

BETWEEN BROADWAY AND THE HUDSON

NACHUM DERSHOWITZ

Canal street, running across Broadway to the Hudson, near the centre of the city, is a spacious street, principally occupied by retail stores. . . .

The streets are generally well paved, with good side walks, lighted at night with lamps, and some of them supplied with gas lights.

—*The Treasury of Knowledge, and Library of Reference* (1834)

ABSTRACT. A substantial generalization of the equinumeracy of grand-Dyck paths and Dyck-path prefixes, constrained within a band, is presented.

1. INTRODUCTION

We are interested in enumerating lattice paths that remain within a band of height h , sometimes called *corridor paths*.¹ Sort of like walking in Manhattan, sticking west of Broadway (Figure 1).

Let $i \overset{n}{\rightsquigarrow}_h \ell$, or just $i \overset{n}{\rightsquigarrow} \ell$ (fixing h), denote the number of monotonic lattice paths from $\langle 0, i \rangle$ to $\langle n, \ell \rangle$ with n steps that stay within (but may touch) the boundaries $y = 0$ and $y = h$, for some given (maximum) height h .² Let $H = [0 : h]$ be the ordinate bounds within which steps are permissible. Steps are diagonal, NE (northeast, ↗), taking $\langle x, y \rangle \mapsto \langle x + 1, y + 1 \rangle$, and SE (southeast, ↘), taking $\langle x, y \rangle \mapsto \langle x + 1, y - 1 \rangle$, both with the proviso that the new ordinate position $y \pm 1 \in H$, as the case may be. It is easy to see that one always has $n + i \equiv \ell \pmod{2}$, or else there are zero n -step paths starting at level i and ending at ℓ . See Figure 2 for a sample path.

The basic recurrence is

$$i \overset{n}{\rightsquigarrow} \ell = \begin{cases} 0 & \text{if } i \notin H \text{ or } \ell \notin H \\ [i = \ell] & \text{if } n = 0 \\ (i \overset{n-1}{\rightsquigarrow} \ell - 1) + (i \overset{n-1}{\rightsquigarrow} \ell + 1) & \text{otherwise} \end{cases}$$

where the bracketed condition $[i = \ell]$ is Iverson's notation for a characteristic function (1 when true; 0 when false), and the conditions are taken in order.

Date: June 10, 2020.

¹See Shaun V. Ault and Charles Kicey, "Counting paths in corridors using circular Pascal arrays", *Discrete Mathematics* **332**(6):45–54, October 2014.

²*Height* here is the maximum length of a unidirectional path (just NE or just SE). Some might prefer to say that the *width* of the corridor is $h + 1$, since $h + 1$ ordinate values are allowed.



FIGURE 1. Manhattan neighborhoods, with East-West streets and North-South avenues, bounded by Broadway on the East and the Hudson River on the West, with Union Square serving as origin.

The ends of the paths we are interested in fall within a range, J , not just a single point ℓ . For example, the window $J = [5 : 10]$ has 6 possible landing spots, but only half of them are feasible, depending on whether $n + i$ is odd or even. Only those $\ell \in J$ with the same parity as $n + i$ are relevant. Our goal is to count

$$i \rightsquigarrow J = \sum_{\ell \in J} i \rightsquigarrow \ell = \sum_{\substack{\ell \in J \\ \ell \equiv n+i \pmod{2}}} i \rightsquigarrow \ell$$

the number of paths constrained to any corridor $H = [0 : h]$ and ending at any (feasible) ordinate in the window J .

