Notations for Rewriting

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The best notation is no notation.
—Paul Halmos

Appropriate notations are important for stating complex results in a way that can be easily understood. Oftentimes, notation is crucial to carrying out correct and simple proofs. Our purpose here is to contribute to the development of good notations for term rewriting and related areas. To that end, we have engaged in many discussions with numerous colleagues, including algebraists. The term-rewriting community has already reached agreement on many of them; some others are controversial or new. Freese, McKenzie, McNulty and Taylor use (or will use) similar notations (with only minor variations) in their series of books. We do not, of course, expect that everybody will agree with all our suggestions, nor that they will all become a standard that must be followed to have a paper accepted for an EATCS conference. But we do hope that everyone will consider them (and the justifications we give for our choices) and compare them with what he or she is accustomed to. We, ourselves, enjoy using them, as do our students, and have adopted them in our recent work.

1 Linear algebra is an illuminating example.
2 Particularly those in “book-writing mode”.
3 This is the place to express our appreciation to Leo Bachmair, Claude Kirchner, Pierre Lescanne, George McNulty, David Plaisted, Wayne Snyder, and all the others for their many constructive suggestions. (This is not to imply that they necessarily concur with our decisions.)
6 To keep this note reasonably self-contained, we include inobvious definitions in footnotes.
7 At least those who have not been scared away.
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<td>$\leftrightarrow^*$</td>
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<td>$\mathop{\leftrightarrow}^*$</td>
<td>reflexive-symmetric-transitive closure of any arrow-like binary relation $\rightarrow^4$</td>
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$^1$Called derivability for arrow-like relations and reachability, in general. This notation is more adaptable than $\rightarrow$. A sequence $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_i \cdots$ is called a derivation of $\rightarrow$ issuing from $s_0$.

$^2$This seems more intuitive than $\rightarrow^*$.

$^3$That is, $s \rightarrow^! t$ if $s \rightarrow^* t$, but $t \not\rightarrow^* u$ for any $u$.

$^4$I.e. the smallest equivalence relation containing $\rightarrow$, called convertibility.
\begin{align*}
\uparrow & \quad \uparrow \downarrow \quad \text{common ancestor relation}^5 \\
\downarrow & \quad \downarrow \quad \text{common descendant relation}^6 \\
R(t) & \quad R(t) \quad R(t) \quad \text{set of normal forms}^7 \text{ of } t \text{ for binary relation } R \\
R(T) & \quad R(T) \quad R(T) \quad \text{normal forms of set } T \text{ for binary relation } R \\
T(F,X) & \quad T(F,X) \quad \{\text{cal } T(F,X)\} \quad \text{set of (finite, first-order) terms with function symbols } F^8 \text{ and variables } X \\
T & \quad T \quad \{\text{cal } T\} \quad \ldots \text{ for short} \\
T^\infty(F,X) & \quad T^\infty(F,X) \quad \{\text{cal } T\}^\infty\{\text{cal}(F,X)\} \quad \text{set of finite and infinite terms with function symbols } F \text{ and variables } X \\
T^\infty & \quad T^\infty \quad \{\text{cal } T\}^\infty \quad \ldots \text{ for short} \\
T^Q(F,X) & \quad T^Q(F,X) \quad \{\text{cal } T^Q\{\text{bf } Q\}(F,X)\} \quad \text{set of rational terms}^9 \text{ with function symbols } F \text{ and variables } X \\
T^Q & \quad T^Q \quad \{\text{cal } T^Q\{\text{bf } Q\}\} \quad \ldots \text{ for short}
\end{align*}

^5\text{I.e. the composed relation } \rightarrow^* \circ \rightarrow^*, \text{ also called “meetability”}.

