge base into a *canonical form* that makes efficient reasoning possible (e.g., [13, 8, 11]). In automated reasoning the knowledge base is often the "presentation" of a theory, where we use "presentation" to mean a set of formulæ, reserving "theory" for a presentation with all its theorems. From the perspective taken here, canonicity of a presentation depends on the availability of the best proofs, or *normal-form proofs*. Proofs are measured by *proof orderings*, and the most desirable are the minimal proofs. Since a minimal proof in a certain presentation may not be minimal in a larger presentation, normal-form proofs are the minimal proofs in the largest presentation, that is, a deductively-closed presentation. However, what is a deductively-closed presentation depends on the choice of *deduction mechanism*. Thus, the choices of normal form and deduction mechanism are intertwined.

An archetypal instance of knowledge compilation is *completion* of *equational theories*, where normal-form proofs are *valley proofs*: a given presentation E

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is transformed into an equivalent, *canonical* presentation $E^\sharp$, such that for all theorems $\forall\bar{x}\ u \simeq v$, $E^\sharp$ offers a *valley proof* of $\widetilde{u} \simeq \widetilde{v}$, where $\widetilde{u}$ and $\widetilde{v}$ are u and v with their variables $\bar{x}$ replaced by Skolem constants. If $E^\sharp$ is *finite*, it serves as *decision procedure*, because validity can be decided by "blind" rewriting. Otherwise, completion semi-decides validity by working refutationally on E and $\widetilde{u} \not\simeq \widetilde{v}$ (see, e.g., [1, 6], also for more references). More generally, the notion of *canonicity* can be articulated into three properties of increasing strength (e.g., [4]): a presentation is *complete*, if it affords a normal-form proof for each theorem, *saturated*, if it supports all normal-form proofs for all theorems,[3] and *canonical*, if it is both saturated and *contracted*, that is, it contains no redundancies.

This paper studies canonicity for implicational systems. An *implicational system* is a set of implications, whose family of models is a *Moore family*, meaning that it is closed under intersection (see [3, 2]). A Moore family defines a *closure operator* that associates with any set the least element of the Moore family that includes it. Moore families, closure operators and implicational systems have played a rôle in a variety of fields in computer science, including relational databases, data mining, artificial intelligence, logic programming, lattice theory and abstract interpretations. We refer to [7] and [2] for surveys, including applications, related formalisms and historical notes.

An implicational systems can be regarded as a Horn presentation of its Moore family. Since a Moore family may be presented by different implicational systems, it makes sense to define and generate implicational systems that are "optimal", or "minimal", or "canonical" in a suitable sense, and allow one to compute their associated closure operator efficiently. Bertet and Nebut [3] proposed the notions of *directness* of implicational systems, optimizing computation by forward chaining, and *direct-optimality* of implicational systems, which adds an optimization step based on a symbol count. Bertet and Monjardet [2] considered other candidates and proved them all equal to direct-optimality, which, therefore, earned the appellation *canonical-directness*.

We investigate correspondences between "optimal" implicational systems (direct, direct-optimal) and canonical rewrite systems. This requires us to establish an equivalence between implicational systems and associative-commutative rewrite systems, and to define and compare their respective deduction mechanisms. The rewriting framework turns out to be more flexible, because it allows one to compute during saturation the image of a given set according to the closure operator associated with the implicational system. Computing the closure amounts to generating minimal models, which may have practical applications. Comparisons of presentations and inferences are complemented at a deeper level by comparisons of the underlying proof orderings. We observe that direct-optimality can be simulated by normalization with respect to a different proof ordering than the one assumed by rewriting, and this discrepancy leads us to introduce a new notion of *rewrite-optimality*. Thus, while directness corresponds to saturation in an expansion-oriented deduction mechanism, rewrite-optimality corresponds to canonicity.

[3] If minimal proofs are unique, *complete* and *saturated* coincide.

2 Background

Let V be a vocabulary of propositional variables. For $a \in V$, a and $\neg a$ are positive and negative literals, respectively; a *clause* is a disjunction of literals, that is *positive* (*negative*), if all its literals are, and *unit*, if it is made of a single literal. A *Horn clause* has at most one positive literal, so positive unit clauses and purely negative clauses are special cases. A *Horn presentation* is a set of non-negative Horn clauses. It is customary to write a Horn clause $\neg a_1 \vee \cdots \vee \neg a_n \vee c$, $n \geq 0$, as the implication or *rule* $a_1 \cdots a_n \Rightarrow c$. A Horn clause is *trivial* if the *conclusion* c is the same as one of the *premises* a_i.

An *implicational system* (see, e.g., [3, 2]) S is a binary relation $S \subseteq \mathcal{P}(V) \times \mathcal{P}(V)$, read as a set of implications $a_1 \cdots a_n \Rightarrow c_1 \cdots c_m$, for $a_i, c_j \in V$, with both sides understood as conjunctions of distinct propositions. Using upper case Latin letters for sets, such an implication is written $A \Rightarrow_S B$ to specify that $A \Rightarrow B \in S$. If all right-hand sides are singletons, S is a *unary implicational system*. Clearly, any non-negative Horn clause is such a unary implication and vice-versa, and any non-unary implication can be decomposed into m unary implications, one for each c_i.