These lattice paths are equivalent to walks along a path graph, forward and backward. When $i = 0$ (at the bottom) and $J = H$ (anywhere), walks for $h = 0, 1, 2, 3, 4, 5, 6, 7, 8$ are enumerated at [A000007](#) (constant 0), [A000012](#) (constant 1),

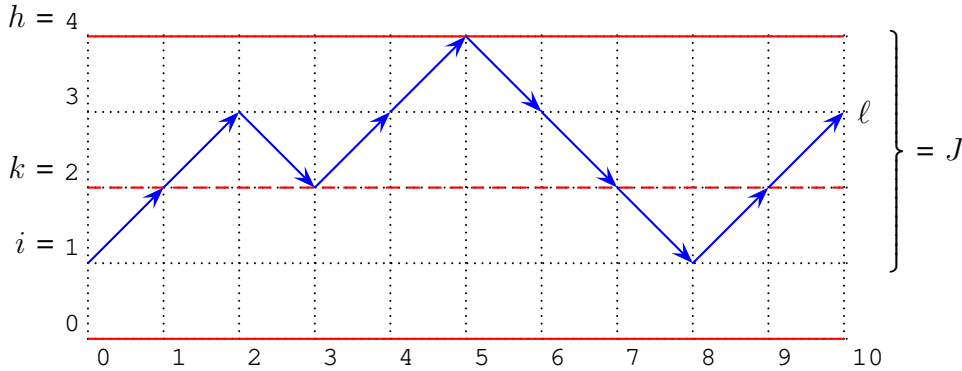


FIGURE 2. A diagonal path $1 \overset{10}{\rightsquigarrow} 3$, from $i = 1$ to $\ell = 3$ consisting of 6 NE-steps and 4 SE-steps, with bound $h = 4$. The target region J is $\llbracket 2 \pm 1 \rrbracket = [1 : 4]$.

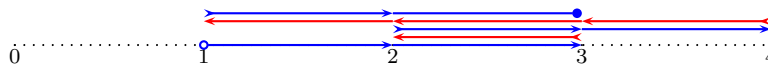


FIGURE 3. A right-left version of the constrained path $1 \overset{10}{\rightsquigarrow} 3$, consisting of 6 R-steps (colored blue) and 4 L-steps (red) along a 5-vertex point graph ($h = 4$), starting at vertex $i = 1$. The path in this representation is RRLRLLLLRR, based at 1. It is an accordion fold of the blue path in Figure 2.

[A016116](#) ($2^{\lfloor n/2 \rfloor}$), [A000045](#) (Fibonacci), [A038754](#) ($\{1, 2\}3^n$), [A028495](#), [A030436](#), [A061551](#), [A178381](#), respectively,³ in Sloane’s *Encyclopedia of Integer Sequences*.⁴ In general, walks can start anywhere in H ($0 \leq i \leq h$), with the position always staying within the range $0 : h$. See Figure 3.

Table 1 lists values for the number of paths through a corridor of height $h = 4$, with one subtable for each starting point ($i = 0, 1, 2, 3, 4$); Table 2 exhibits $h = 5$. These may be viewed as constrained versions of Pascal’s triangle, with each entry the sum of two prior entries.

2. MAIN RESULTS

We will use the notation $\llbracket k \pm j \rrbracket$ as shorthand for a range $[k - j : k + j + 1]$, which we make of even size, viz. $2j + 2$, by stretching the upper end one spot, to $k + j + 1$. Thus, the window $\llbracket k \pm j \rrbracket$ covers $j + 1$ feasible endpoints – the odd ones or the even ones, as the case may be – centered about k .

³Compiled mainly by Jonathon Bryant; see Ault and Kicey.

⁴Neil J. A. Sloane, ed., “The on-line encyclopedia of integer sequences”, <http://oeis.org>.

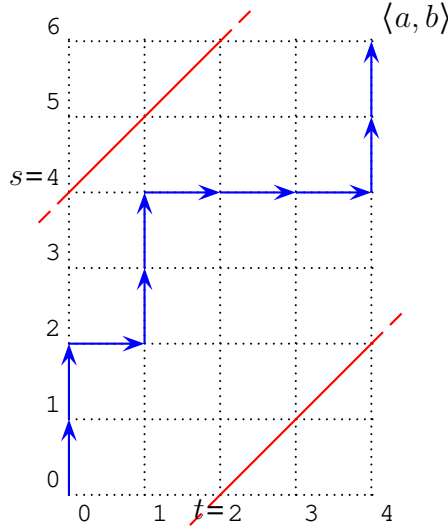


FIGURE 4. An orthogonal path, starting at origin and ending at $\langle a, b \rangle = \langle 4, 6 \rangle$, consisting of 6 N-steps and 4 E-steps, staying strictly within bounds $s = 4$ (below $y = x + 4$) and $t = 2$ (above $y = x - 2$). This path and its constraints are analogues of those in Figure 2; see Section 3.