^6\text{I.e. the composed relation } \rightarrow^* \circ \rightarrow^*, \text{ also called “joinability”. A binary relation } \rightarrow \text{ is said to be } \text{confluent} \text{ if any two elements with a common ancestor have a common descendent. This is equivalent to the Church-Rosser property: any two convertible elements have a common descendent.}

^7\text{A normal form for } t \text{ is any element } s \text{ such that } t \rightarrow^1 s. \text{ A relation } \rightarrow \text{ is said to be } \text{normalizing} \text{ if every element has at least one normal form, } \text{terminating} \text{ if its graph has no infinite chains, and } \text{convergent} \text{ if it is both terminating and confluent. Several alternatives to “terminating” have been used in the literature, including “Noetherian” (after the algebraist Emily Noether), “Artinian” (after E. Artin), “finitely terminating”, “uniformly terminating”, and “well-founded”. As there is a potential source of confusion between terms used for “no infinite ascending chain” and for “no infinite descending chain”, we chose the more meaningful “terminating”, and reserve well-founded for orderings. Poorer alternatives to “convergent” include “canonical” (cf. note 50) and “complete”.}

^8\text{It is convenient to write } F = \cup_n F_n, \text{ where } F_n \text{ is the set of symbols of } \text{arity} \text{ (or “rank”) } n.

^9\text{Rational terms are possibly infinite terms with finitely many different subterms } (\{t \in T^\infty : \|\{t_t \in F_{\alpha(t)}\} \| < \infty\}). \text{ They are solutions of equations between terms in the sense of Prolog II.}
\[ \mathcal{G}(\mathcal{F}) \quad \mathcal{G}(\mathcal{F}) \quad \{\mathcal{G}(\mathcal{F})\} \quad \text{set of ground terms}^{10} \text{ with function symbols } \mathcal{F}^{11} \]

\[ \mathcal{G} \quad \mathcal{G} \quad \{\mathcal{G}\} \quad \ldots \text{ for short}^{11} \]

\[ \text{Head}(t) \quad \text{Head}(t) \quad \{\text{Head}(t)\} \quad \text{function symbol heading term } t \]

\[ \text{Var}(t) \quad \text{Var}(t) \quad \{\text{Var}(t)\} \quad \text{set of variables occurring in a term } t \]

\[ \text{Pos}(t) \quad \text{Pos}(t) \quad \{\text{Pos}(t)\} \quad \text{set of positions in a term } t^{12} \]

\[ \text{FP\text{Pos}}(t) \quad \text{FP\text{Pos}}(t) \quad \{\text{FP\text{Pos}}(t)\} \quad \text{set of non-variable positions in a term } t^{13} \]

\[ \text{VP\text{Pos}}(t) \quad \text{VP\text{Pos}}(t) \quad \{\text{VP\text{Pos}}(t)\} \quad \text{set of variable positions in a term } t^{14} \]

\[ \Lambda \quad \Lambda \quad \text{top-most position}^{15} \]

\[ p \leq q \quad p \leq q \quad p \ \leq q \quad \text{position } p \text{ is above } q^{16} \]

\[ p \parallel q \quad p \parallel q \quad p \ \parallel q \quad \text{disjoint positions}^{17} \]

\[ p \not\parallel q \quad p \not\parallel q \quad p \ \not\parallel q \quad \text{non-disjoint positions}^{18} \]

\[ t|_p \quad t|_p \quad t|_{\neg p} \quad \text{subterm of } t \text{ at position } p^{19} \]

---

10 Ground terms are terms without variables.

11 \( \mathcal{G} \) can be superscripted to designate infinite, or rational, terms.

12 A position may be represented as a sequence of positive integers pointing to a particular subterm in a term (seen as an ordered labeled tree). Positions have often been called “occurrences” in the literature, thereby confusing a position with the subterm it designates. We prefer to restrict the use of occurrence to the latter. This suggests the use of Pos for the set of all positions in a term, rather than Dom.