Since an implication $a_1 \cdots a_n \Rightarrow c_1 \cdots c_m$ is equivalent to the bi-implication $a_1 \cdots a_n c_1 \cdots c_m \Leftrightarrow a_1 \cdots a_n$, again with both sides understood as conjunctions, it can also be translated into a rewrite rule $a_1 \cdots a_n c_1 \cdots c_m \;\rightarrow\; a_1 \cdots a_n$, where juxtaposition stands for associative-commutative-idempotent conjunction, and the arrow $\rightarrow$ signifies logical *equivalence* (see, e.g., [9, 5]). A positive literal c is translated into a rule $c \rightarrow \mathit{true}$, where *true* is a new constant. We will be making use of a well-founded ordering $\succ$ on $V \cup \{\mathit{true}\}$, wherein *true* is minimal. Conjunctions of propositions are compared by the multiset extension of $\succ$, also denoted $\succ$, so that $a_1 \ldots a_n c_1 \ldots c_m \succ a_1 \ldots a_n$. A rewrite rule $P \rightarrow Q$ is measured by the multiset $\{\!\{P, Q\}\!\}$, and these measures are ordered by a second multiset extension of $\succ$, written $\succ_L$ to avoid confusion. Sets of rewrite rules are measured by multisets of multisets (e.g., containing $\{\!\{P, Q\}\!\}$ for each $P \rightarrow Q$ in the set) compared by the multiset extension of $\succ_L$, denoted $\succ_C$.

A subset $X \subseteq V$ represents the propositional interpretation that assigns *true* to all elements in X. Accordingly, X is said to *satisfy* an implication $A \Rightarrow B$ if either $B \subseteq X$ or else $A \not\subseteq X$. Similarly, we say that X *satisfies* an implicational system S, or is a *model* of S, denoted $X \models S$, if X satisfies all implications in S.

A *Moore family* on V is a family $\mathcal{F}$ of subsets of V that contains V and is closed under intersection. Moore families are in one-to-one correspondence with closure operators, where a *closure operator* on V is an operator $\varphi\colon \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ that is: (i) *monotone*: $X \subseteq X'$ implies $\varphi(X) \subseteq \varphi(X')$; (ii) *extensive*: $X \subseteq \varphi(X)$; and (iii) *idempotent*: $\varphi(\varphi(X)) = \varphi(X)$. The Moore family $\mathcal{F}_\varphi$ associated with a given closure operator φ is the set of all fixed points of φ:

$$\mathcal{F}_\varphi \quad \stackrel{!}{=} \quad \{X \subseteq V : X = \varphi(X)\} \ ,$$

using $\stackrel{!}{=}$ for definitions. The closure operator $\varphi_{\mathcal{F}}$ associated with a given Moore family $\mathcal{F}$ maps any $X \subseteq V$ to the least element of $\mathcal{F}$ that contains X:

$$\varphi_{\mathcal{F}}(X) \quad \stackrel{!}{=} \quad \bigcap\{Y \in \mathcal{F} \colon X \subseteq Y\} \; .$$

The Moore family $\mathcal{F}_S$ associated with a given implicational system S is the family of the *propositional models* of S, in the sense given above:

$$\mathcal{F}_S \quad \stackrel{!}{=} \quad \{X \subseteq V \colon X \models S\} \; .$$

Two implicational systems S and S' that have the same Moore family, $\mathcal{F}_S = \mathcal{F}_{S'}$, are *equivalent*. Combining the notions of closure operator for a Moore family, and Moore family associated with an implicational system, the closure operator φ_S for implicational system S maps any $X \subseteq V$ to the least model of S that satisfies X [3]:

$$\varphi_S(X) \quad \stackrel{!}{=} \quad \bigcap\{Y \subseteq V \colon Y \supseteq X \wedge Y \models S\} \; .$$

Example 1. Let S be $\{a \Rightarrow b,\ ac \Rightarrow d,\ e \Rightarrow a\}$. The Moore family $\mathcal{F}_S$ is $\{\emptyset, b, c, d, ab, bc, bd, cd, abd, abe, bcd, abcd, abde, abcde\}$, and $\varphi_S(ae) = abe$. □

The obvious syntactic correspondence between Horn presentations and implicational systems is matched by a semantic correspondence between Horn theories and Moore families, since Horn theories are those theories whose models are closed under intersection, a fact observed first by McKinsey [12].

A *(one-step) deduction mechanism* $\rightsquigarrow$ is a binary relation over presentations. A deduction step $Q \rightsquigarrow Q \cup Q'$ is an *expansion* provided $Q' \subseteq \mathit{Th}\, Q$, where $\mathit{Th}\, Q$ is the set of theorems of Q. A deduction step $Q \cup Q' \rightsquigarrow Q$ is a *contraction* provided $Q \cup Q' \succsim Q$, which means $\mathit{Th}\, Q \cup Q' = \mathit{Th}\, Q$, and for all theorems, Q offers a proof that is smaller or equal than that in $Q \cup Q'$ in a well-founded proof ordering. A sequence $Q_0 \rightsquigarrow Q_1 \rightsquigarrow \cdots$ is a *derivation*, whose result, or *limit*, is the set of *persisting* formulæ: $Q_\infty \stackrel{!}{=} \cup_j \cap_{i \geq j} Q_i$. A fundamental requirement of derivations is *fairness*, doing all inferences that are needed to achieve the desired degree of proof normalization. A *fair* derivation generates a complete limit and a *uniformly fair* derivation generates a saturated limit (see [4] for more details).

3 Direct Systems

A *direct* implicational system allows one to compute $\varphi_S(X)$ in one single round of *forward chaining*:

Definition 1 (Directness [3, Def. 1]). *An implicational system S is* direct *if $\varphi_S(X) = S(X)$, where $S(X) \stackrel{!}{=} X \cup \bigcup\{B \colon A \Rightarrow_S B \wedge A \subseteq X\}$.*

In general, $\varphi_S(X) = S^*(X)$, where

$$S^0(X) = X, \qquad S^{i+1}(X) = S(S^i(X)), \qquad S^*(X) = \bigcup\nolimits_i S^i(X).$$

Since S, X and V are all finite, $S^*(X) = S^k(X)$ for the smallest k such that $S^{k+1}(X) = S^k(X)$.

