BETWEEN BROADWAY AND THE HUDSON

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Let ${}_iH_j^n$ denote the number of lattice paths from $\langle 0, i\rangle$, i avenues west of Union Square, to $\langle n, j\rangle$, northwest of the starting point, with n (northwest or northeast diagonal) steps that stay within the boundaries $y = 0$ (Broadway) and $y = k$ (the Hudson River) for some fixed k. Each step takes one $\langle m, \ell\rangle \mapsto \langle m+1, \ell \pm 1\rangle$, with the proviso that $0 \le \ell \pm 1 \le k$.

The basic recurrence is

$$ {}_iH_j^{n+1} = {}_iH_{j-1}^n + {}_iH_{j+1}^n \qquad j \neq 0, k \tag{1}$$

$$ {}_iH_0^{n+1} = {}_iH_1^n \tag{2}$$

$$ {}_iH_k^{n+1} = {}_iH_{k-1}^n \tag{3}$$

Let

$$A_j^n := {}_0H_j^n$$
$$B_j^n = {}_1H_j^n$$

The problem raised by Johann Cigler,[1] restated,[2] is to show by bijection, for $k = 3$, that

$$A_0^n + A_1^n + A_2^n + A_3^n = B_1^n + B_2^n \tag{4}$$

or, more specifically,

$$A_0^n + A_2^n = B_1^n \qquad n \text{ even} \tag{5}$$

$$A_1^n + A_3^n = B_2^n \qquad n \text{ odd} \tag{6}$$

By symmetry (left–right and top–bottom), for all k,

$$ {}_iH_j^n = {}_jH_i^n \tag{7}$$

$$ {}_iH_j^n = {}_{k-i}H_{k-j}^n \tag{8}$$

The main point[3] is simply that, for all k,

$$A_i^n = {}_0H_i^n \overset{(7)}{=} {}_iH_0^n \overset{(2)}{=} {}_iH_1^n \overset{(7)}{=} {}_1H_i^{n-1} = B_i^{n-1} \tag{9}$$

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[1]`https://www.researchgate.net/post/Is_there_a_simple_bijection_between_the_following_sets_A_n_and_B_n_which_are_counted_by_the_Fibonacci_numbers`

[2]My sequences of ordinate values for B_n and Cigler's are related by the transformation $j \mapsto 1 - j$.

[3]Which is at the root of the answer given by Thomas Prellberg.

See Tables 1 and 2, which appear like a constrained version of Pascal's triangle.

What this means is just that B tracks A exactly one step behind, because the only step from a border position (viz. 0, whence A begins) is away from it (to 1, where B begins). So every A path of length n is also a B path of length $n-1$ starting one step later. It follows that the relation between the sum of paths A_j^n, for $j = 0, \ldots, k$, and the sum of paths B_j^n is the same as the relation between the sum of A_j^{n+1} and that of A_j^n.

In particular,

$$A_0^n + A_2^n \stackrel{(9)}{=} B_0^{n-1} + B_2^{n-1} = {}_1H_0^{n-1} + {}_1H_2^{n-1} \stackrel{(1)}{=} {}_1H_1^n = B_1^n$$

$$A_1^n + A_3^n \stackrel{(9)}{=} B_1^{n-1} + B_3^{n-1} = {}_1H_1^{n-1} + {}_1H_3^{n-1} \stackrel{(1)}{=} {}_1H_2^n = B_2^n$$

regardless of k (as long as $k \geq 3$).

In general

$$\sum_j A_j^n = \sum_j B_j^{n-1} = \frac{1}{2}\left(\sum_j B_j^n + B_0^{n-1} + B_k^{n-1}\right) \tag{10}$$

Finally, it should be mentioned that it is well known [Mohanty, 1979] that the number of (shortest) lattice paths with horizontal and vertical steps going from $(0,0)$ to (a,b) while avoiding the diagonal boundaries $y = x+s$ and $y = x-t$ is

$$\sum_{\ell \in \mathbb{Z}} \left[\binom{a+b}{b+\ell(s+t)} - \binom{a+b}{b+\ell(s+t)+t} \right] \tag{11}$$

Rephrasing in our terms (letting $a = \frac{n+i-j}{2}$, $b = \frac{n-i+j}{2}$, $s = k-i+1$, $t = i+1$):

$${}_iH_j^n = \sum_{\ell \in \mathbb{Z}} \left[\binom{n}{\frac{n-i+j}{2} + \ell(k+2)} - \binom{n}{\frac{n-i+j}{2} + \ell(k+2) + i + 1} \right] \tag{12}$$

The sum can be bounded to the range $-\left\lceil \frac{(n-i+j)/2+i+1}{k+2} \right\rceil \leq \ell \leq \left\lceil \frac{(n-i+j)/2+n+1}{k+2} \right\rceil$.

TABLE 1. A^n vs. B^{n-1} for $k = 3$.

			1		3		8		21		55		144		377		987		2584		6765		17711	
		1		3		8		21		55		144		377		987		2584		6765		17711		46368
	1		2		5		13		34		89		233		610		1597		4181		10946		28657	
1		1		2		5		13		34		89		233		610		1597		4181		10946		28657

		1		3		8		21		55		144		377		987		2584		6765		17711	
	1		3		8		21		55		144		377		987		2584		6765		17711		46368
1		2		5		13		34		89		233		610		1597		4181		10946		28657	
	1		2		5		13		34		89		233		610		1597		4181		10946		28657

TABLE 2. A^n vs. B^{n-1} for $k = 4$.

				1		4		13		40		121		364		1093		3280		9841		29524		88573
			1		4		13		40		121		364		1093		3280		9841		29524		88573	
		1		3		9		27		81		243		729		2187		6561		19683		59049		177147
	1		2		5		14		41		122		365		1094		3281		9842		29525		88574	
1		1		2		5		14		41		122		365		1094		3281		9842		29525		88574

			1		4		13		40		121		364		1093		3280		9841		29524		88573
		1		4		13		40		121		364		1093		3280		9841		29524		88573	
	1		3		9		27		81		243		729		2187		6561		19683		59049		177147
1		2		5		14		41		122		365		1094		3281		9842		29525		88574	
	1		2		5		14		41		122		365		1094		3281		9842		29525		88574