13 I.e. \( \{p : t|_p \notin \mathcal{X}\} \) in the notation below.

14 I.e. \( \{p : t|_p \in \mathcal{X}\} \) in the notation below.

15 The position of the root or outermost symbol. Alternate symbols, e.g. \( \epsilon \), are overused.

16 A position \( p \) is above position \( q \) (and \( q \) is below \( p \)), if \( p \) is a prefix of \( q \).

17 Two positions are disjoint, or parallel, if neither is above the other.

18 Two positions are non-disjoint if one is above the other. That is, \( p \not\parallel q \).

19 This notation for subterm possesses a nice symmetry with the notations that follow: \( t|_{t|_p|_p} = t = s|_{t|_p} \).
\[
\begin{align*}
t[\cdot]_p & \quad t[\cdot]_p & \quad t[\cdot]_p \\
t[s]_p & \quad t[s]_p & \quad t[s]_p \\
t[s] & \quad t[s] & \quad t[s] \\
t[s_1, \ldots, s_n]_{p_1, \ldots, p_n} & \quad t[s_1, \ldots, s_n]_{p_1, \ldots, p_n} \\
t[s]_\Pi & \quad t[s]_\Pi & \quad t[s]_\Pi \\
\Dom(\sigma) & \quad \Dom(\sigma) & \quad \{\cal D\}om(\sigma) \\
\VRan(\sigma) & \quad \VRan(\sigma) & \quad \{\cal VR\}an(\sigma) \\
\{x_1 \mapsto s_1, \ldots\} & \quad \{x_1 \mapsto s_1, \ldots\} & \quad \{x_1 \mapsto s_1, \ldots\} \\
\sigma|_V & \quad \sigma|_V & \quad \sigma|_V \\
t\sigma & \quad t\sigma & \quad t\sigma \\
\sigma\rho & \quad \sigma\rho & \quad \sigma\rho
\end{align*}
\]

context \(t\) with designated position \(p\)\(^{20}\)  
subterm of \(t\) at position \(p\) is replaced by \(s\)\(^{21}\)  
... for short  
subterms of \(t\) at disjoint positions \(p_i\) are replaced by \(s_i\)  
subterms of \(t\) at set \(\Pi\) of disjoint positions are replaced by \(s\)  
variable-domain of substitution \(\sigma\)\(^{22}\)  
variable-range of substitution \(\sigma\)\(^{23}\)  
substitution with finite domain \{\ldots x_i \ldots\}\(^{24}\)  
substitution \(\sigma\) restricted to variables in set \(V\)  
application of substitution \(\sigma\) to term \(t\)\(^{25}\)  
composition of substitutions \(\sigma\) and \(\rho\)\(^{26}\)

\(^{20}\)A context is a term with a bound variable. The above notation is preferable to the more precise lambda-notation \(\lambda x.t[x]_p\).

\(^{21}\)This notation avoids extra arrows and allows for convenient abbreviation.

\(^{22}\)By variable-domain, or just domain, is meant the variables in \(X\) for which something else is substituted, i.e. \(\{x \in X : x\sigma \neq x\}\).

\(^{23}\)By variable-range, or just range, is meant the variables in \(X\) that are introduced by the substitution, i.e. \(\VRan(\sigma) = \cup_{x \in \Dom(\sigma)} \Var(x\sigma)\).

\(^{24}\)The symbol \(\mapsto\) is exactly what’s called for here. A substitution is called a renaming (“conversion” is also used) when all \(s_i\) are distinct variables.
Many authors prefer prefix notation. Sometimes, as here, intuitiveness should take precedence over prevalent mathematical usage.

That is, $\sigma$ followed by $\rho$. 

\footnote{Many authors prefer prefix notation. Sometimes, as here, intuitiveness should take precedence over prevalent mathematical usage.}{Many authors prefer prefix notation. Sometimes, as here, intuitiveness should take precedence over prevalent mathematical usage.}

\footnote{That is, $\sigma$ followed by $\rho$.}{That is, $\sigma$ followed by $\rho$.}
s = t  \quad s = t  \quad s = t \quad \text{syntactic equality of } s \text{ and } t^{27}

P^3(\cdots) \quad P^3(\cdots) \quad P^\approx(\cdots) \quad \text{satisfiability of a predicate } P(\cdots)

s = t^? \quad s = t^? \quad s \mathop{=}^? t \quad \text{syntactic unifiability of } s \text{ and } t^{28}

l \approx r \quad l \approx r \quad l \approx r \quad \text{equational axiom with left-hand side } l \text{ and right-hand side } r^{29}

l \approx r \quad l \approx r \quad l \approx r \quad \text{equational axiom with one side } l \text{ and other side } r^{30}

s \mathop{ightarrow^p} t \quad s \mathop{\leftarrow^p} t \quad s \mathop{\leftarrow^p} t \quad \text{application of axiom } e \text{ at position } p^{31}

s \mathop{\leftarrow^E} t \quad s \mathop{\leftarrow^E} t \quad s \mathop{\leftarrow^E} t \quad \text{... for short}

s =_E t \quad s =_E t \quad s =_E t \quad \text{equality of } s \text{ and } t \text{ in the models of the axioms } E^{32}