Our main result is the following intriguing equivalence:

Theorem 1. For all $n, h \in \mathbb{N}$, $k \in [-1 : h]$, $i, j \in [0 : \min\{k + 1, h - k\}]$:

$$i \overset{n}{\rightsquigarrow}_h \llbracket k \pm j \rrbracket = j \overset{n}{\rightsquigarrow}_h \llbracket k \pm i \rrbracket \quad (1)$$

For example, $2 \overset{9}{\rightsquigarrow}_4 \llbracket 2 \pm 1 \rrbracket = 162 = 1 \overset{9}{\rightsquigarrow}_4 \llbracket 2 \pm 2 \rrbracket$; see Table 1. The bounds on i and j ensure that the starting points are in $H = [0 : h]$ and that the endpoints in the target windows $\llbracket k \pm j \rrbracket$ and $\llbracket k \pm i \rrbracket$ do not extend more than one row beyond the corridor H (one row above or below). Were i or j too big, $k \pm i$ or $k \pm j$ could extend too far beyond H , and the equality would not hold, as is the case for $2 \overset{9}{\rightsquigarrow}_4 \llbracket 3 \pm 1 \rrbracket = 81 \neq 1 \overset{9}{\rightsquigarrow}_4 \llbracket 3 \pm 2 \rrbracket = 121$. When $k \leq h/2$, we need only that $i, j \leq k$ for the theorem to hold.

This is a significant generalization of an equality due to Johann Cigler:⁵

$$0 \overset{n}{\rightsquigarrow}_h H = h \div 2 \overset{n}{\rightsquigarrow}_h h \div 2^+ \quad (2)$$

for all heights h , where we are using the notation $\ell^+ = [\ell : \ell + 1]$ to include also $\ell + 1$ for when parity demands the adjacent target spot. Paths $h \div 2 \overset{n}{\rightsquigarrow}_h h \div 2^+$ start in the middle of the swath and end either in the middle – when the number of steps is even, or just above – when odd. These are called “grand-Dyck” paths in the

⁵“Some remarks and conjectures related to lattice paths in strips along the x-axis”, <https://arxiv.org/abs/1501.04750>, Jan. 2015–June 2016.

even case. The paths $0 \rightsquigarrow_h^n H$ start at the bottom and end anywhere within the swath; these are prefixes of Dyck paths. Cigler’s case (2) is a particular instance of our more general result (1) with $i = 0$ and $j = k = h \div 2$; in other words:

$$0 \rightsquigarrow_h^n \llbracket h \div 2 \pm h \div 2 \rrbracket = h \div 2 \rightsquigarrow_h^n \llbracket h \div 2 \pm 0 \rrbracket$$

Cigler solicited alternative proofs of his result. More specifically, he asked for a *bijective* proof of the fact that⁶

$$0 \rightsquigarrow_3^n [0 : 3] = 1 \rightsquigarrow_3^n 1^+$$

The wished-for bijective solution to this very special case was discovered by Thomas Prellberg.⁷

For when $i > h \div 2$, we can combine the following lemma with the above theorem.

Lemma 2. *For all $n, h \in \mathbb{N}$, $i, j, k \in [0 : h]$,*

$$i \rightsquigarrow_h^n \llbracket k \pm j \rrbracket = (h - i) \rightsquigarrow_h^n \llbracket h - k - 1 \pm j \rrbracket \tag{3}$$

This is just up-down symmetry.