Example 2. The implicational system $S = \{ac \Rightarrow d, e \Rightarrow a\}$ is not direct. Indeed, for $X = ce$, the computation of $\varphi_S(X) = \{acde\}$ requires two rounds of forward chaining, because only after a has been added by $e \Rightarrow a$, can d be added by $ac \Rightarrow d$. That is, $S(X) = \{ace\}$ and $\varphi_S(X) = S^2(X) = S^*(X) = \{acde\}$. $\square$

Generalizing this example, it is sufficient to have two implications $A \Rightarrow_S B$ and $C \Rightarrow_S D$ such that $A \subseteq X$ but $C \nsubseteq X$ for $\varphi_S(X)$ to require more than one iteration of forward chaining. If $A \subseteq X$, $A \Rightarrow_S B$ adds B in the first round. If, additionally, $C \subseteq X \cup B$, then $C \Rightarrow_S D$ adds D in a second round. In the above example, $A \Rightarrow B$ is $e \Rightarrow a$ and $C \Rightarrow D$ is $ac \Rightarrow d$. The conditions $A \subseteq X$ and $C \subseteq X \cup B$ are equivalent to $A \cup (C \setminus B) \subseteq X$, because $C \subseteq X \cup B$ means that whatever is in C and not in B must be in X. Thus, to collapse the two iterations of forward chaining into one, it is sufficient to add the implication $A \cup (C \setminus B) \Rightarrow_S D$. In the example $A \cup (C \setminus B) \Rightarrow_S D$ is $ce \Rightarrow d$. This mechanism can be defined in more abstract terms as the following inference rule:

Implicational overlap

$$\frac{A \Rightarrow BO \quad CO \Rightarrow D}{AC \Rightarrow D} \quad B \cap C = \emptyset \neq O$$

One inference step of this rule will be denoted $\vdash_I$. The condition $O \neq \emptyset$ is included, because otherwise $AC \Rightarrow D$ is subsumed by $C \Rightarrow D$. Also, if $B \cap C$ is not empty, then an alternate inference is more general. Thus, directness can be characterized as follows:

Definition 2 (Generated direct system [3, Def. 4]). *Given an implicational system S, the* direct *implicational system $I(S)$ generated from S is the smallest implicational system containing S and closed with respect to implicational overlap.*

A main theorem of [3] shows that indeed $\varphi_S(X) = I(S)(X)$. Let $\rightsquigarrow_I$ be the deduction mechanism that generates and adds implications by implicational overlap: clearly, $\rightsquigarrow_I$ steps are expansion steps. Thus, we can rephrase Definition 2:

Definition 3 (Generated direct system). *Given an implicational system S, the* direct *implicational system $I(S)$ generated from S is the limit S_∞ of a fair derivation $S = S_0 \rightsquigarrow_I S_1 \rightsquigarrow_I \cdots$.*

By applying the translation of implications into rewrite rules (cf. Sect. 2), we define:

Definition 4 (Associated rewrite system). *Given $X \subseteq V$, its* associated rewrite system *is $R_X \stackrel{!}{=} \{x \rightarrow true : x \in X\}$. For an implicational system S, its* associated rewrite system *is $R_S \stackrel{!}{=} \{AB \rightarrow A : A \Rightarrow_S B\}$. Given S and X we can also form the rewrite system $R^S_X \stackrel{!}{=} R_X \cup R_S$.*

Example 3. If $S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$, then $R_S = \{ab \to a, acd \to ac, ae \to e\}$. If $X = ae$, then $R_X = \{a \to \mathit{true}, e \to \mathit{true}\}$. Thus, $R_X^S = \{a \to \mathit{true}, e \to \mathit{true}, ab \to a, acd \to ac, ae \to e\}$. □

We show that there is a correspondence between implicational overlap and the classical notion of overlap between monomials in Boolean rewriting (e.g.,[5]):

Equational overlap

$$\frac{AO \to B \quad CO \to D}{M \to N} \quad A \cap C = \emptyset \neq O,\ M \succ N$$

where M and N are the normal-forms of BC and AD with respect to $\{AO \to B, CO \to D\}$. One inference step of this rule will be denoted $\vdash_E$.

Equational overlap combines expansion, the generation of $BC \leftrightarrow AD$, with contraction – its normalization to $M \to N$. This sort of contraction applied to normalize a newly generated formula, is called *forward contraction*. The contraction applied to reduce an already established equation is called *backward contraction*. Let $\leadsto_E$ be the deduction mechanism of equational overlap: then, $\leadsto_E$ features expansion and forward contraction.

Example 4. For $S = \{ac \Rightarrow d, e \Rightarrow a\}$ as in Ex. 2, we have $R_S = \{acd \to ac, ae \to e\}$, and the overlap of the two rewrite rules gives $ace \leftarrow acde \to cde$. Since $ace \to ce$, equational overlap yields the rewrite rule $cde \to ce$, which corresponds to the implication $ce \Rightarrow d$ generated by implicational overlap. □

Since it is designed to produce a direct system, implicational overlap "unfolds" the forward chaining in the implicational system. Since forward chaining is complete for Horn logic, it is coherent to expect that the only non-trivial equational overlaps are those corresponding to implicational overlaps:

Lemma 1. *If $A \Rightarrow B$ and $C \Rightarrow D$ are two non-trivial Horn clauses ($|B| = |D| = 1$, $B \not\subseteq A$, $D \not\subseteq C$), and $A \Rightarrow B, C \Rightarrow D \vdash_I E \Rightarrow D$ by implicational overlap, then $AB \to A, CD \to C \vdash_E DE \to E$ by equational overlap, and vice-versa. Furthermore, all other equational overlaps are trivial.*

Proof. (If direction.) Assume $A \Rightarrow B, C \Rightarrow D \vdash_I E \Rightarrow D$. Since B is a singleton by hypothesis, it must be $C = BF$, or the consequent of the first implication and the antecedent of the second one overlap on B. Thus, $C \Rightarrow D$ is $BF \Rightarrow D$ and the implicational overlap of $A \Rightarrow B$ and $BF \Rightarrow D$ generates $AF \Rightarrow D$. The corresponding rewrite rules are $AB \to A$ and $BFD \to BF$, that also overlap on B yielding the equational overlap

$$AFD \leftarrow ABFD \to ABF \to AF\ ,$$

which generates the corresponding rule $AFD \to AF$.
(Only if direction.) If $AB \to A, CD \to C \vdash_E DE \to E$, the rewrite rules $AB \to A$ and $CD \to C$ can overlap in four ways: $B \cap C \neq \emptyset$, $A \cap D \neq \emptyset$, $A \cap C \neq \emptyset$ and $B \cap D \neq \emptyset$, which we consider in order.