\vdash l \quad \vdash l \quad \vdash l \quad \text{inference using system } I

s \mathop{\rightarrow^s} t \quad s \mathop{\leftarrow^s} t \quad s \mathop{\leftarrow^s} t \quad \text{provability of equality of } s \text{ and } t \text{ with equational axioms } E^{33}

\text{Footnotes:

27 I.e. } s \text{ and } t \text{ are identical terms in } T.

28 \text{i.e. } \exists \sigma \ s \sigma = t \sigma. \text{ This notation treats unification as satisfiability of the equality predicate.}

29 \text{Here, equational axioms are ordered pairs; } l \text{ and } r \text{ do not play the same role. Their usual application, however, is symmetric: the equational theory associated with a set of axioms is the reflexive, symmetric, transitive closure of the relation } s[l\sigma] \approx s[r\sigma] \text{, for all contexts } s[\cdot]_p, \text{ substitutions } \sigma \text{ and axioms } l \approx r.

30 \text{Here, the axioms are viewed as unordered pairs, which is also very useful. In this case, the equational theory is simply the reflexive-transitive closure of the same relation as in the previous footnote. Accordingly, we use } \approx (\text{and not } \approx) \text{ when we wish to stress that } l \text{ and } r \text{ play the same role.}

31 \text{If } e = l \approx r, \text{ then } s|_p = l\sigma \text{ and } t = s[r\sigma]_p \text{ for some substitution } \sigma. \text{ If } e = l \approx r, \text{ then either } s|_p = l\sigma \text{ and } t = s[r\sigma]_p \text{ or } s|_p = r\sigma \text{ and } t = s[l\sigma]_p, \text{ for some substitution } \sigma.

32 \text{i.e. } \vdash s \approx t.

33 \text{i.e. } \vdash_E s \approx t \text{ (} s \approx t \text{ is in the theory of } E). \text{ By Birkhoff's Completeness Theorem for equational logic, } =_E \text{ and } \vdash^*_E \text{ coincide.}
\[ s_0 \overset{p_0}{\rightarrow} s_1 \cdots s_n \]

\[
I(E) \quad I(E) \quad \{\text{cal } I\}(E)
\]

\[
s = I(E) t \quad s = t \quad s \mathrel{\{\text{cal } I\}(E)} t
\]

\[
s = E t \quad s = t \quad s \mathrel{E} t
\]

\[ l \rightarrow r \quad l \rightarrow r \quad 1 \rightarrow r \]

\[ l \leftarrow r \quad l \leftarrow r \quad 1 \leftarrow r \]

\[ l = r \quad l = r \quad 1 \leftrightarrow r \]

\[ c \mid l \rightarrow r \quad c \mid l \rightarrow r \quad c \mid 1 \rightarrow r \]

\[ \overset{p}{\rightarrow} r \quad \overset{p}{\rightarrow} r \quad \mathrel{\{\text{rightarrow}\} \_ \_ \_ \_ \_ \_ r \_ \_ \_ \_ \_ \_ p}
\]

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34 That is, a proof is a finite derivation of the relation \( \Leftarrow \), “justified” by axiom and position. When necessary, the specific instance \( e; \sigma \) of an axiom can be indicated.

35 By definition, an equation is true in \( I(E) \) if all its ground instances are true in \( E \).

36 That is, satisfiability of \( s = E t \), or \( \exists \sigma \ s \sigma = E t \sigma \).

37 Ordered rewriting uses equational axioms in one direction or the other, depending on the instance under consideration. For example, one may use the associativity axiom \( (x+y)+z \approx x+(y+z) \) from left to right to rewrite \((c+b)+a \approx c+(b+a)\), or from right-to-left to rewrite \(a+(a+b) \approx (a+a)+b\). See note 43.