Corollary 3. *The equivalence*

$$i \rightsquigarrow_h^n \llbracket k \pm j \rrbracket = j \rightsquigarrow_h^n \llbracket h - k - 1 \pm h - i \rrbracket$$

holds for all $n, h \in \mathbb{N}$, $k \in [0 : h]$, $i \in [\max\{h - k - 1, h - k\} : h]$, $j \in [0 : \min\{k + 1, h - k\}]$.

Finally, the closed-form formula for the paths of interest is as follows:

Theorem 4. *The number of corridor paths $i \rightsquigarrow_h^n \llbracket k \pm j \rrbracket$ is*

$$\sum_{z=\lfloor -n/4 \rfloor}^{\lfloor n/4 \rfloor} \sum_{\substack{s=0 \\ 0 \leq k-j+2s \leq h}}^j \left[\binom{n}{\lfloor \frac{n+k-i-j}{2} \rfloor + z(h+2) + s} - \binom{n}{\lfloor \frac{n+k+i-j}{2} \rfloor + z(h+2) + s + 1} \right]$$

for all $n, h, j, k \in \mathbb{N}$, $i \in [0 : h]$.

3. COMBINATORIAL PROOF

One can derive the enumeration of Theorem 4 using a standard result for bounded lattice paths. Our main theorem will then follow as a corollary.⁸

The number $M(a, b, s, t)$ of “monotonic” paths from $\langle 0, 0 \rangle$ to $\langle a, b \rangle$, taking a steps to the east (E, \rightarrow) and b steps to the north (N, \uparrow), while totally avoiding

⁶https://www.researchgate.net/post/Is_there_a_simple_bijection_between_the_following_sets_A_n_and_B_n_which_are_counted_by_the_Fibonacci_numbers, Dec. 2015. This problem was brought to my attention by Christian Rinderknecht.

⁷See reference in previous footnote.

⁸Alternative proofs remain to be explored.

(not touching or crossing) the boundaries $y = x + s$ and $y = x - t$ ($s, t \in \mathbb{Z}^+$, $t < b - a < s$) is known (by a reflection argument) to be⁹

$$M(a, b, s, t) = \sum_{z \in \mathbb{Z}} \left[\binom{a+b}{b+z(s+t)} - \binom{a+b}{b+z(s+t)+t} \right] \quad (4)$$

with the (nonstandard) convention that $\binom{n}{m} = 0$ whenever $m \notin \mathbb{N}$. See Figure 4.

There is a straightforward relationship between these constrained N/E paths $\langle 0, 0 \rangle \rightsquigarrow \langle a, b \rangle$ and those NE/SE paths $\langle 0, i \rangle \rightsquigarrow \langle n, \ell \rangle$ that we have set out to study (as illustrated in Figure 2):

$$\begin{aligned} n &= a + b & \ell - i &= b - a \\ t &= i + 1 & s + t &= h + 2 \end{aligned}$$

Plugging the solution

$$\begin{aligned} a &= \frac{n+i-\ell}{2} & b &= \frac{n-i+\ell}{2} \\ s &= h-i+1 & t &= i+1 \end{aligned}$$

into (4), we get¹⁰

$$i \overset{n}{\rightsquigarrow}_h \ell = \sum_{z \in \mathbb{Z}} \left[\binom{n}{\frac{n-i+\ell}{2} + z(h+2)} - \binom{n}{\frac{n-i+\ell}{2} + z(h+2) + i + 1} \right] \quad (5)$$

as long as $0 \leq i, \ell \leq h$. For those ℓ for which $\frac{n-i+\ell}{2}$ is not a whole number, the binomial coefficients are all 0.