1. $B \cap C \neq \emptyset$: Since B is a singleton, it must be $B \cap C = B$ or $C = BF$ for some F. Thus, $CD \to C$ is $BFD \to BF$, and the overlap of $AB \to A$ and $BFD \to BF$ is the same as above, yielding $AFD \to AF$. The corresponding implications $A \Rightarrow B$ and $BF \Rightarrow D$ generate $AF \Rightarrow D$ by implicational overlap.
2. $A \cap D \neq \emptyset$: This case is symmetric to the previous one.
3. $A \cap C \neq \emptyset$: Let $A = FO$ and $C = OG$, so that the rules are $FOB \to FO$ and $OGD \to OG$, with $O \neq \emptyset$ and $F \cap G = \emptyset$. The resulting equational overlap is trivial: $FOG \leftarrow FOGD \leftarrow FBOGD \to FBOG \to FOG$.
4. $B \cap D \neq \emptyset$: Since B and D are singletons, it must be $B \cap D = B = D$, and rules $AB \to A$ and $CB \to C$ produce the trivial overlap $AC \leftarrow ABC \to AC$. □

Lemma 1 yields the following correspondence between deduction mechanisms:

Lemma 2. *For all implicational systems S, $S \leadsto_I S'$ if and only if $R_S \leadsto_E R_{S'}$.*

Proof. If $S \leadsto_I S'$ then $R_S \leadsto_E R_{S'}$ follows from the if direction of Lemma 1. If $R_S \leadsto_E R'$ then $S \leadsto_I S'$ and $R' = R_{S'}$ follows from the only-if direction of Lemma 1. □

The next theorem shows that for fair derivations the process of completing S with respect to implicational overlap, and turning the result into a rewrite system, commutes with the process of translating S into the rewrite system R_S, and then completing it with respect to equational overlap.

Theorem 1. *For every implicational system S, and for all fair derivations $S = S_0 \leadsto_I S_1 \leadsto_I \cdots$ and $R_S = R_0 \leadsto_E R_1 \leadsto_E \cdots$, we have*

$$R_{(S_\infty)} = (R_S)_\infty \ .$$

Proof.

(a) $R_{(S_\infty)} \subseteq (R_S)_\infty$: for any $AB \to A \in R_{(S_\infty)}$, $A \Rightarrow B \in S_\infty$ by Definition 4; then $A \Rightarrow B \in S_j$ for some $j \geq 0$. Let j be the smallest such index. If $j = 0$, or $S_j = S$, $AB \to A \in R_S$ by Definition 4, and $AB \to A \in (R_S)_\infty$, because $\leadsto_E$ features no backward contraction. If $j > 0$, $A \Rightarrow B$ is generated at stage j by implicational overlap. By Lemma 2 and by fairness of $R_0 \leadsto_E R_1 \leadsto_E \cdots$, $AB \to A \in R_k$ for some $k > 0$. Then $AB \to A \in (R_S)_\infty$, since $\leadsto_E$ features no backward contraction.

(b) $(R_S)_\infty \subseteq R_{(S_\infty)}$: for any $AB \to A \in (R_S)_\infty$, $AB \to A \in R_j$ for some $j \geq 0$. Let j be the smallest such index. If $j = 0$, or $R_j = R_S$, $A \Rightarrow B \in S$ by Definition 4, and $A \Rightarrow B \in S_\infty$, because $\leadsto_I$ features no backward contraction. Hence $AB \to A \in R_{(S_\infty)}$. If $j > 0$, $AB \to A$ is generated at stage j by equational overlap. By Lemma 2 and by fairness of $S_0 \leadsto_I S_1 \leadsto_I \cdots$, $A \Rightarrow B \in S_k$ for some $k > 0$. Then $A \Rightarrow B \in S_\infty$, since $\leadsto_I$ features no backward contraction, and $AB \to A \in R_{(S_\infty)}$ by Definition 4. □

Since the limit of the $\leadsto_I$-derivation is $I(S)$, it follows that:

Corollary 1. *For every implicational system S, and for all fair derivations $S = S_0 \rightsquigarrow_I S_1 \rightsquigarrow_I \cdots$ and $R_S = R_0 \rightsquigarrow_E R_1 \rightsquigarrow_E \cdots$, we have*

$$R_{(I(S))} = (R_S)_\infty \, .$$

4 Computing Minimal Models

The motivation for generating $I(S)$ from S is to be able to compute, for any subset $X \subseteq V$, its minimal S-model $\varphi_S(X)$ in one round of forward chaining. In other words, one envisions a two-stage process: in the first stage, S is saturated with respect to implicational overlap to generate $I(S)$; in the second stage, forward chaining is applied to $I(S) \cup X$ to generate $\varphi_{I(S)}(X) = \varphi_S(X)$. In the rewrite-based framework, these two stages can be replaced by one. For any $X \subseteq V$ we can compute $\varphi_S(X) = \varphi_{I(S)}(X)$, by giving the rewrite system R^S_X as input to completion and extracting rules of the form $x \rightarrow \mathit{true}$. For this purpose, the deduction mechanism is enriched with contraction rules, for which we employ a double inference line:

Simplification

$$\frac{AC \rightarrow B \quad C \rightarrow D}{\overline{AD \rightarrow B \quad C \rightarrow D}}\; AD \succ B \qquad \frac{AC \rightarrow B \quad C \rightarrow D}{\overline{B \rightarrow AD \quad C \rightarrow D}}\; B \succ AD$$

$$\frac{B \rightarrow AC \quad C \rightarrow D}{\overline{B \rightarrow AD \quad C \rightarrow D}}\,,$$

where A can be empty, and

Deletion

$$\frac{A \leftrightarrow A}{=\!=\!=}\,,$$

which eliminates trivial equivalences.

