BETWEEN BROADWAY AND THE HUDSON

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Canal street, running across Broadway to the Hudson, near the centre of
the city, is a spacious street, principally occupied by retail stores. . . .
The streets are generally well paved, with good side walks, lighted at
night with lamps, and some of them supplied with gas lights.
—The Treasury of Knowledge, and Library of Reference (1834)

ABSTRACT. A substantial generalization of the equinumeracy of grand-Dyck
paths and Dyck-path prefixes, constrained within a band, is presented.

1. INTRODUCTION

We are interested in enumerating lattice paths that remain within a band of
height \( h \), sometimes called corridor paths.\(^1\) Sort of like walking in Manhattan,
sticking west of Broadway (Figure 1).

Let \( i \xmapsto{n}{ \ell} \), or just \( i \xmapsto{n}{ \ell} \) (fixing \( h \)), denote the number of monotonic lattice
paths from \( \langle 0, i \rangle \) to \( \langle n, \ell \rangle \) with \( n \) steps that stay within (but may touch) the
boundaries \( y = 0 \) and \( y = h \), for some given (maximum) height \( h \).\(^2\) Let \( H = [0:h] \)
be the ordinate bounds within which steps are permissible. Steps are diagonal,
NE (northeast, \( \nearrow \)), taking \( \langle x, y \rangle \mapsto \langle x + 1, y + 1 \rangle \), and SE (southeast, \( \searrow \)), taking
\( \langle x, y \rangle \mapsto \langle x + 1, y - 1 \rangle \), both with the proviso that the new ordinate position
\( y \pm 1 \in H \), as the case may be. It is easy to see that one always has \( n + i \equiv \ell \)
(mod 2), or else there are zero \( n \)-step paths starting at level \( i \) and ending at \( \ell \).
See Figure 2 for a sample path.

The basic recurrence is

\[
i \xmapsto{n}{ \ell} = \begin{cases}
0 & \text{if } i \notin H \text{ or } \ell \notin H \\
[i = \ell] & \text{if } n = 0 \\
(i \xmapsto{n-1}{ \ell - 1}) + (i \xmapsto{n-1}{ \ell + 1}) & \text{otherwise}
\end{cases}
\]

where the bracketed condition \([i = \ell]\) is Iverson’s notation for a characteristic
function (1 when true; 0 when false), and the conditions are taken in order.

\(^{1}\)See Shaun V. Ault and Charles Kicey, “Counting paths in corridors using circular Pascal

\(^{2}\)Height here is the maximum length of a unidirectional path (just NE or just SE). Some might
prefer to say that the width of the corridor is \( h + 1 \), since \( h + 1 \) ordinate values are allowed.
Figure 1. Manhattan neighborhoods, with East-West streets and North-South avenues, bounded by Broadway on the East and the Hudson River on the West, with Union Square serving as origin.

The ends of the paths we are interested in fall within a range, \( J \), not just a single point \( \ell \). For example, the window \( J = [5 : 10] \) has 6 possible landing spots, but only half of them are feasible, depending on whether \( n+i \) is odd or even. Only those \( \ell \in J \) with the same parity as \( n+i \) are relevant. Our goal is to count

\[
i \overset{\sim}{\longrightarrow} J = \sum_{\ell \in J} i \overset{\sim}{\longrightarrow} \ell = \sum_{\ell \in J} i \overset{\sim}{\longrightarrow} \ell_{\equiv n+i \pmod{2}}
\]

the number of paths constrained to any corridor \( H = [0 : h] \) and ending at any (feasible) ordinate in the window \( J \).

These lattice paths are equivalent to walks along a path graph, forward and backward. When \( i = 0 \) (at the bottom) and \( J = H \) (anywhere), walks for \( h = 0, 1, 2, 3, 4, 5, 6, 7, 8 \) are enumerated at \( A000007 \) (constant 0), \( A000012 \) (constant 1),
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Figure 2. A diagonal path $1 \rightsquigarrow 3$, from $i = 1$ to $\ell = 3$ consisting of 6 NE-steps and 4 SE-steps, with bound $h = 4$. The target region $J$ is $[2 \pm 1] = [1 : 4]$.