Note that formula (5) also yields 0 for $\ell = -1$ and for $\ell = h + 1$. In the first case, since the sum is over all z :

$$\begin{aligned} & \sum_{z \in \mathbb{Z}} \left[\binom{n}{\frac{n-i-1}{2} + z(h+2)} - \binom{n}{\frac{n-i-1}{2} + z(h+2) + i + 1} \right] \\ &= \sum_{z \in \mathbb{Z}} \binom{n}{\frac{n-i-1}{2} + z(h+2)} - \sum_{z \in \mathbb{Z}} \binom{n}{\frac{n+i+1}{2} - z(h+2)} = 0 \end{aligned}$$

⁹Robert Dutton Fray and David Paul Roselle, “Weighted lattice paths”, *Pacific Journal of Mathematics* **37**(1):85–96, 1971; Sri Gopal Mohanty, *Lattice Path Counting and Applications*, volume 37 of *Probability and Mathematical Statistics*, Academic Press, New York, 1979, p. 6.

¹⁰Cf. Ault and Kicey.

Similarly, in the second case ($\ell = h + 1$):

$$\begin{aligned} & \sum_{z \in \mathbb{Z}} \left[\binom{n}{\frac{n-i+h+1}{2} + z(h+2)} - \binom{n}{\frac{n-i+h+1}{2} + z(h+2) + i + 1} \right] \\ &= \sum_{z \in \mathbb{Z}} \binom{n}{\frac{n-i+h+1}{2} + z(h+2)} - \sum_{z \in \mathbb{Z}} \binom{n}{\frac{n+i-h-1}{2} - z(h+2)} = 0 \end{aligned}$$

Letting ℓ move along the window from $k - j$ to $k + j + 1$, we get from (5) that

$$i \overset{n}{\rightsquigarrow}_h \llbracket k \pm j \rrbracket = \sum_{\ell=\max\{0, k-j\}}^{\min\{k+j+1, h\}} \sum_{z \in \mathbb{Z}} \left[\binom{n}{\frac{n-i+\ell}{2} + z(h+2)} - \binom{n}{\frac{n-i+\ell}{2} + z(h+2) + i + 1} \right]$$

The sum for z can be restricted to the range $[-n/4]:[n/4]$. Skipping over the impossible odd or even values (for which the denominators of the binomial coefficients are fractional), we arrive at the stated formula of Theorem 4:

$$i \overset{n}{\rightsquigarrow}_h \llbracket k \pm j \rrbracket = \sum_{z=\lfloor -n/4 \rfloor}^{\lfloor n/4 \rfloor} \sum_{\substack{s=0 \\ 0 \leq k-j+2s \leq h}}^j \left[\binom{n}{\lceil \frac{n-i+k-j}{2} \rceil + z(h+2) + s} - \binom{n}{\lceil \frac{n+i+k-j}{2} \rceil + z(h+2) + s + 1} \right] \quad (6)$$

Consider now only the cases considered in Theorem 1, which guarantee that $k - j \geq -1$ and that $k + j \leq h + 1$, so s may run from 0 to j without exception – bearing in mind (as shown above) that any instances when $k - j + 2s = 0, h + 1$ have no impact on the sum. Reversing the order of the second sum in (6), replacing s with $j - s$, we get

$$i \overset{n}{\rightsquigarrow}_h \llbracket k \pm j \rrbracket = \sum_{z \in \mathbb{Z}} \sum_{s=0}^j \left[\binom{n}{\lceil \frac{n-i+k-j}{2} \rceil + z(h+2) + s} - \binom{n}{\lceil \frac{n+i+k+j}{2} \rceil + z(h+2) - s + 1} \right]$$

When $j > i$, the inner sums overlap (for $s > i$) and cancel each other. So the above sum is always equal to

$$\sum_{z \in \mathbb{Z}} \left[\sum_{s=0}^{\min\{i, j\}} \binom{n}{r + z(h+2) + s} - \binom{n}{r + z(h+2) + i + j - s + 1} \right]$$

where $r = \lceil (n + k - i - j)/2 \rceil$. This is symmetric in i and j ; hence Theorem 1.