Let $\rightsquigarrow_R$ denote the deduction mechanism that extends $\rightsquigarrow_E$ with simplification and deletion. Thus, in addition to the simplification applied as forward contraction within equational overlap, there is simplification applied as backward contraction to any rule. The following theorem shows that the completion of R^S_X with respect to $\rightsquigarrow_R$ generates a limit that includes the least S-model of X:

Theorem 2. *For all $X \subseteq V$, implicational systems S, and fair derivations $R^S_X = R_0 \rightsquigarrow_R R_1 \rightsquigarrow_R \cdots$, if $Y = \varphi_S(X) = \varphi_{I(S)}(X)$, then*

$$R_Y \subseteq (R^S_X)_\infty \, .$$

Proof. By Definition 4, $R_Y = \{x \rightarrow \mathit{true}\colon x \in Y\}$. The proof is by induction on the construction of $Y = \varphi_S(X)$.
Base case: If $x \in Y$ because $x \in X$, then $x \rightarrow \mathit{true} \in R_X$, $x \rightarrow \mathit{true} \in R_X^S$ and $x \rightarrow \mathit{true} \in (R_X^S)_\infty$, since a rule in the form $x \rightarrow \mathit{true}$ is irreducible by simplification.
Inductive case: If $x \in Y$ because for some $A \Rightarrow_S B$, $B = x$ and $A \subseteq Y$, then $AB \rightarrow A \in R_S$ and $AB \rightarrow A \in R_X^S$. By the induction hypothesis, $A \subseteq Y$ implies that, for all $z \in A$, $z \in Y$ and $z \rightarrow \mathit{true} \in (R_X^S)_\infty$. Let $j > 0$ be the smallest index in the derivation $R_0 \rightsquigarrow_E R_1 \rightsquigarrow_E \cdots$ such that for all $z \in A$, $z \rightarrow \mathit{true} \in R_j$. Then there is an $i > j$ such that $x \rightarrow \mathit{true} \in R_i$, because the rules $z \rightarrow \mathit{true}$ simplify $AB \rightarrow A$ to $x \rightarrow \mathit{true}$. It follows that $x \rightarrow \mathit{true} \in (R_X^S)_\infty$, since a rule in the form $x \rightarrow \mathit{true}$ is irreducible by simplification. $\square$

Then, the least S-model of X can be extracted from the saturated set:

Corollary 2. *For all $X \subseteq V$, implicational systems S, and fair derivations $R_X^S = R_0 \rightsquigarrow_R R_1 \rightsquigarrow_R \cdots$, if $Y = \varphi_S(X) = \varphi_{I(S)}(X)$, then*

$$R_Y = \{x \rightarrow \mathit{true}\colon x \rightarrow \mathit{true} \in (R_X^S)_\infty\} \ .$$

Proof. If $x \rightarrow \mathit{true} \in (R_X^S)_\infty$, then $x \in R_Y$ by the soundness of equational overlap and simplification. The other direction was established in Th. 2. $\square$

Example 5. Let $S = \{ac \Rightarrow d, e \Rightarrow a, bd \Rightarrow f\}$ and $X = ce$. Then $Y = \varphi_S(X) = acde$, and $R_Y = \{a \rightarrow \mathit{true}, c \rightarrow \mathit{true}, d \rightarrow \mathit{true}, e \rightarrow \mathit{true}\}$. On the other hand, for $R_S = \{acd \rightarrow ac, ae \rightarrow e, bdf \rightarrow bd\}$ and $R_X = \{c \rightarrow \mathit{true}, e \rightarrow \mathit{true}\}$, completion gives $\left(R_X^S\right)_\infty \{c \rightarrow \mathit{true}, e \rightarrow \mathit{true}, a \rightarrow \mathit{true}, d \rightarrow \mathit{true}, bf \rightarrow b\}$, where $a \rightarrow \mathit{true}$ is generated by simplification of $ae \rightarrow e$ with respect to $e \rightarrow \mathit{true}$, $d \rightarrow \mathit{true}$ is generated by simplification of $acd \rightarrow ac$ with respect to $c \rightarrow \mathit{true}$ and $a \rightarrow \mathit{true}$, and $bf \rightarrow b$ is generated by simplification of $bdf \rightarrow bd$ with respect to $d \rightarrow \mathit{true}$. So, $\left(R_X^S\right)_\infty$ includes R_Y, which is made exactly of the rules in the form $x \rightarrow \mathit{true}$ of $\left(R_X^S\right)_\infty$. The direct system $I(S)$ contains the implication $ce \Rightarrow d$, generated by implicational overlap from $ac \Rightarrow d$ and $e \Rightarrow a$. The corresponding equational overlap of $acd \rightarrow ac$ and $ae \rightarrow e$ gives $e \leftarrow ace \leftarrow acde \rightarrow cde$ and hence generates the rule $cde \rightarrow ce$. However, this rule is redundant in the presence of $\{c \rightarrow \mathit{true}, e \rightarrow \mathit{true}, d \rightarrow \mathit{true}\}$ and simplification. $\square$

5 Direct-Optimal Systems

Direct-optimality is defined by adding to directness a requirement of optimality, with respect to a measure $|\,S\,|$ that counts the sum of the number of occurrences of symbols on each of the two sides of each implication in a system S:

Definition 5 (Optimality [3, Sect. 2]). *An implicational system S is* optimal *if, for all equivalent implicational system S', $|\,S\,| \leq |\,S'\,|$ where*

$$|\,S\,| \quad \stackrel{!}{=} \quad \sum_{A \Rightarrow_S B} |A| + |B| \ ,$$

where $|A|$ is the cardinality of set A.