Figure 3. A right-left version of the constrained path $1 \rightsquigarrow 3$, consisting of 6 R-steps (colored blue) and 4 L-steps (red) along a 5-vertex point graph ($h = 4$), starting at vertex $i = 1$. The path in this representation is RRLRRLLLRR, based at 1. It is an accordion fold of the blue path in Figure 2.

A016116 ($2^\lfloor n/2 \rfloor$), A000045 (Fibonacci), A038754 ($\{1, 2\}^n$), A028495, A030436, A061551, A178381, respectively, in Sloane’s Encyclopedia of Integer Sequences. In general, walks can start anywhere in $H (0 \leq i \leq h)$, with the position always staying within the range $0 : h$. See Figure 3.

Table 1 lists values for the number of paths through a corridor of height $h = 4$, with one subtable for each starting point ($i = 0, 1, 2, 3, 4$); Table 2 exhibits $h = 5$. These may be viewed as constrained versions of Pascal’s triangle, with each entry the sum of two prior entries.

2. Main Results

We will use the notation $[k \pm j]$ as shorthand for a range $[k - j : k + j + 1]$, which we make of even size, viz. $2j + 2$. by stretching the upper end one spot, to $k + j + 1$. Thus, the window $[k \pm j]$ covers $j + 1$ feasible endpoints – the odd ones or the even ones, as the case may be – centered about $k$.

3 Compiled mainly by Jonathon Bryant; see Ault and Kicey.

Our main result is the following intriguing equivalence:

**Theorem 1.** For all \( n, h \in \mathbb{N}, k \in [-1 : h], i, j \in [0 : \min\{k + 1, h - k\}] \):

\[
i \overset{n}{\rightsquigarrow}^h [k \pm j] = j \overset{n}{\rightsquigarrow}^h [k \pm i]
\]

For example, \( 2 \overset{0}{\rightsquigarrow}^4 [2 \pm 1] = 162 = 1 \overset{0}{\rightsquigarrow}^4 [2 \pm 2] \); see Table 1. The bounds on \( i \) and \( j \) ensure that the starting points are in \( H = [0 : h] \) and that the endpoints in the target windows \( [k \pm j] \) and \( [k \pm i] \) do not extend more than one row beyond the corridor \( H \) (one row above or below). Were \( i \) or \( j \) too big, \( k \pm i \) or \( k \pm j \) could extend too far beyond \( H \), and the equality would not hold, as is the case for \( 2 \overset{0}{\rightsquigarrow}^4 [3 \pm 1] = 81 \neq 1 \overset{0}{\rightsquigarrow}^4 [3 \pm 2] = 121 \). When \( k \leq h/2 \), we need only that \( i, j \leq k \) for the theorem to hold.

This is a significant generalization of an equality due to Johann Cigler:

\[
0 \overset{n}{\rightsquigarrow}^h H = h + 2 \overset{n}{\rightsquigarrow}^h h + 2^\dagger
\]

for all heights \( h \), where we are using the notation \( \ell^\dagger = [\ell : \ell + 1] \) to include also \( \ell + 1 \) for when parity demands the adjacent target spot. Paths \( h + 2 \overset{n}{\rightsquigarrow}^h h + 2^\dagger \) start in the middle of the swath and end either in the middle – when the number of steps is even, or just above – when odd. These are called “grand-Dyck” paths in the

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even case. The paths $0 \sim^n H$ start at the bottom and end anywhere within the swath; these are prefixes of Dyck paths. Cigler’s case (2) is a particular instance of our more general result (1) with $i = 0$ and $j = k = h/2$; in other words:

\[ 0 \sim^n H [h \div 2 \pm h \div 2] = h \div 2 \sim^n H [h \div 2 \pm 0] \]

Cigler solicited alternative proofs of his result. More specifically, he asked for a bijective proof of the fact that

\[ 0 \sim^n H [0 : 3] = 1 \sim^n H ^+ \]

The wished-for bijective solution to this very special case was discovered by Thomas Prellberg.