38 By definition, \( s \overset{l \rightarrow r}{\rightarrow} t \) if \( s \mid_p = l \sigma \) and \( t = s \mid r \sigma \), for some substitution \( \sigma \). The term \( s \mid_p \) is the redex. This notation adapts conveniently to give the relative position of a redex: \( \overset{p}{\rightarrow} r \) if the redex is other than \( p \); \( \overset{p}{\rightarrow} r \) if it’s strictly above \( p \); \( \overset{p}{\rightarrow} r \) if it’s strictly below; \( \overset{p}{\rightarrow} r \) if it’s either above or below; \( \overset{p}{\rightarrow} r \) if it’s neither. To specify the substitution \( \sigma \), called the match of pattern \( l \) to the redex,
we subscript by rule instance instead: $s \rightarrow_{\sigma}^{p} t$. For two-way rules, $s \rightarrow_{l-r}^{p} t$ if $s \rightarrow_{l-r}^{p} t$ or $s \rightarrow_{r-l}^{p} t$. 
The rewrite closure \( \rightarrow_R \) is the smallest relation containing \( R \) having the replacement property (\( s \rightarrow t \) implies \( u[s]_p \rightarrow u[t]_p \), for all terms \( s \) and \( t \) and contexts \( u[\cdot]_p \)) and full invariance property (\( s \rightarrow t \) implies \( s\sigma \rightarrow t\sigma \), for all terms \( s \) and \( t \) and for all substitutions \( \sigma \)). There is no reason to use the overabused adjective “monotonic” for relations with these closure properties. A term \( t \) is reducible (with respect to a system \( R \)) if a subterm of \( t \) is a redex, and is ground-reducible if every ground instance is reducible.

In in-line usage, we use the regular-length arrow; in displayed equations, we usually use long ones or variable length ones as here.

The notions of derivation and proof as justified derivation carry over to sequences of rewrites.

In parallel rewriting, zero or more disjoint redexes may be rewritten in one step. The Parallel Moves Lemma can accordingly be phrased: \( \rightarrow \parallel \circ \rightarrow \parallel \subseteq \rightarrow \parallel \circ \rightarrow \parallel \) when \( r \) and \( s \) are orthogonal rules. Rules are orthogonal (formerly called “regular”) if they are left-linear and their left-hand sides do not overlap.

By definition, \( s \rightarrow_{r^\triangleright} t \) if \( s \rightarrow_{r^\triangleright} t \) and \( s \triangleright t \). (This is the preferred definition.) We’re tempted to suggest the symbol \( \rightarrow_{r^\triangleright} \) instead, but this arrow requires a complicated definition in \( \text{BT}_{\text{ET}} \).

By definition, \( s \rightarrow_{R/S} t \) if \( s \rightarrow_S \circ \rightarrow_R \circ \rightarrow_S t \). This is (congruence-) class rewriting.
to the stipulation that all substitutions:

\[ s \triangleright R = s \triangleright R \]

is more meaningful than encompassment combines those for subterm and subsumption. Encompassment is used to define that

\[ \text{normal literal similarity} \]

By definition, \( s \triangleright R \) is the most general unifier of \( s \triangleright R \), and substitution

\[ s \triangleright R \]

subsumption ordering on terms or substitutions.

\[ s \equiv \]

 literal similarity of terms or substitutions

\[ \triangleright E \]

 subsumption ordering modulo \( E \)

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45 By definition, \( s \triangleright R \) if \( s \triangleright R \) is the most general unifier of \( s \triangleright R \). The suggested notation is meant to draw attention to the stipulation that all \( S \)-steps take place before the rewrite and at positions at or below it; it is more meaningful than \( R \).

46 By definition, \( s \triangleright R \) if \( s \triangleright R \) is the most general unifier of \( s \triangleright R \). Many of the same variations as used for \( \rightarrow \) apply to \( \triangleright \); e.g., \( s \triangleright R \) is the most general unifier of \( l \) and \( s \triangleright L \), for some \( s \triangleright R \), \( s \triangleright R \) means that \( \triangleright \) is a quasi-ordering on substitutions: \( s \triangleright R \) if \( s \triangleright R \) and \( s \triangleright R \) for \( \sigma \) as above.