From an implicational system S, one can generate an equivalent implicational system that is both direct and optimal, denoted $D(S)$, with the following necessary and sufficient properties (cf. [3, Thm. 2]):

- *extensiveness*: for all $A \Rightarrow_{D(S)} B$, $A \cap B = \emptyset$;
- *isotony*: for all $A \Rightarrow_{D(S)} B$ and $C \Rightarrow_{D(S)} D$, if $C \subset A$, then $B \cap D = \emptyset$;
- *premise*: for all $A \Rightarrow_{D(S)} B$ and $A \Rightarrow_{D(S)} B'$, $B = B'$;
- *non-empty conclusion*: for all $A \Rightarrow_{D(S)} B$, $B \neq \emptyset$.

This leads to the following characterization:

Definition 6 (Direct-optimal system [3, Def. 5]). *Given a direct system S, the* direct-optimal *system $D(S)$ generated from S contains precisely the implications*

$$A \Rightarrow \bigcup\{B \colon A \Rightarrow_S B\} \setminus \{C \colon D \Rightarrow_S C \wedge D \subset A\} \setminus A\ ,$$

for each set A of propositions – provided the conclusion is non-empty.

From the above four properties, we can define an *optimization* procedure, applying – in order – the following rules:

Premise

$$\frac{A \Rightarrow B,\ A \Rightarrow C}{A \Rightarrow BC}\ ,$$

Isotony

$$\frac{A \Rightarrow B,\ AD \Rightarrow BE}{A \Rightarrow B,\ AD \Rightarrow E}\ ,$$

Extensiveness

$$\frac{AC \Rightarrow BC}{AC \Rightarrow B}\ ,$$

Definiteness

$$\frac{A \Rightarrow \emptyset}{}\ .$$

The first rule merges all rules with the same antecedent A into one and implements the *premise* property. The second rule removes from the consequent thus generated those subsets B that are already implied by subsets A of AD, to enforce *isotony*. The third rule makes sure that antecedents C do not themselves appear in the consequent to enforce *extensiveness*. Finally, implications with empty consequent are eliminated. This latter rule is called *definiteness*, because it eliminates negative clauses, which, for Horn theories, represent queries and are not “definite” (i.e., non-negative) clauses. Clearly, the changes wrought by the optimization rules do not affect the theory. Application of this optimization to the direct implicational system $I(S)$ yields the direct-optimal system $D(S)$ of S.

The following example shows that this notion of optimization does *not* correspond to elimination of redundancies by contraction in completion:

Example 6. Let $S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$. Then, $I(S) = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a, e \Rightarrow b, ce \Rightarrow d\}$, where $e \Rightarrow b$ is generated by implicational overlap of $e \Rightarrow a$ and $a \Rightarrow b$, and $ce \Rightarrow d$ is generated by implicational overlap of $e \Rightarrow a$ and $ac \Rightarrow d$. Next, optimization replaces $e \Rightarrow a$ and $e \Rightarrow b$ by $e \Rightarrow ab$, so that $D(S) = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow ab, ce \Rightarrow d\}$. If we consider the rewriting side, we have $R_S = \{ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e\}$. Equational overlap of $ae \rightarrow e$ and $ab \rightarrow a$ generates $be \rightarrow e$, and equational overlap of $ae \rightarrow e$ and $acd \rightarrow ac$ generates $cde \rightarrow ce$, corresponding to the two implicational overlaps. Thus, $(R_S)_\infty = \{ab \rightarrow a, acd \rightarrow ac, ae \rightarrow e, be \rightarrow e, cde \rightarrow ce\}$. The rule corresponding to $e \Rightarrow ab$, namely $abe \rightarrow e$, would be redundant if added to $(R_S)_\infty$, because it would be reduced to a trivial equivalence by $ae \rightarrow e$ and $be \rightarrow e$. Thus, the optimization consisting of replacing $e \Rightarrow a$ and $e \Rightarrow b$ by $e \Rightarrow ab$ does not correspond to a rewriting inference. □

The reason for this discrepancy is the different choice of ordering. Seeking direct-optimality means optimizing the overall size of the system. For Ex. 6, we have $|\{e \Rightarrow ab\}| = 3 < 4 = |\{e \Rightarrow a, e \Rightarrow b\}|$. The corresponding proof ordering measures a proof of a from a set X and an implicational system S by a multiset of pairs $\langle |B|, \#_B S\rangle$, for each $B \Rightarrow_S aC$ such that $B \subseteq X$, where $\#_B S$ is the number of implications in S with antecedent B. A proof of a from $X = \{e\}$ and $\{e \Rightarrow ab\}$ will have measure $\{\!\{\langle 1, 1\rangle\}\!\}$, which is smaller than the measure $\{\!\{\langle 1, 2\rangle, \langle 1, 2\rangle\}\!\}$ of a proof of a from $X = \{e\}$ and $\{e \Rightarrow a, e \Rightarrow b\}$.

Completion, on the other hand, optimizes with respect to the ordering $\succ$. For $\{abe \rightarrow e\}$ and $\{ae \rightarrow e, be \rightarrow e\}$, we have $ae \prec abe$ and $be \prec abe$, so $\{\!\{ae, e\}\!\} \prec_L \{\!\{abe, e\}\!\}$ and $\{\!\{be, e\}\!\} \prec_L \{\!\{abe, e\}\!\}$ in the multiset extension $\succ_L$ of $\succ$, and $\{\!\{\{\!\{ae, e\}\!\}, \{\!\{be, e\}\!\}\}\!\} \prec_C \{\!\{\{\!\{abe, e\}\!\}\}\!\}$ in the multiset extension $\succ_C$ of $\succ_L$. Indeed, from a rewriting point of view, it is better to have $\{ae \rightarrow e, be \rightarrow e\}$ than $\{abe \rightarrow e\}$, since rules with smaller left hand side are more applicable.

