Let $iH^n_j$ denote the number of lattice paths from $\langle 0, i \rangle$, $i$ avenues west of Union Square, to $\langle n, j \rangle$, northwest of the starting point, with $n$ (northwest or northeast diagonal) steps that stay within the boundaries $y = 0$ (Broadway) and $y = k$ (the Hudson River) for some fixed $k$. Each step takes one $\langle m, \ell \rangle \mapsto \langle m + 1, \ell \pm 1 \rangle$, with the proviso that $0 \leq \ell \pm 1 \leq k$.

The basic recurrence is

(1) $iH^n_{j+1} = iH^n_{j-1} + iH^n_{j+1}$ \hspace{1cm} j \neq 0, k

(2) $iH^n_0 = iH^n_1$

(3) $iH^n_{k+1} = iH^n_{k-1}$

Let

$A^n_j := iH^n_j$

$B^n_j = iH^n_j$

The problem raised by Johann Cigler,\(^1\) restated,\(^2\) is to show by bijection, for $k = 3$, that

(4) $A^n_0 + A^n_1 + A^n_2 + A^n_3 = B^n_1 + B^n_2$

or, more specifically,

(5) $A^n_0 + A^n_2 = B^n_1$ \hspace{1cm} n even

(6) $A^n_1 + A^n_3 = B^n_2$ \hspace{1cm} n odd

By symmetry (left–right and top–bottom), for all $k$,

(7) $iH^n_j = iH^n_i$

(8) $iH^n_j = k-iH^n_{k-j}$

The main point\(^3\) is simply that, for all $k$,

(9) $A^n_i = iH^n_i = iH^n_0 = iH^n_1 = iH^n_{n-1} = B^n_{i-1}$

\(^1\)https://www.researchgate.net/post/Is_there_a_simple_bijection_between_the_following_sets_A_n_and_B_n_which_are_counted_by_the_Fibonacci_numbers

\(^2\)My sequences of ordinate values for $B_n$ and Cigler’s are related by the transformation $j \mapsto 1 - j$.

\(^3\)Which is at the root of the answer given by Thomas Prellberg.

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Date: March 17, 2016.
See Tables 1 and 2, which appear like a constrained version of Pascal’s triangle. What this means is just that \( B \) tracks \( A \) exactly one step behind, because the only step from a border position (viz. 0, whence \( A \) begins) is away from it (to 1, where \( B \) begins). So every \( A \) path of length \( n \) is also a \( B \) path of length \( n - 1 \) starting one step later. It follows that the relation between the sum of paths \( A^n_j \), for \( j = 0, \ldots, k \), and the sum of paths \( B^n_j \) is the same as the relation between the sum of \( A^{n+1}_j \) and that of \( A^n_j \).

In particular,

\[
A^n_0 + A^n_2 = B^{n-1}_0 + B^{n-1}_2 = 1H^{n-1}_0 + 1H^{n-1}_2 = 1H^n_1 = B^n_1
\]

\[
A^n_1 + A^n_3 = B^{n-1}_1 + B^{n-1}_3 = 1H^{n-1}_1 + 1H^{n-1}_3 = 1H^n_2 = B^n_2
\]

regardless of \( k \) (as long as \( k \geq 3 \)).

In general

\[
\sum_j A^n_j = \sum_j B^{n-1}_j = \frac{1}{2} \left( \sum_j B^n_j + B^{n-1}_0 + B^{n-1}_k \right)
\]

Finally, it should be mentioned that it is well known [Mohanty, 1979] that the number of (shortest) lattice paths with horizontal and vertical steps going from \((0,0)\) to \((a,b)\) while avoiding the diagonal boundaries \(y = x + s\) and \(y = x - t\) is

\[
\sum_{\ell \in \mathbb{Z}} \left[ \binom{a + b}{b + \ell(s + t)} - \binom{a + b}{b + \ell(s + t) + t} \right]
\]

Rephrasing in our terms (letting \( a = \frac{n-i-j}{2}, b = \frac{n-i+j}{2}, s = k - i + 1, t = i + 1 \)):

\[
iH^n_j = \sum_{\ell \in \mathbb{Z}} \left[ \binom{n}{\frac{n-i+j}{2} + \ell(k+2)} - \binom{n}{\frac{n-i+j}{2} + \ell(k+2) + i + 1} \right]
\]

The sum can be bounded to the range

\[
-\left[ \frac{(n-i+j)/2+i+1}{k+2} \right] \leq \ell \leq \left[ \frac{(n-i+j)/2+n+1}{k+2} \right].
\]
### Table 1. $A^n$ vs. $B^{n-1}$ for $k = 3$.

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### Table 2. $A^n$ vs. $B^{n-1}$ for $k = 4$.

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BETWEEN BROADWAY AND THE HUDSON