47 Backward motions \( \triangleright \) and \( \triangleright \) may be fine-tuning situations, like footnotes.

48 By definition, \( t \triangleright E \) if \( t \triangleright E \) is admissible for \( s \). Subsumption is extended to a quasi-ordering on substitutions: \( s \triangleright E \) if \( s \triangleright E \) is admissible for \( s \). This explains the dot the two notations share.

49 Literal similarity is the equivalence associated with \( \triangleright \). That is \( s \equiv t \) and sometimes read as \( s \triangleright t \), for some \( \sigma \). This explains the dot the two notations share.

50 The encompassment ordering is defined as the subterm ordering composed with the subsumption ordering, i.e., \( t \triangleright E \) if \( t \triangleright E \) is admissible for \( s \). “Encompassment” conveys the notion that \( s \) “appears” in \( t \), but with context \( t \triangleright E \) above” and substitution \( \sigma \) “below” (and is therefore a better term than “containment” or “specialization”). This is why the notation for encompassment combines those for subterm and subsumption. Encompassment is used to define (inter-) reduced sets of rules, that is sets \( R \) of rules such that for every rule \( l \rightarrow r \), \( r \) is in normal form, as are all terms smaller than \( l \) in the encompassment ordering. (This coincides with the usual definition of “reduced” when \( R \) is convergent.) Since reduced convergent sets of rules enjoy
a uniqueness property, we term them canonical.

51 By definition, if and only if for some . This ordering, too, extends to an ordering on substitutions. So, the subsumption ordering modulo amounts to , respectively. By definition, if all , respectively. By definition, if is well-founded, then so is the strict part of . We write for the associated strict partial-ordering.

55 The strict part of this ordering is not well-founded, since sequences may be of arbitrary length.

56 By definition, is the smallest partial ordering containing the following relation between multisets: for and . If is well-founded, so is .

57 By definition, if and only if or if and there exist such that . By Kruskal’s Tree Theorem, homeomorphic embedding is a well-quasi ordering of if is a well-quasi-ordering of .
<table>
<thead>
<tr>
<th>$\succ_{\text{lpo}}$</th>
<th>$\succ_{\text{mpo}}$</th>
<th>$\succ_{\text{rpo}}$</th>
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<tbody>
<tr>
<td>$\mathop{\text{succ}}_{\text{lpo}}$</td>
<td>$\mathop{\text{succ}}_{\text{mpo}}$</td>
<td>$\mathop{\text{succ}}_{\text{rpo}}$</td>
</tr>
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| lexicographic path ordering with precedence $\succ^{58}$ | multiset path ordering with precedence $\succ^{59}$ | recursive path ordering with precedence $\succ^{60}$ |

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58 By definition, $s \succ t$ if $s \succ t |_{p \in P_{os}(t)}$ and one of the following holds: $\text{Head}(s) \succ \text{Head}(t)$, or $s |_{p \in P_{os}(s)} \succ t |_{p \in P_{os}(t)}$, or $\text{Head}(s) \sim \text{Head}(t)$ and $(s|_1, \ldots, s|_n) (\succ^{\text{rpo}}_{\text{lex}} (t|_1, \ldots, t|_n))$, where $n = \text{arity}(\text{Head}(s)) = \text{arity}(\text{Head}(t))$.

59 This is the original “recursive path ordering”, with multiset “status” of operators, for which the last case in note 58 is replaced by a comparison of multisets: $\{s|_1, \ldots, s|_{\text{arity}(\text{Head}(s))}\} (\succ^{\text{mpo}}_{\sim})_{\text{mult}} \{t|_1, \ldots, t|_{\text{arity}(\text{Head}(t))}\}$. Equivalence of multisets means that they are the same up to equivalence of elements under $\sim_{\text{mpo}}$ (that is, under the intersection of $\sim_{\text{mpo}}$ and its inverse $\preceq_{\text{mpo}}$).

60 With multiset and/or lexicographic “status” of operators.