i	$n = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ℓ
4	1		1		2		5		14		41		122		365		1094	4
3		1		2		5		14		41		122		365		1094		3
2			1		3		9		27		81		243		729		2187	2
1				1		4		13		40		121		364		1093		1
0					1		4		13		40		121		364		1093	0
4		1		2		5		14		41		122		365		1094		4
3	1		2		5		14		41		122		365		1094		3281	3
2		1		3		9		27		81		243		729		2187		2
1			1		4		13		40		121		364		1093		3280	1
0				1		4		13		40		121		364		1093		0
4			1		3		9		27		81		243		729		2187	4
3		1		3		9		27		81		243		729		2187		3
2	1		2		6		18		54		162		486		1458		4374	2
1		1		3		9		27		81		243		729		2187		1
0			1		3		9		27		81		243		729		2187	0
4				1		4		13		40		121		364		1093		4
3			1		4		13		40		121		364		1093		3280	3
2		1		3		9		27		81		243		729		2187		2
1	1		2		5		14		41		122		365		1094		3281	1
0		1		2		5		14		41		122		365		1094		0
4					1		4		13		40		121		364		1093	4
3				1		4		13		40		121		364		1093		3
2			1		3		9		27		81		243		729		2187	2
1		1		2		5		14		41		122		365		1094		1
0	1		1		2		5		14		41		122		365		1094	0

TABLE 1. The number of paths $i \overset{n}{\rightsquigarrow}_4 \ell$ for $i, \ell \in [0 : 4]$, $n \in [0 : 16]$. For example, $2 \overset{16}{\rightsquigarrow} 2 = 0 \overset{16}{\rightsquigarrow}_4 [0 : 4] = 4374$ and $3 \overset{16}{\rightsquigarrow} 3 = 4 \overset{16}{\rightsquigarrow} [2 : 4] = 3281$. Like for a bishop on a chessboard, half the squares are unreachable from any given starting point. The few squares that require backward steps are likewise inaccessible. The particular path of Figures 2–4 is highlighted in **blue boldface**.

i	$n = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	ℓ	
5		1		1		2		5		14		42		131		417		1341	5
4			1		2		5		14		42		131		417		1341		4
3				1		3		9		28		89		286		924		2993	3
2					1		4		14		47		155		507		1652		2
1						1		5		19		66		221		728		2380	1
0							1		5		19		66		221		728		0
5			1		2		5		14		42		131		417		1341		5
4		1		2		5		14		42		131		417		1341		4334	4
3			1		3		9		28		89		286		924		2993		3
2				1		4		14		47		155		507		1652		5373	2
1					1		5		19		66		221		728		2380		1
0						1		5		19		66		221		728		2380	0
5				1		3		9		28		89		286		924		2993	5
4			1		3		9		28		89		286		924		2993		4
3		1		2		6		19		61		197		638		2069		6714	3
2			1		3		10		33		108		352		1145		3721		2
1				1		4		14		47		155		507		1652		5373	1
0					1		4		14		47		155		507		1652		0
5					1		4		14		47		155		507		1652		5
4				1		4		14		47		155		507		1652		5373	4
3			1		3		10		33		108		352		1145		3721		3
2		1		2		6		19		61		197		638		2069		6714	2
1			1		3		9		28		89		286		924		2993		1
0				1		3		9		28		89		286		924		2993	0
5						1		5		19		66		221		728		2380	5
4					1		5		19		66		221		728		2380		4
3				1		4		14		47		155		507		1652		5373	3
2			1		3		9		28		89		286		924		2993		2
1		1		2		5		14		42		131		417		1341		4334	1
0			1		2		5		14		42		131		417		1341		0
5							1		5		19		66		221		728		5
4					1		5		19		66		221		728		2380		4
3					1		4		14		47		155		507		1652		3
2				1		3		9		28		89		286		924		2993	2
1			1		2		5		14		42		131		417		1341		1
0		1		1		2		5		14		42		131		417		1341	0

TABLE 2. Paths $i \overset{n}{\rightsquigarrow}_5 \ell$ constrained to height 5, $n \in [0 : 16]$.