6 Rewrite-Optimality

It is apparent that the differences between direct-optimality and completion arise because of the application of the *premise* rule. Accordingly, we propose an alternative definition of optimality, one that does not require the *premise* property, because symbols in repeated antecedents are counted only once:

Definition 7 (Rewrite-optimality). *An implicational system S is* rewrite-optimal *if $\| S \| \leq \| S' \|$ for all equivalent implicational system S', where the measure $\| S \|$ is defined by:*

$$\| S \| \stackrel{!}{=} |Ante(S)| + |Cons(S)| \, ,$$

for $Ante(S) \stackrel{!}{=} \{c \colon c \in A,\ A \Rightarrow_S B\}$, the set *of symbols occurring in antecedents, and $Cons(S) \stackrel{!}{=} \{\!\{c \colon c \in B,\ A \Rightarrow_S B\}\!\}$, the* multiset *of symbols occurring in consequents.*

Unlike Definition 5, where antecedents and consequents contribute equally, here symbols in antecedents are counted only once, because $Ante(S)$ is a set, while symbols in consequents are counted as many times as they appear, since $Cons(S)$ is a multiset. Rewrite-optimality appears to be appropriate for Horn clauses, because the *premise* property conflicts with the decomposition of non-unary implications into Horn clauses. Indeed, if S is a non-unary implicational system, and S_H is the equivalent Horn system obtained by decomposing non-unary implications, the application of the *premise* rule to S_H undoes the decomposition.

Example 7. Applying rewrite optimality to $S = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a\}$ of Ex. 6, we have $\|\{e \Rightarrow ab\}\| = 3 = \|\{e \Rightarrow a, e \Rightarrow b\}\|$, so that replacing $\{e \Rightarrow a, e \Rightarrow b\}$ by $\{e \Rightarrow ab\}$ is no longer justified. Thus, $D(S) = I(S) = \{a \Rightarrow b, ac \Rightarrow d, e \Rightarrow a, e \Rightarrow b, ce \Rightarrow d\}$, and the rewrite system associated with $D(S)$ is $\{ab \to a, acd \to ac, ae \to e, be \to e, cde \to ce\} = (R_S)_\infty$. A proof ordering corresponding to rewrite optimality would measure a proof of a from a set X and an implicational system S by the set of the cardinalities $|B|$, for each $B \Rightarrow_S aC$ such that $B \subseteq X$. Accordingly, a proof of a from $X = \{e\}$ and $\{e \Rightarrow ab\}$ will have measure $\{\!\{1\}\!\}$, which is the same as the measure of a proof of a from $X = \{e\}$ and $\{e \Rightarrow a, e \Rightarrow b\}$. □

Let $\leadsto_O$ denote the deduction mechanism that includes implicational overlap and the optimization rules except *premise*, namely *isotony*, *extensiveness* and *definiteness*. We deem *canonical*, and denote by $O(S)$, the implicational system obtained from S by closure with respect to implicational overlap, isotony, extensiveness and definiteness:

Definition 8 (Canonical system). *Given an implicational system S, the* canonical *implicational system $O(S)$ generated from S is the limit S_∞ of any fair derivation $S = S_0 \leadsto_O S_1 \leadsto_O \cdots$.*

The following lemma shows that every inference by $\leadsto_O$ is covered by an inference in $\leadsto_R$:

Lemma 3. *For all implicational systems S, if $S \leadsto_O S'$, then $R_S \leadsto_R R_{S'}$.*

Proof. We consider four cases, corresponding to the four inference rules in $\leadsto_O$:

1. *Implicational overlap:* If $S \leadsto_O S'$ by an implicational overlap step, then $R_S \leadsto_R R_{S'}$ by equational overlap, by Lemma 2.
2. *Isotony:* For an application of this rule, $S = S'' \cup \{A \Rightarrow B, AD \Rightarrow BE\}$ and $S' = S'' \cup \{A \Rightarrow B, AD \Rightarrow E\}$. Then, $R_S = R_{S''} \cup \{AB \to A, ADBE \to AD\}$. Simplification applies to R_S using $AB \to A$ to rewrite $ADBE \to AD$ to $ADE \to AD$, yielding $R_{S''} \cup \{AB \to A, ADE \to AD\} = R_{S'}$.
3. *Extensiveness:* When this rule applies, $S = S'' \cup \{AC \Rightarrow BC\}$ and $S' = S'' \cup \{AC \Rightarrow B\}$. Then, $R_S = R_{S''} \cup \{ACBC \to AC\}$. By mere idempotence of juxtaposition, $R_S = R_{S''} \cup \{ABC \to AC\} = R_{S'}$.

4. *Definiteness:* If $S = S' \cup \{A \Rightarrow \emptyset\}$, then $R_S = R_{S'} \cup \{A \leftrightarrow A\}$ and an application of deletion eliminates the trivial equation, yielding $R_{S'}$. $\square$

However, the other direction of this lemma does not hold, because $\rightsquigarrow_R$ features simplifications that do not correspond to inferences in $\rightsquigarrow_O$:

Example 8. Assume that the implicational system S includes $\{de \Rightarrow b, b \Rightarrow d\}$. Accordingly, R_S contains $\{deb \rightarrow de, bd \rightarrow b\}$. A simplification inference applies $bd \rightarrow b$ to reduce $deb \rightarrow de$ to $be \leftrightarrow de$, which is oriented into $be \rightarrow de$, if $b \succ d$, and into $de \rightarrow be$, if $d \succ b$. (Were $\rightsquigarrow_R$ equipped with a cancellation inference rule, $be \leftrightarrow de$ could be rewritten to $b \leftrightarrow d$, whence $b \rightarrow d$ or $d \rightarrow b$.) The deduction mechanism $\rightsquigarrow_O$ can apply implicational overlap to $de \Rightarrow b$ and $b \Rightarrow d$ to generate $de \Rightarrow d$. However, $de \Rightarrow d$ is reduced to $de \Rightarrow \emptyset$ by the extensiveness rule, and $de \Rightarrow \emptyset$ is deleted by the definiteness rule. Thus, $\rightsquigarrow_O$ does not generate anything that corresponds to $be \leftrightarrow de$. $\square$