For when $i > h \div 2$, we can combine the following lemma with the above theorem.

\section*{Lemma 2.}

\textit{For all $n, h \in \mathbb{N}, i, j, k \in [0 : h]$,}

\[ i \sim^n H [k \pm j] = (h - i) \sim^n H [h - k - 1 \pm j] \tag{3} \]

This is just up-down symmetry.

\section*{Corollary 3.}

\textit{The equivalence}

\[ i \sim^n H [k \pm j] = j \sim^n H [h - k - 1 \pm h - i] \]

\textit{holds for all $n, h \in \mathbb{N}, k \in [0 : h], i \in [\max\{h - k - 1, h - k\} : h], j \in [0 : \min\{k + 1, h - k\}]$.}

Finally, the closed-form formula for the paths of interest is as follows:

\section*{Theorem 4.}

\textit{The number of corridor paths $i \sim^n H [k \pm j]$ is}

\[ \sum_{z=\lceil -n/4 \rceil}^{\lfloor n/4 \rfloor} \sum_{s=0}^{i} \left[ \binom{n}{\frac{n+k-i-j}{2}} + z(h+2) + s \right] - \left[ \binom{n}{\frac{n+k+i-j}{2}} + z(h+2) + s + 1 \right] \]

\textit{for all $n, h, j, k \in \mathbb{N}, i \in [0 : h]$.}

\section*{3. Combinatorial Proof}

One can derive the enumeration of Theorem 4 using a standard result for bounded lattice paths. Our main theorem will then follow as a corollary.\footnote{https://www.researchgate.net/post/Is_there_a_simple_bijection_between_the_following_sets_A_n_and_B_n_which_are_counted_by_the_Fibonacci_numbers, Dec. 2015. This problem was brought to my attention by Christian Rinderknecht.}

The number $M(a, b, s, t)$ of “monotonic” paths from $\langle 0, 0 \rangle$ to $\langle a, b \rangle$, taking $a$ steps to the east (E, $\rightarrow$) and $b$ steps to the north (N, $\uparrow$), while totally avoiding...
(not touching or crossing) the boundaries \( y = x + s \) and \( y = x - t \) \((s, t \in \mathbb{Z}^+, t < b - a < s)\) is known (by a reflection argument) to be\(^9\)

\[
M(a, b, s, t) = \sum_{z \in \mathbb{Z}} \left[ \left( \frac{a + b}{b + z(s + t)} \right) - \left( \frac{a + b}{b + z(s + t) + t} \right) \right]
\] (4)

with the (nonstandard) convention that \( \binom{n}{m} = 0 \) whenever \( m \notin \mathbb{N} \). See Figure 4.

There is a straightforward relationship between these constrained N/E paths \( \langle 0, 0 \rangle \rightarrow \langle a, b \rangle \) and those NE/SE paths \( \langle 0, i \rangle \rightarrow \langle n, \ell \rangle \) that we have set out to study (as illustrated in Figure 2):

\[
\begin{align*}
n &= a + b \\
t &= i + 1
\end{align*}
\]

Plugging the solution

\[
\begin{align*}
a &= \frac{n + i - \ell}{2} \\
b &= \frac{n - i + \ell}{2} \\
s &= h - i + 1 \\
t &= i + 1
\end{align*}
\]

into (4), we get\(^{10}\)

\[
i \overset{\sim}{\overset{\sim}{\underset{h}{\rightarrow}}} \ell = \sum_{z \in \mathbb{Z}} \left[ \left( \frac{n - i + \ell}{2} + z(h + 2) \right) - \left( \frac{n - i}{2} + z(h + 2) + i + 1 \right) \right]
\] (5)

as long as \( 0 \leq i, \ell \leq h \). For those \( \ell \) for which \( \frac{n - i + \ell}{2} \) is not a whole number, the binomial coefficients are all 0.

