Cayley's Formula: A Page From The Book

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Abstract. We present a simple proof of Cayley's formula.

We give a short elementary proof of Cayley's famous formula for the enumeration T_n of free, unrooted trees with $n \geq 1$ labeled nodes. We first count $F_{n,k}$, the number of n-node forests composed of k rooted, directed trees, for $1 \leq k \leq n$. For the history of the formula, including Jim Pitman's use of directed forests, see [**1**, pp. 221–226].

The crux of the proof is simple double counting. There are two equivalent ways of counting the number of k-tree forests with one designated internal (nonroot) node, which shows, for all $k = 1, \dots, n-1$, that

$$(n-k)F_{n,k} = knF_{n,k+1}. \tag{*}$$

- For the left side of (*): Consider one of the $F_{n,k}$ forests with k trees. Designate any one of its $n-k$ internal nodes.
- For the right side: Consider one of the $F_{n,k+1}$ forests with $k+1$ trees. Choose any one of the n nodes, and hang from it any one of the k trees not containing that node. The root of that grafted subtree is the designated internal node.

Iterating (*) $n-1$ times gives

$$F_{n,1} = \tfrac{1}{n-1}\, nF_{n,2} = \tfrac{1}{n-1}\tfrac{2}{n-2}\, n^2 F_{n,3} = \cdots = \tfrac{1}{n-1}\tfrac{2}{n-2}\cdots\tfrac{n-1}{1}\, n^{n-1}F_{n,n}.$$

The k and $n-k$ factors all cancel each other out. Because there is precisely one way of turning n nodes into n distinct trees (each root being a whole tree), we have $F_{n,n} = 1$. Thus, the number $F_{n,1}$ of n-node rooted trees is n^{n-1}. Since any of the n nodes in a tree can be the root, $F_{n,1} = nT_n$; Cayley's formula, $T_n = n^{n-2}$, follows.

Applying (*) only $n-k$ times, yields $F_{n,k} = \binom{n}{k}kn^{n-k-1}$, for $k = 1, \dots, n$.

Alternatively, the relation $(k+1)R_{n,k} = knR_{n,k+1}$ for the number $R_{n,k}$ of n-node forests with k designated roots leads to $R_{n,k} = kn^{n-k-1}$ and to $T_n = R_{n,1} = n^{n-2}$.

As a final remark, there are $(n+1)^{n-1}$ rooted trees with $n+1$ nodes that all share the same root. Each corresponds to a rooted forest with n nodes—just chop off the root node. Therefore, the limit of the ratio of rooted labeled forests to rooted labeled trees, as their size grows, is $\lim_{n\to\infty}(n+1)^{n-1}/n^{n-1} = e$.

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REFERENCE

1. M. Aigner, G. M. Ziegler, *Proofs from THE BOOK*. Fifth edition. Illustrations by K. H. Hofmann. Springer-Verlag, Berlin, 2014.

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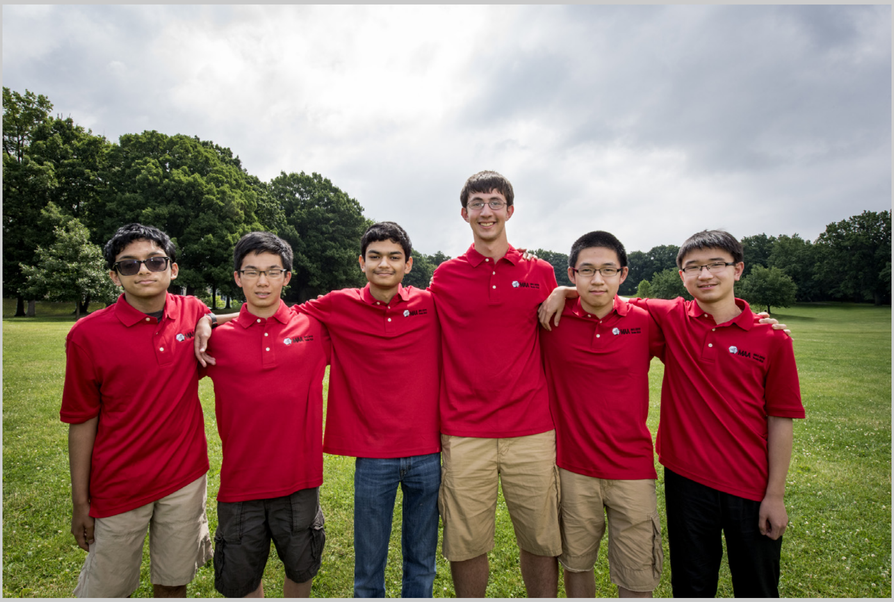

Congratulations to the U.S. team, Ankan Bhattacharya, Michael Kural, Allen Liu, Junyao Peng, Ashwin Sah, and Yuan Yao for their second consecutive win at the 57th International Mathematical Olympiad in Hong Kong! Not only did the team bring home first place for the U.S., but students Allen Liu and Yuan Yao earned perfect scores on the exam, and all six U.S. students took home a gold medal for their individual high scores.