This example can be generalized to provide a simple analysis of simplification steps, one that shows which steps correspond to $\rightsquigarrow_O$-inferences and which do not. Assume we have two rewrite rules $AB \rightarrow A$ and $CD \rightarrow C$, corresponding to non-trivial Horn clauses ($|B| = 1$, $B \not\subseteq A$, $|D| = 1$, $D \not\subseteq C$), and such that $CD \rightarrow C$ simplifies $AB \rightarrow A$. We distinguish three cases:

1. In the first one, CD appears in AB because CD appears in A. In other words, $A = CDE$ for some E. Then, the simplification step is
$$\frac{CDEB \rightarrow CDE,\ CD \rightarrow C}{\overline{CEB \rightarrow CE,\ CD \rightarrow C}}$$
(where simplification is actually applied to both sides). The corresponding implications are $A \Rightarrow B$ and $C \Rightarrow D$. Since $A \Rightarrow B$ is $CDE \Rightarrow B$, implicational overlap applies to generate the implication $CE \Rightarrow B$ that corresponds to $CEB \rightarrow CE$:
$$\frac{C \Rightarrow D,\ CDE \Rightarrow B}{CE \Rightarrow B}\,.$$
The isotony rule applied to $CE \Rightarrow B$ and $CDE \Rightarrow B$ reduces the latter to $CDE \Rightarrow \emptyset$, which is deleted by definiteness: a combination of implicational overlap, isotony and definiteness simulates the effects of simplification.
2. In the second case, CD appears in AB because C appears in A, that is, $A = CE$ for some E, and $D = B$. Then, the simplification step is
$$\frac{CEB \rightarrow CE,\ CB \rightarrow C}{\overline{CE \leftrightarrow CE,\ CB \rightarrow C}}\,,$$
and $CE \leftrightarrow CE$ is removed by deletion. The isotony inference
$$\frac{C \Rightarrow B,\ CE \Rightarrow B}{\overline{C \Rightarrow B,\ CE \Rightarrow \emptyset}}\,,$$
generates $CE \Rightarrow \emptyset$ which gets deleted by definiteness.

3. The third case is the generalization of Ex. 8: CD appears in AB because D appears in A, and C is made of B and some F that also appears in A, that is, $A = DEF$ for some E and F, and $C = BF$. The simplification step is

$$\frac{DEFB \to DEF,\ BFD \to BF}{\overline{BFE \leftrightarrow DEF,\ BFD \to BF}}\ .$$

Implicational overlap applies

$$\frac{DEF \Rightarrow B,\ BF \Rightarrow D}{DEF \Rightarrow D}$$

to generate an implication that is first reduced by extensiveness to $DEF \Rightarrow \emptyset$ and then eliminated by definiteness. Thus, nothing corresponding to $BFE \leftrightarrow DEF$ gets generated.

It follows that whatever is generated by $\leadsto_O$ is generated by $\leadsto_R$, but may become redundant eventually:

Theorem 3. *For every implicational system S, for all fair derivations $S = S_0 \leadsto_O S_1 \leadsto_O \cdots$ and $R_S = R_0 \leadsto_R R_1 \leadsto_R \cdots$, for all $FG \to F \in R_{(S_\infty)}$, either $FG \to F \in (R_S)_\infty$ or $FG \to F$ is redundant in $(R_S)_\infty$.*

Proof. For all $FG \to F \in R_{(S_\infty)}$, $F \Rightarrow G \in S_\infty$ by Definition 4, and $F \Rightarrow G \in S_j$ for some $j \geq 0$. Let j be the smallest such index. If $j = 0$, or $S_j = S$, $FG \to F \in R_S = R_0$ by Definition 4. If $j > 0$, $F \Rightarrow G$ was generated by an application of implicational overlap, the isotony rule or extensiveness. By Lemma 3 and the fairness of the $\leadsto_R$-derivation, $FG \to F \in R_k$ for some $k > 0$. If $FG \to F$ persists, then $FG \to F \in (R_S)_\infty$. Otherwise, $FG \to F$ gets rewritten by simplification and is, therefore, redundant in $(R_S)_\infty$. □

Since the limit of the $\leadsto_O$-derivation is $O(S)$, it follows that:

Corollary 3. *For every implicational system S, for all fair derivations $S = S_0 \leadsto_O S_1 \leadsto_O \cdots$ and $R_S = R_0 \leadsto_R R_1 \leadsto_R \cdots$, and for all $FG \to F \in R_{O(S)}$, either $FG \to F$ is in $(R_S)_\infty$ or else $FG \to F$ is redundant in $(R_S)_\infty$.*

7 Discussion

We analyzed the notions of direct and direct-optimal implicational system in terms of completion and canonicity. We found that a direct implicational system corresponds to the canonical limit of a derivation by completion that features expansion by equational overlap and contraction by forward simplification. When completion also features backward simplification, it computes the image of a given set with respect to the closure operator associated with the given implicational system. In other words, it computes the minimal model that satisfies both the implicational system and the set. On the other hand, a direct-optimal implicational system does not correspond to the limit of a derivation by completion, because the underlying proof orderings are different and, therefore, normalization induces two different notions of optimization. Thus, we introduced a new notion of optimality for implicational systems, termed *rewrite optimality*, that corresponds to canonicity defined by completion up to redundancy.

Directions for future work include generalizing this analysis beyond propositional Horn theories, studying enumerations of Moore families and related structures (see [10] and Sequences A102894–7 and A108798–801 in [14]), and exploring connections between canonical systems and decision procedures, or the rôle of canonicity of presentations in specific contexts where Moore families occur, such as in the abstract interpretations of programs.

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