Note that formula (5) also yields 0 for \( \ell = -1 \) and for \( \ell = h + 1 \). In the first case, since the sum is over all \( z \):

\[
\sum_{z \in \mathbb{Z}} \left[ \left( \frac{n - i - 1}{2} + z(h + 2) \right) - \left( \frac{n - i - 1}{2} + z(h + 2) + i + 1 \right) \right]
\]

\[
= \sum_{z \in \mathbb{Z}} \left( \frac{n - i - 1}{2} + z(h + 2) \right) - \sum_{z \in \mathbb{Z}} \left( \frac{n + i + 1}{2} - z(h + 2) \right) = 0
\]


\(^{10}\)Cf. Ault and Kicey.
Similarly, in the second case ($\ell = h + 1$):

$$\sum_{z \in \mathbb{Z}} \left( \frac{n-i+h+1}{2} + z(h+2) \right) - \left( \frac{n-i+h+1}{2} + z(h+2) + i + 1 \right)$$

$$= \sum_{z \in \mathbb{Z}} \left( \frac{n-i+h+1}{2} + z(h+2) \right) - \sum_{z \in \mathbb{Z}} \left( \frac{n+i-h-1}{2} - z(h+2) \right) = 0$$

Letting $\ell$ move along the window from $k - j$ to $k + j + 1$, we get from (5) that

$$i \sim_h [k \pm j] = \sum_{\ell = \max(0,k-j)}^{\min(k+j+1,h)} \sum_{z \in \mathbb{Z}} \left( \frac{n-i+\ell}{2} + z(h+2) \right) - \left( \frac{n-i+\ell}{2} + z(h+2) + i + 1 \right)$$

The sum for $z$ can be restricted to the range $[-n/4: n/4]$. Skipping over the impossible odd or even values (for which the denominators of the binomial coefficients are fractional), we arrive at the stated formula of Theorem 4:

$$i \sim_h [k \pm j] = \sum_{z = \lceil -n/4 \rceil}^{\lfloor n/4 \rfloor} \sum_{s = 0}^{+j} \left( \frac{n}{\lceil n+k-j \rceil + z(h+2) + s} \right) - \left( \frac{n}{\lceil n+k-j \rceil + z(h+2) + s + 1} \right)$$

Consider now only the cases considered in Theorem 1, which guarantee that $k - j \geq -1$ and that $k + j \leq h + 1$, so $s$ may run from 0 to $j$ without exception – bearing in mind (as shown above) that any instances when $k - j + 2s = 0, h + 1$ have no impact on the sum. Reversing the order of the second sum in (6), replacing $s$ with $j - s$, we get

$$i \sim_h [k \pm j] = \sum_{z \in \mathbb{Z}} \sum_{s = 0}^{j} \left( \frac{n}{\lceil n+k-j \rceil + z(h+2) + s} \right) - \left( \frac{n}{\lceil n+k-j \rceil + z(h+2) - s + 1} \right)$$

When $j > i$, the inner sums overlap (for $s > i$) and cancel each other. So the above sum is always equal to

$$\sum_{z \in \mathbb{Z}} \left( \frac{n}{\min(i,j)} \left( r + z(h+2) + s \right) - \left( r + z(h+2) + i + j - s + 1 \right) \right)$$

where $r = \lceil (n + k - i - j)/2 \rceil$. This is symmetric in $i$ and $j$; hence Theorem 1.

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Table 1. The number of paths $i \mapsto^n \ell$ for $i, \ell \in [0 : 4]$, $n \in [0 : 16]$. For example, $2 \mapsto^6 2 = 0 \mapsto^0 [0 : 4] = 4374$ and $3 \mapsto^10 3 = 4 \mapsto^4 [2 : 4] = 3281$. Like for a bishop on a chessboard, half the squares are unreachable from any given starting point. The few squares that require backward steps are likewise inaccessible. The particular path of Figures 2–4 is highlighted in blue boldface.
Table 2. Paths \( i \sim n \rightarrow \ell \) constrained to height 5, \( n \in [0:16] \).